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Discrete Chaos in Fractional Hénon Maps

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Abstract: Recently the discrete fractional calculus has been attracted lots of attention due to its potential applications to the natural science with memory effect. In this paper, the discrete fractional Hénon map is proposed in the left Caputo discrete delta's sense. We obtained the numerical formula of discrete fractional Hénon map by utilizing the discrete fractional calculus. The chaos behavior are numerically discussed when difference order is a fractional one. The phase diagrams and the bifurcation portraits are presented,respectively.

Keywords: Discrete fractional calculus;Chaos;Fractional Hénon maps;Caputo-like delta difference compression

1 Introduction

Fractional calculus is a classical mathematical concept, with a history as long as calculus itself. It is a generalization of ordinary differentiation and integration to arbitrary order, and is the fundamental theories of fractional order dynamical systems. In the past, it was considered that this technique is only a mathematical concept [1]. But, in recent years, applications of fractional order systems in many areas of science and engineering have been presented,such as viscoelastic system [2], dielectric polarization[3], electrode-electrolyte polarization [4] and electromagnetic wave [5], and so on.

The continuous fractional order chaotic systems in natural science and technology field have been extensively demonstrated and is well known [6-10]. Typically, chaotic systems often remain chaotic when their equations become fractional [11-13]. For example, it has been shown that the fractional order Chua's circuit with an appropriate cubic nonlinearity and with an order as low as 2.7 can produce a chaotic attractor [14].

In 1989, Miller and Ross [15] began the theory of the fractional difference, and the fractional integral was given as a fractional summation. Furthermore, this topic became more attractive and extensive discussion were proposed[16-19], such as the Taylor series[20], the definitions of the fractional differences and their properties[21], the Laplace transform[22] and the existence results[23]. However, the relevant investigation, particularly the bifurcation dynamics and emergence of chaos in the discrete fractional difference equations keep open, and those topics deserve further investigation, into the systems described by discrete fractional difference equations is still dull.

The Hénon map [24] was introduced by Michel Hénon as a simplified model of the Poincaré section of the Lorenz model. It is one of the most studied examples of dynamical systems that exhibit chaotic behavior, and reads

$$\begin{cases} x_{n+1} &= 1 - ax_n^2 + y_n, \\ y_{n+1} &= bx_n. \end{cases} \tag{1}$$

The map depends on two parameters, a and b, which for the classical Hénon map have values of $a = 1.4$ and $b = 0.3$. For the classical values the Hénon map is chaotic. For other values of a and b the map may be chaotic, intermittent, or converge to a periodic orbit. An overview of the behavior transition of the map at different parameter values may be obtained from its orbit diagram. Naturally, one question may be proposed: whether there is a discrete fractional Hénon map which has a generalized chaotic behavior.

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In this paper, we use the theories of delta difference equations to reveal the discrete chaotic behavior of fractionalized Hénon map via bifurcation analysis. The paper is organized as follows. In Section 2, we introduce some of basic definitions and the preliminaries of the discrete fractional calculus. In Section 3, we applied the tool of the discrete fractional calculus to the discrete fractional Hénon map and the numerical formula is obtained. Then the chaotic behaviors are reproduced by calculating the phase diagrams, the bifurcation portraits under various difference orders. Conclusions in Section 5 close the paper.

2 Basic definitions and preliminaries about discrete fractional calculus

We start with some necessary definitions from discrete fractional calculus theory and review the preliminary results so that paper is self-contained. In the following we use two notations $\mathbb{N}_p$ and Δ^m. $\mathbb{N}_p$ denotes the isolated time scale and $\mathbb{N}_p = \{p, p+1, p+2, \cdots\}, p \in \mathbb{R}$. Δ^m is the mth order forward difference operator

$$(\Delta^m f)(t) = \sum_{k=0}^{m} \binom{m}{k} (-1)^{m-k} f(t+k). \tag{2}$$

When $m = 1, \Delta f(t) = f(t+1) - f(t)$.

Definition 2.1 [25] Let $v > 0$. The vth fractional sum f is defined by

$$\Delta_p^{-v} f(t) = \frac{1}{\Gamma(v)} \sum_{s=p}^{t-v} (t-s-1)^{(v-1)} f(s). \tag{3}$$

Where p is the starting point, f is defined for $s = p \mod 1$ and $\Delta^{-v} f$ is defined for $t = (p+v) \mod 1$, in fact, Δ^{-v} maps functions defined on $\mathbb{N}_p$ to functions defined on $\mathbb{N}_{p+v}$. In addition,

$$t^{(v)} = \frac{\Gamma(t+1)}{\Gamma(t-v+1)}. \tag{4}$$

Definition 2.2 [26] Let $v > 0$ and $m-1 < v < m$, where m denotes a positive integer, $m = \lceil \mu \rceil$, $\lceil \cdot \rceil$ denotes the ceiling of number. The vth fractional Caputo like difference is defined as

$${}^C\Delta_p^v f(t) = \Delta^{v-m}(\Delta^m f(t)) = \frac{1}{\Gamma(m-v)} \sum_{s=p}^{t-(m-v)} (t-s-1)^{(m-v-1)} (\Delta^m f)(s), \qquad t \in \mathbb{N}_{p+m-v}. \tag{5}$$

