ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.18(2014) No.3,pp.189-194

Regularization Method for the Two-dimensional Fredholm Integral Equations of the First Kind

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(Received 6 May 2014, accepted 28 September 2014)

Abstract: Recently, Wazwaz has proposed the regularization method to the one-dimentional linear Fredholm integral equations of the first kind in [1]. In this paper, we develop this method to the linear and nonlinear two-dimensional integral equations of the first kind. Indeed, the regularization method is used for linear equations directly. But nonlinear equations of the first kind are transformed to linear equations of the first kind by a change of variable, then the regularization method is applied. Some examples will be used to highlight the reliablity of the generalized regularization method.

Keywords: regularization method; two-dimetional integral equations of the first kind; nonlinear integarl equations.

1 Introduction

Integral equations of the first kind in one-dimensional case have been studied in many papers (see [2-11]). But although, this equations in two-dimensional case have many interesting applications in Mechanical engineering, Physical sciences and other applied sciences (see [12 , 13]), only a few papers have been written about them (see [13 - 17]).
In this paper, we consider general form of the two-dimensional Fredholm integral equations of the form

$$f(x,t) = \lambda \int_c^d \int_a^b K(x,t,y,z)G(h(y,z))dydz, \quad x\epsilon[a,b], \quad t\epsilon[c,d], \tag{1}$$

where f and K are continuous functions and λ is a constant. Also, G is a continuous function which has continuous inverse and finally u is the unknown function of the equation (1) to be find.

Obviously, if G is linear then the Eq. (1) wil be linear. As mentioned above, we develop the regularization method of [1] to linear case directly and in nonlinear case, we first set $u(x,t) = G(h(x,t))$ to convert (1) to linear form, then the regularization method combined with the generalized appropriate technique is applied to solve equation.

2 The regularization method

The regularization method was established independently by Tikhonov [18,19] and Philips [20]. This method consists of transforming first kind integral equations to second kind equations. In this section, we develop the regularization method to the two-dimensional integral equations of the first kind.

The regularization method transforms the two-dimensional Fredholm integral equation of the first kind as

$$f(x,t) = \int_c^d \int_a^b K(x,t,y,z)u(y,z)dydz, \tag{2}$$

to a two-dimensional linear Fredholm integral equation of the second kind as

$$\alpha u_\alpha(x,t) = f(x,t) - \int_c^d \int_a^b K(x,t,y,z)u_\alpha(y,z)dydz, \tag{3}$$

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IJNS.2014.12.15/840

where α is a small positive parameter. It is clear that the Eq. (3) can be rewritten as

$$u_\alpha(x,t) = \frac{1}{\alpha} f(x,t) - \frac{1}{\alpha}\int_c^d \int_a^b K(x,t,y,z) u_\alpha(y,z)dydz, \tag{4}$$

and the solution $u_\alpha(x,t)$ of the Eq. (4) converges to the solution u of (2) as $\alpha \to 0$. In other words, it was shown that [21]

$$u(x,t) = \lim_{\alpha\to 0} u_\alpha(x,t). \tag{5}$$

Now we give some results about existence and uniqueness of solution.
To this end, we define the integral operator $A : \mathcal{C}([a,b]\times[c,d]) \to \mathcal{C}([a,b]\times[c,d])$ as

$$Au(x,t) = \int_c^d \int_a^b K(x,t,y,z)u(y,z)dydz, \quad x \in [a,b], t \in [c,d]. \tag{6}$$

Theorem 1 *Let $K : \mathcal{C}([a,b]\times[c,d]\times[a,b]\times[c,d]) \to R$ be continuous, then the operator A defined by (6) is bounded.*

Proof. See[22]. ■

Theorem 2 *For any $\alpha > 0$, $f \in \mathcal{C}([a,b]\times[c,d])$ and bounded linear operator A, the equation (4) is solvable and has a unique solution.*

Proof. See[21]. ■

It is important to note that the two-dimensional Fredholm integral equations of the first kind are ill-posed problems. The solution for an ill-posed problem may not exist, and if it exists it may be non-unique. We will apply the regularization method to convert the first kind Fredholm integral equation to the second kind integral equation. Then, the resulting second kind integral equaion will be solved by the generalized appropriate technique that will be presented in the next section.

