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Solving System of Fractional Order Partial Differential Equations by the Reduced Differential Transform Method

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Abstract: In this paper, we illustrate application of the reduced differential transform method (RDTM) for solving systems of fractional order partial differential equations. The illustration will be based on formulation of the method on a general form of the fractional order PDE systems and using it on some examples. One of the most important advantages of this method is its simplicity in using.

Keywords: differential transform; system of fractional order partial differential equations; reduced Fractional Differential transform

1 Introduction

We consider a fractional order partial differential equation of theform

$$a(x,t)D_x u(x,t) + b(x,t)D_t^{\gamma} u(x,t) + F(x,t,u(x,t)) = f(x,t), x,t \in [0,1] \tag{1}$$

where D_x and D_t^{γ} are the fractional and ordinary differential operators respectively with respect to x and y and

$$a(x,t)D_x = \begin{pmatrix} a_{11}(x,t)D_x^{m_{11}} & a_{12}(x,t)D_x^{m_{12}} & \dots & a_{1n}(x,t)D_x^{m_{1n}} \\ a_{21}(x,t)D_x^{m_{21}} & a_{22}(x,t)D_x^{m_{22}} & \dots & a_{2n}(x,t)D_x^{m_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}(x,t)D_x^{m_{n1}} & a_{n2}(x,t)D_x^{m_{n2}} & \dots & a_{nn}(x,t)D_x^{m_{nn}} \end{pmatrix},$$

$$b(x,t)D_t^{\gamma} = \begin{pmatrix} b_{11}(x,t)D_t^{\gamma_{11}} & b_{12}(x,t)D_t^{\gamma_{12}} & \dots & b_{1n}(x,t)D_t^{\gamma_{1n}} \\ b_{21}(x,t)D_t^{\gamma_{21}} & b_{22}(x,t)D_t^{\gamma_{22}} & \dots & b_{2n}(x,t)D_t^{\gamma_{2n}} \\ \vdots & \vdots & \ddots & \vdots \\ b_{n1}(x,t)D_t^{\gamma_{n1}} & b_{n2}(x,t)D_t^{\gamma_{n2}} & \dots & b_{nn}(x,t)D_t^{\gamma_{nn}} \end{pmatrix},$$

with $m_{ij} \in \mathbb{Z}^+$ and $q-1 \le \gamma_{ij} < q, \quad q \in \mathbb{Z}^+$ for $i,j = 1,\cdots,n$,

$$f(x,t) = \begin{pmatrix} f_1(x,t) \\ \vdots \\ f_n(x,t) \end{pmatrix} \quad and \quad u(x,t) = \begin{pmatrix} u_1(x,t) \\ \vdots \\ u_n(x,t) \end{pmatrix}.$$

We consider the initial conditions for the problem (1) of the form

$$u(x,0) = C(x) = \begin{pmatrix} c_1(x) \\ \vdots \\ c_n(x) \end{pmatrix}, \quad u(0,t) = E(t) = \begin{pmatrix} e_1(t) \\ \vdots \\ e_n(t) \end{pmatrix}.$$

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Note that the functions $a(x,t)$, $b(x,t)$, $f(x,t)$, $C(x)$ and $E(t)$ are given continuous functions, and $F(x,t,u(x,t))$ is generally a non-linear term in terms of the unknown function $u(x,t)$.

The fractional integro-differential equations and fractional differential equations have been the focus of many studies due to their frequent appearance in various fields such as physics, chemistry, biology, engineering and so on [1–3]. The concept of the differential transform method was first proposed in [1], and then applied for solving differential equations and systems of differential equations in [2]. Arikoglu [3] used this method for solving fractional differential equations. Shahmorad and et al. [4] used this method for solving fractional-order integro-differential equations with nonlocal boundary conditions. In recent years, some numerical methods have been used successfully to find the solution of various types of ordinary and partial fractional differential equations. Some of these methods are: the homotopy analysis method [6, 7], Adomian decomposition method [8, 9], variational iteration method [10] and fractional difference method [11, 12]. Recently a reduced form of the differential transform has been introduced and developed for solving ordinary and partial equations such as gas dynamics equation [13], generalized KDV equation [14] and so on. In this paper, we solve a system of fractional order partial differential equation of the form (1) by using Reduced Fractional differential Transform Method (RFDTM). The main advantages of this method are: its simplicity and directly using it to the problems without linearizing and discretizing.

2 Reduced differential transform method

In this section, we recall the fractional derivative and fractional RDTM definitions and summarize properties of the RDT in the proposition 1.