3 Some solution techniques

As we know, there are many analitic methods to solve integral equations [23], particularly in one-dimensional case. In this section, we extend some availble techniques for one-dimensional equations to the two-dimensional Fredholm integral equations as:

$$u(x,t) = f(x,t) + \lambda \int_c^d \int_a^b K(x,t,y,z)u(y,z)dydz. \tag{7}$$

and the two-dimensional Volttera integarl equations as:

$$u(x,t) = f(x,t) + \lambda \int_c^t \int_a^x K(x,t,y,z)u(y,z)dydz. \tag{8}$$

3.1 Successive approximations method

The successive approximations method is an analitical method to solve the Fredholm integral equations. This method by starting with an intial guess, which is called zeroth approximation, produces a sequence of approximations by a recurrence relation as

$$u_{n+1}(x,t) = f(x,t) + \int_c^d \int_a^b K(x,t,y,z)u_n(y,z)dydz, \quad n = 0,1,\cdots, \tag{9}$$

where the zeroth approximation $u_0(x,t)$ can be any selective real valued function, but mostly it is selected as $u_0(x,t) = 0, 1$. It can be shown that the sequence $\{u_n\}_{n=0}^{\infty}$ converge to the solution of the equation (7).

3.2 Direct computation method

The direct computation method is one of the most important methods to solve Fredholm integral equations. It is important to point out that, this method is only applicable for equations with degenerate or separable kernel as:

$$K(x,t,y,z)=\sum_{i=1}^{n} g_i(x,t)h_i(y,z). \tag{10}$$

To apply the direct computation method to (7), substituting from (10) to (7) implies

$$u(x,t)=f(x,t)+\lambda\sum_{i=1}^{n} g_i(x,t)\int_c^d\int_a^b h_i(y,z)u(y,z)dydz. \tag{11}$$

or

$$u(x,t)=f(x,t)+\lambda g_1(x,t)\int_c^d\int_a^b h_1(y,z)u(y,z)dydz+\lambda g_2(x,t)\int_c^d\int_a^b h_2(y,z)u(y,z)dydz$$
$$+\cdots+\lambda g_n(x,t)\int_c^d\int_a^b h_n(y,z)u(y,z)dydz.$$

Each integral at the right side depends only on the variables y and z, so by setting

$$\alpha_i=\int_c^d\int_a^b h_i(y,z)u(y,z)dydz, \quad i=1,2,\cdots,n, \tag{12}$$

we obtain

$$u(x,t)=f(x,t)+\lambda\left(\alpha_1 g_1(x,t)+\alpha_2 g_2(x,t)+\cdots+\alpha_n g_n(x,t)\right). \tag{13}$$

Substituting (13) into (12) yeilds

$$\alpha_i-\lambda\sum_{j=1}^{n}\left(\int_c^d\int_a^b h_i(y,z)g_j(y,z)dydz\right),\alpha_i=\int_c^d\int_a^b h_i(y,z)f(y,z)dydz,\ i=1,\cdots,n \tag{14}$$

which is a system of n algebraic equations with unknowns $\alpha_1,\alpha_2,\cdots,\alpha_n$, so by solving the system (14) and using the obtained numerical valuse of $\alpha_1,\alpha_2,\cdots,\alpha_n$ into (13), the solution $u(x,t)$ of the Fredholm integral equation (7) is obtained.

3.3 Adomian decomposition method

In the Adomian decomposition method, which is applicable for both Fredholm and Volterra integral equations, the solution $u(x,t)$ of equation is assumed as

$$u(x,t)=\sum_{n=0}^{\infty} u_n(x,t),$$

in which the zeroth component $u_0(x,t)$ is identified by all terms that are not included under the integral sign and the other components $u_j(x,t)$, $j=1,2,\cdots$ are completely determined by setting the recurrence relation:

$u_0(x,t)=f(x,t), \quad u_{n+1}(x,t)=f(x,t)+\lambda\int_c^d\int_a^b K(x,t,y,z)u_n(y,z)dydz, \quad n=0,1,\cdots.$

4 Numerical examples

In what follows, we apply the regularization method combined with a generalized appropriate technique to illustrate the presented method.

Example 1 *Use the regularization method to solve the two-dimensional Fredholm integral equation of the first kind*

$$\frac{7}{12}(x+t)=\int_0^1\int_0^1 (x+t)yu(y,z)dydz. \tag{15}$$

The regularization method convert Eq. (15) *to*

$$\alpha u_\alpha(x,t) = \frac{7}{12}(x+t) - \int_0^1 \int_0^1 (x+t) y u_\alpha(y,z) dydz, \tag{16}$$

or

$$u_\alpha(x,t) = \frac{7}{12\alpha}(x+t) - \frac{1}{\alpha} \int_0^1 \int_0^1 (x+t) y u_\alpha(y,z) dydz. \tag{17}$$

The resulting two-dimensional Fredholm integral equation of the second kind can be solved by the direct computation method. To do this, Eq. (17) *can be written as*

$$u_\alpha(x,t) = \frac{7}{12\alpha}(x+t) - \frac{1}{\alpha}(x+t)C, \tag{18}$$

where C is a constant given by

$$C = \int_0^1 \int_0^1 y u_\alpha(y,z) dydz. \tag{19}$$

Substituting (18) *into* (19) *implies*

$$C = \frac{(\frac{7}{12})^2}{\frac{7}{12} + \alpha}. \tag{20}$$

So we obtain

$$u_\alpha(x,t) = \frac{7}{7 + 12\alpha}(x+t). \tag{21}$$

Finally the solution $u(x,t)$ of Eq. (15) *can be obtained as*

$$u(x,t) = \lim_{\alpha \to 0} u_\alpha(x,t) = x + t. \tag{22}$$

Example 2 *In this example, we consider a nonlinear equation as follows*

$$2(x+t) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (x+t) \sin(g(y,z)) dydz. \tag{23}$$