Definition 1 *The fractional differentiation in the Riemann-Liouville sense is defined by*

$$D_{t_0}^{\beta}u(x,t) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dt^m}\int_{t_0}^{t}(t-\zeta)^{m-\beta-1}u(x,\zeta)d\zeta, \ \ x > x_0 \tag{2}$$

for $m-1 \le \beta < m, m \in \mathbb{Z}^+$ *and* $t > t_0$.

In order to avoid fractional initial and boundary conditions, the fractional derivative is defined in the Caputo sense:

Definition 2 *[5] The fractional differentiation in the Caputo sense is defined by*

$$D_{*t_0}^{\beta}u(x,t) = \frac{1}{\Gamma(m-\beta)}\frac{d^m}{dt^m}\left\{\int_{t_0}^{t}\left[\frac{u(x,\zeta)-\sum_{k=0}^{m-1}\frac{1}{k!}(\zeta-t_0)^k u^{(k)}(x,t_0)}{(t-\zeta)^{1+\beta-m}}\right]d\zeta\right\}. \tag{3}$$

We expand the analytical function $u(x,t)$ as the fractional power series :

$$u(x,t) = \sum_{k=0}^{\infty}U_k(x)(t-t_0)^{k/\alpha}, \tag{4}$$

where α is the order of fraction and

$$U_k(x) = \frac{1}{\Gamma(\beta k+1)}[(D_{*t_0}^{\beta})^k u(x,t)]_{t=t_0} \tag{5}$$

is the reduced fractional differential transform of $u(x,t)$.

Since the initial conditions are implemented for the integer order derivatives, the transformation of the initial conditions are defined as follows:

$$U_k(x) = \begin{cases} \frac{1}{(k/\alpha)!}\left[\frac{d^{k/\alpha}}{dt^{k/\alpha}}u(x,t)\right]_{t=t_0}, & (k/\alpha) \in \mathbb{Z}^+, k = 0,1,\ldots(\alpha\,\gamma-1) \\ 0, & (k/\alpha) \notin \mathbb{Z}^+, \end{cases} \tag{6}$$

where γ is the order of fractional differential operator D_t^{γ}.

Proposition 1 *[15] Let $W_k(x)$, $U_k(x)$ and $V_k(x)$ be reduced differential transforms of the functions $w(x,t)$, $u(x,t)$ and $v(x,t)$ respectively. Then*

(a). *If $w(x,t) = au(x,t) \pm bv(x,t)$ for $a, b \in R$, then*

$$W_k(x) = aU_k(x) \pm bV_k(x).$$

(b). *If $w(x,t) = x^m t^n$, then*

$$W_k(x) = x^m \delta(k-n),$$

where

$$\delta(k) = \begin{cases} 1 & k = 0 \\ 0 & k \neq 0. \end{cases}$$

(c). *If $w(x,t) = x^m t^n u(x,t)$, then*

$$W_k(x) = x^m U_{k-n}(x).$$

(d). *If $w(x,t) = u(x,t)v(x,t)$, then*

$$W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x) = \sum_{r=0}^{k} U_{k-r}V_r(x).$$

(e). *If $w(x,t) = u_1(x,t)u_2(x,t)...u_{n-1}(x,t)u_n(x,t)$, then*

$$W_k(x) = \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} ... \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} U_{1,k_1}(x)U_{2,k_2-k_1}(x)...U_{n-1,k_{n-1}-k_{n-2}}(x)U_{n,k-k_{n-1}}(x).$$

(f). *If $w(x,t) = \frac{\partial^m}{\partial x^m}u(x,t)$, then*

$$W_k(x) = \frac{\partial^m}{\partial x^m} U_k(x).$$

(g). *If $w(x,t) = \frac{\partial^r}{\partial t^r}u(x,t)$, then*

$$W_k(x) = (k+1)(k+2)...(k+r)U_{k+r}(x) = \frac{(k+r)!}{k!}U_{k+r}(x).$$

(h). *If $w(x,t) = \frac{\partial^\beta}{\partial t^\beta}u(x,t)$, then*

$$W_k(x) = \frac{\Gamma(\beta + 1 + k/\alpha)}{\Gamma(1 + k/\alpha)} U_{k+\alpha\beta}(x).$$

(i). *If $w(x,t) = \frac{\partial^{\beta_1}}{\partial t^{\beta_1}}[u_1(x,t)] \frac{\partial^{\beta_2}}{\partial t^{\beta_2}}[u_2(x,t)] \cdots \frac{\partial^{\beta_{n-1}}}{\partial t^{\beta_{n-1}}}[u_{n-1}(x,t)] \frac{\partial^{\beta_n}}{\partial t^{\beta_n}}[u_n(x,t)]$, then*

$$\begin{aligned} W_k(x) &= \sum_{k_{n-1}=0}^{k} \sum_{k_{n-2}=0}^{k_{n-1}} \cdots \sum_{k_2=0}^{k_3} \sum_{k_1=0}^{k_2} \frac{\Gamma(\beta_1 + 1 + k_1/\alpha)}{\Gamma(1 + k_1/\alpha)} \\ &\times \frac{\Gamma(\beta_2 + 1 + (k_2 - k_1)/\alpha)}{\Gamma(1 + (k_2 - k_1)/\alpha)} \cdots \frac{\Gamma(\beta_{n-1} + 1 + (k_{n-1} - k_{n-2})/\alpha)}{\Gamma(1 + (k_{n-1} - k_{n-2})/\alpha)} \\ &\times \frac{\Gamma(\beta_{n-1} + 1 + (k_{n-1} - k_{n-2})/\alpha)}{\Gamma(1 + (k_{n-1} - k_{n-2})/\alpha)} \frac{\Gamma(\beta_n + 1 + (k_n - k_{n-1})/\alpha)}{\Gamma(1 + (k_n - k_{n-1})/\alpha)} U_{1,k_1+\alpha\beta_1}(x) \\ &\times U_{2,k_2-k_1+\alpha\beta_2}(x) \cdots U_{n-1,k_{n-1}-k_{n-2}+\alpha\beta_{n-1}}(x) U_{n,k-k_{n-1}+\alpha\beta_n}(x) \end{aligned}$$

where $\alpha\beta_i \in \mathbb{Z}^+$ for $i = 1, 2, \cdots n$.

3 Numerical examples

In this section, we illustrate by the following examples

Example 1 *Consider the partial system of differential equation*

$$\begin{aligned}\frac{\partial^{1/2}u_1(x,t)}{\partial t^{1/2}} &+ \frac{\partial^2 u_2(x,t)}{\partial x^2} = 3u_1(x,t)+x^2t\\ \frac{\partial^{1/3}u_2(x,t)}{\partial t^{1/3}} &+ \frac{\partial^3 u_1(x,t)}{\partial x^3} = u_2^2(x,t)-x^2t^2\end{aligned} \tag{7}$$

subject to the initial conditions

$$u_1(x,0)=\sin(x),\quad u_2(x,0)=\cos(x). \tag{8}$$

By choosing $\gamma_{11}=2$, $\gamma_{21}=3$ and applying the RDT to the both sides of Eq.(7) and using proposition 1, it transforms to

$$\begin{aligned}U_{1,k+1}(x) &= \frac{\Gamma(\frac{k}{2}+1)}{\Gamma(\frac{k}{2}+\frac{3}{2})}\left(-\frac{\partial^2}{\partial x^2}U_{2,k}(x)+3U_{1,k}(x)+x^2\delta(k-1)\right)\\ U_{2,k+1}(x) &= \frac{\Gamma(\frac{k}{3}+1)}{\Gamma(\frac{k}{3}+\frac{4}{3})}\left(-\frac{\partial^3}{\partial x^3}U_{1,k}(x)+\sum_{r=0}^{k}U_{2,r}(x)U_{2,k-r}(x)-x^2\delta(k-2)\right)\end{aligned}$$

The reduced differential transform of the initial conditions are given by

$$u_1(k,0)=\sin(x)\quad U_2(k,0)=\cos(x)$$

Thus the approximate solutions, corresponding to $N=2$ and $N=3$, are listed as follows.
$N=2:$

$$\begin{aligned}u_1(x,t) &= \sin(x)+t^{\frac{1}{2}}(3.38513\sin(x)+1.128379\cos(x))+t(8.007561\sin(x)+3\cos(x)\\ &+ 0.886226x^2+3.96975\cos(x)^2-1.984876)\\ u_2(x,t) &= \cos(x)-0.1000000000\times10^{-8}t^{\frac{1}{3}}(0.111984652\times10^{10}\sin(x)-0.111984\times10^{10}\cos(x)^2\\ &+ 0.110773\times10^{10}\cos(x)t^{\frac{1}{3}}+0.1107732\times10^{11}t^{\frac{1}{3}}\cos(x)\sin(x)-0.221546\times10^{10}t^{\frac{1}{3}})\end{aligned}$$