By using a change of variable as $u(y,z) = \sin(g(y,z))$, the Eq. (23) *yeilds*

$$2(x+t) = \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (x+t) u(y,z) dydz. \tag{24}$$

So by using the regularization method in Eq. (24)*, we obtain*

$$\alpha u_\alpha(x,t) = 2(x+t) - \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (x+t) u_\alpha(y,z) dydz. \tag{25}$$

or

$$u_\alpha(x,t) = \frac{2}{\alpha}(x+t) - \frac{1}{\alpha} \int_0^{\frac{\pi}{2}} \int_0^{\frac{\pi}{2}} (x+t) u_\alpha(y,z) dydz. \tag{26}$$

The resulting two-dimensional Fredholm integral equation of the second kind will be solved by the Adomian decomposition method. To this end, we set

$$u_\alpha(x,t) = \sum_{n=0}^{\infty} u_{\alpha_n}(x,t) \tag{27}$$

where

$$u_{\alpha_0}(x,t) = \frac{2}{\alpha}(x+t), u_{\alpha_n}(x,t) = \frac{2(-1)^n}{\alpha^{n+1}}(x+t)\frac{\pi^3}{8}, \tag{28}$$

substituting this result into (27) *gives the approximate solution*

$$u_\alpha(x,t) = \frac{32}{8\alpha + \pi^3}(x+t), \tag{29}$$

so the solution $u(x,t)$ *of Eq.* (24) *can be obtained as*

$$u(x,t) = \lim_{\alpha \to 0} u_\alpha(x,t) = \frac{32}{\pi^3}(x+t). \tag{30}$$

So the solution of the Eq. (23) *is*

$$g(x,t) = \arcsin\left(\frac{32}{\pi^3}(x+t)\right). \tag{31}$$

Example 3 *We consider a two-dimentional linear Volterra integral equation as*

$$\cosh x \cosh t = \int_0^t \int_0^x \coth x \coth t u(y,z) dydz. \tag{32}$$

Using the regularization method to Eq. (32) *implies*

$$\alpha u_\alpha(x,t) = \cosh x \cosh t - \int_0^t \int_0^x \coth x \coth t u_\alpha(y,z) dydz, \tag{33}$$

so

$$u_\alpha(x,t) = \frac{1}{\alpha} \cosh x \cosh t - \frac{1}{\alpha} \int_0^t \int_0^x \coth x \coth t u_\alpha(y,z) dydz. \tag{34}$$

The resulting two-dimensional Volterra integral equation of the second kind will be solved by the generalized Adomian decomposition method. The generalized Adomian decomposition method admits the use of

$$u_\alpha(x,t) = \sum_{n=0}^{\infty} u_{\alpha_n}(x,t), \tag{35}$$

where

$$u_{\alpha_0}(x,t) = \frac{1}{\alpha} \cosh x \cosh t, \tag{36}$$

$$u_{\alpha_1}(x,t) = -\frac{1}{\alpha} \int_0^t \int_0^x \frac{1}{\alpha} \coth x \coth t \cosh y \cosh z dydz = -\frac{1}{\alpha^2} \cosh x \cosh t. \tag{37}$$

$$u_{\alpha_2}(x,t) = -\frac{1}{\alpha} \int_0^t \int_0^x \frac{-1}{\alpha^2} \coth x \coth t \cosh y \cosh z dydz = \frac{1}{\alpha^3} \cosh x \cosh t. \tag{38}$$

$$\vdots$$

so that

$$u_{\alpha_n}(x,t) = \frac{(-1)^n}{\alpha^{n+1}} \cosh x \cosh t, \tag{39}$$

substituting this result into (35) *gives the approximate solution*

$$u_\alpha(x,t) = \left(\frac{1}{\alpha} - \frac{1}{\alpha^2} + \frac{1}{\alpha^3} - \cdots\right) \cosh x \cosh t, \tag{40}$$

and the solution $u(x,t)$ *of Eq.* (32) *can be obtained as[1]*

$$u(x,t) = \lim_{\alpha \to 0} u_\alpha(x,t) = \cosh x \cosh t, \tag{41}$$

which is the exact solution.

5 Conclusion

In this paper, we used the regularization method combined with extended techniques to solve linear and nonlinear two-dimetional integral equations of the first kind. As examples showed, this method is simple and it often gives the exact solution. It seems that, this simple method may be used to solve multiple integral equations of the first kind.

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