$N=3:$

$$\begin{aligned}u_1(x,t) &= \sin(x)+t(8.007562\sin(x)+0.886227x^2+3.969752\cos(x)^2+3\cos(x)-1.984876)\\ &+ t^{\frac{1}{2}}(3.385137\sin(x)+1.128379\cos(x))+t^{\frac{3}{2}}(18.071131\sin(x)-33.33178\cos(x)\sin(x)\\ &\quad 1.99999x^3+14.999302\cos(x)^3+8.95877\cos(x)^2-4.06255\cos(x)-4.4793860)\\ u_2(x,0) &= \cos(x)+t^{\frac{1}{3}}(-1.119846\sin(x)+1.119846\cos(x)^2+t^{\frac{2}{3}}(-11.077321\cos(x)\sin(x)\\ &+ 2.21546\cos(x)^3)-1.1077321\cos(x))+t(+41.1320933+13\sin(x)-76.26418\sin(x)\cos(x)^2\\ &- 83.132093\cos(x)^2+5.132093\cos(x)^4-0.9027452x^2)\end{aligned}$$

Example 2 *Consider a nonlinear system of fractional order partial differential equation of the form*

$$\begin{aligned}\frac{\partial^{1/4}u_1(x,t)}{\partial t^{4/3}} &- 6u_2(x,t)u_3(x,t)=xt+\exp(x)\\ \frac{\partial^{1/3}u_2(x,t)}{\partial t^{1/3}} &+ \frac{\partial^2 u_3(x,t)}{\partial x^2}=\frac{1}{2}xt+x^3t^2\\ \frac{\partial^{1/2}u_3(x,t)}{\partial t^{1/2}} &+ \frac{\partial^2 u_2(x,t)}{\partial x^2}=xt^3\end{aligned} \tag{9}$$

subject to the initial conditions

$$u_1(x,0) = \sin(x) \quad u_2(x,0) = \cos(x) \quad u_3(x,0) = exp(x) \tag{10}$$

By choosing $\gamma_{11} = 4$, $\gamma_{21} = 3$, $\gamma_{31} = 2$ and applying the RDT to the both sides of Eq.(3.3) and using proposition 1, it transforms to

$$\begin{aligned}
U_{1,k+1}(x) &= \frac{\Gamma(\frac{k}{4}+1)}{\Gamma(\frac{k}{4}+\frac{5}{4})}\left(6\sum_{r=0}^{k} U_{2,r}(x)U_{3,k-r}(x) + x\delta(k-1) + \delta(k)exp(x)\right)\\
U_{2,k+1}(x) &= \frac{\Gamma(\frac{k}{3}+1)}{\Gamma(\frac{k}{3}+\frac{4}{3})}\left(-\frac{\partial^2}{\partial x^2}U_{3,k}(x) + \frac{1}{2}x\delta(k-1) + x^3\delta(k-2)\right)\\
U_{3,k+1}(x) &= \frac{\Gamma(1+\frac{k}{2})}{\Gamma(\frac{3}{2}+\frac{k}{2})}\left(-\frac{\partial^2}{\partial x^2}U_{2,k}(x) + x\delta(k-3)\right)
\end{aligned}$$

The reduced differential transform of the initial condition is given by

$$U_1(k,0) = \sin(x), \quad U_2(k,0) = \cos(x), \quad U_3(k,0) = \exp(x),$$

The approximate solutions, corresponding to $N = 3$ and $N = 4$, are listed as follows.
$N = 3$*:*

$$u_1(x,t) = \sin(x) + t^{\frac{1}{4}}(1.103262 + 6\cos(x)e^{(x)} + e^x) + t^{\frac{1}{2}}(1.022765 + 6.77027\cos(x)^2 - 6.719079e^{2x} + x)$$
$$+ t^{\frac{3}{4}}(0.964272 + 5.069994\cos(x)e^{(x)} + 2.967546e^{(x)}x)$$

$$u_2(x,t) = \cos(x) + t^{\frac{1}{3}}(-1.119846e^x) + t^{\frac{2}{3}}(1.1161725\cos(x) + 0.494591x) + t(0.8959188e^{(x)} + 0.9027452x^3)$$

$$u_3(x,t) = e^{(x)} + 1.128379\cos(x)t^{\frac{1}{2}} + 0.9924381e^{(x)}t + 0.839643\cos(x)t^{\frac{3}{2}}$$

$N = 4$*:*

$$\begin{aligned}
u_1(x,t) &= \sin(x) + t^{\frac{1}{4}}(1.103262 + 6\cos(x)e^{(x)} + e^x) + t^{\frac{1}{2}}(1.022765 + 6.77027\cos(x)^2\\
&- 6.719079e^{2x} + x) + t^{\frac{3}{4}}(0.964272 + 5.069994\cos(x)e^{(x)} + 2.967546e^{(x)}x)\\
&+ t(0.9190625 + 12.59465\cos(x)^2 - 12.04378e^{(2x)} + 3.348517\cos(x)x + 5.41641e^{(x)}x^3)\\
u_2(x,t) &= \cos(x) + t^{\frac{1}{3}}(-1.119846e^x) + t^{\frac{2}{3}}(1.1161725\cos(x) + 0.494591x) + t(0.8959188e^{(x)}\\
&+ 0.9027452x^3 + \frac{4}{3}(0.7052042\cos(x)))\\
u_3(x,t) &= e^x + 1.128379\cos(x)t^{\frac{1}{2}} + 0.9924381e^{(x)}t + 0.839643\cos(x)t^{\frac{3}{2}} + t(0.595490e^{(x)} - 2.93549x)
\end{aligned}$$

4 Conclusion

In this paper we analyzed applicability of the reduced differential transform method for solving system of partial differential equations of fractional order. We showed simplicity and reliability of the method by handling examples of linear and nonlinear partial differential equations of fractional order. It can be applied for many similar equations without linearization, discretization and perturbation.

References

[1] J.K. Zhou. Differential Transfomation and its Application for Electrical Circuits. *Huazhong University Press*, Wuhan, China, (in Chinese).1986.

[2] F. Ayaz. Solution of the system of differential equations by differential transform. *Applied Mathematics and Computation*, 147(2004): 547-567.

[3] A. Arikoglu, I Ozkol. olution of fractional differential equations by using differential transform method. *Chaos, Solitons and Fractals*, 34 (2007): 1473-148.

[4] D. Nazari, S. Shahmorad, Application of the fractional differential transform method to fractional-order integro-differential equations with nonlocal boundary conditions. *Journal of Computational and Applied Mathematics*, 234 (2010): 883-891.

[5] S. P. Lin. Finite amplitude side-band stability of a viscous film, *J. Fluid Mech*, 63(1974): 417-429.

[6] H. Jafari, S. Seifi. Solving a system of nonlinear fractional partial differential equations using homotopy analysis method, *Commun. Nonlinear. Sci. Numer. Simulat*, 14 (5) (2009): 1962-1969.

[7] H. Xu, S.J. Liao, X.C. You. Analysis of nonlinear fractional partial differential equations with the homotopy analysis method, *Commun. Nonlinear. Sci. Numer. Simulat*, 14 (4) (2009): 1152-1156.

[8] S.R. Santanu. Analytical solution for the space fractional diffusion equation by two-step Adomian decomposition method, *Commun. Nonlinear. Sci. Numer.Simulat*, 14 (4) (2009): 1295-1306.

[9] C.P. Li, Y.H. Wang. Numerical algorithm based on Adomian decomposition for fractional differential equations, *Comput. Math. Appl*, 57 (10) (2009): 1672-1681.

[10] I. Mustafa. The approximate and exact solutions of the spaceand time-fractional Burgers equations with initial conditions by variational iteration method, *J. Math. Anal. Appl*, 345 (1) (2008): 476-484.

[11] S. Chen, F. Liu, P. Zhuang, V. Anh. Finite difference approximations for the fractional FokkerPlanck equation, *Appl. Math. Model*, 33 (1) (2009): 256-273.

[12] Y. Zhang. A finite difference method for fractional partial differential equation, *Appl. Math. Comput*, 215 (2) (2009): 524-529.

[13] Y. Keskin. Application of reduced differential transform method for solving gas dynamics equation, *Int. J. Contemp. Math. Sci*, 5(22), (2010), 1091-1096.

[14] Y. Keskin and G. Qturanc. Reduced differential transform method for solving generalized KDV equation, *Math. Comput. Appl* , 15(3), (2010): 382-393.

[15] P.K.Gupta. Approximate analytical solutions of fractional Benney-Lin equation by reduced differential transform method and the homotopy perturbation method, *Computers and Mathematics with Applications*, 61 (2011): 2829-2842.