Remark 2.1 If $0 < v < 1$, then $m = 1$. The μth fractional Caputo like difference is defined as

$${}^C\Delta_p^v f(t) = \Delta^{v-1}(\Delta f(t)) = \frac{1}{\Gamma(1-v)} \sum_{s=p}^{t-(1-v)} (t-s-1)^{(-v)} (\Delta f)(s), \qquad t \in \mathbb{N}_{p+1-v}. \tag{6}$$

Theorem 2.1 [23] For the delta fractional difference equation

$$\begin{cases} {}^C\Delta_p^v x(t) &= f(t+v-1, x(t+v-1)). \\ \Delta^k x(p) &= x_k, \qquad m = [v]+1, \qquad k = 0, 1, 2, \cdots, m-1, \end{cases} \tag{7}$$

the equivalent discrete integral equation can be obtained as

$$x(t) = x_0(t) + \frac{1}{\Gamma(v)} \sum_{s=p+m-v}^{t-v} (t-s-1)^{(v-1)} f(s+v-1, x(s+v-1)), \qquad t \in \mathbb{N}_{p+m}, \tag{8}$$

where the initial iteration reads

$$x_0(t) = \sum_{k=0}^{m-1} \frac{(t-p)^{(k)}}{k!} \Delta^k x(p). \tag{9}$$

Remark 2.2 If $0 < v < 1$, then equation (8) can be written as

$$x(t) = x(p) + \frac{1}{\Gamma(v)} \sum_{s=p+1-v}^{t-v} (t-s-1)^{(v-1)} f(s+v-1, x(s+v-1)), \qquad t \in \mathbb{N}_{p+1}, \tag{10}$$

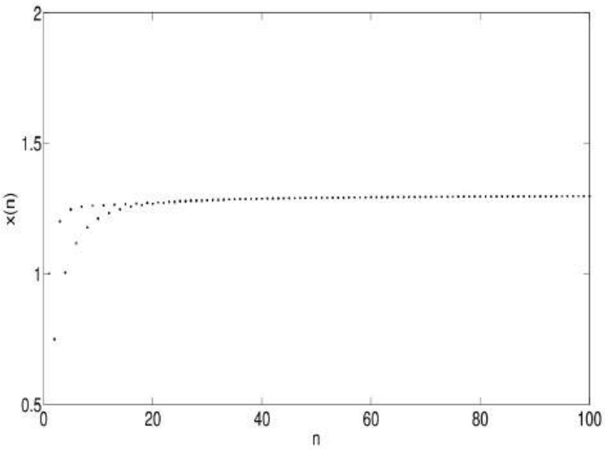

Figure 1: Stable solution of the discrete fractional Hénon map for $a = 0.05, b = 0.3, v = 0.8$

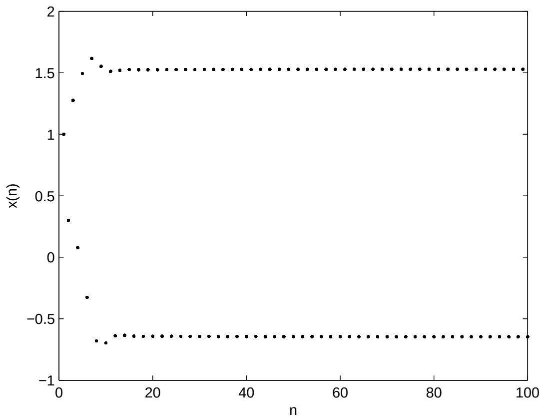

Figure 2: Vibration solution of the discrete fractional Hénon map for $a = 0.5, b = 0.3, v = 0.8$

3 Chaos of the discrete fractional Hénon map

Considering the fractional generalization of the momentum $x(n)$, we modify the Hénon map as a fractional one

$$\begin{cases} {}^C\Delta_p^v x(t) & = 1 - x(t+v-1)(1-ax(t+v-1)) + y(t+v-1), \qquad 0 < v \leq 1, \qquad t \in \mathbb{N}_{p+1-v}, \\ y(n) & = bx(n-1). \end{cases} \tag{11}$$

From Theorem 2.1, we can obtain the following equivalent discrete integral form for $0 < v < 1$

$$\begin{cases} x(t) & = x(p) + \frac{1}{\Gamma(v)} \sum_{s=p+1-v}^{t-v} \frac{\Gamma(t-s)}{\Gamma(t-s+1-v)} (1 - x(s+v-1)(1-ax(s+v-1)) + y(s+v-1)), \qquad t \in \mathbb{N}_{p+1}, \\ y(n) & = bx(n-1). \end{cases} \tag{12}$$

As a result, the numerical formula can be presented explicitly

$$\begin{cases} x(n) & = x(p) + \frac{1}{\Gamma(v)} \sum_{j=p+1}^{n} \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} (1 - x(j-1)(1-ax(j-1)) + y(j-1)), \\ y(n) & = bx(n-1), \qquad n \geq 1. \end{cases} \tag{13}$$

Especially, if $p = 0$, (13) can be written

$$\begin{cases} x(n) & = x(0) + \frac{1}{\Gamma(v)} \sum_{j=1}^{n} \frac{\Gamma(n-j+v)}{\Gamma(n-j+1)} (1 - x(j-1)(1-ax(j-1)) + y(j-1)), \\ y(n) & = bx(n-1), \qquad n \geq 1. \end{cases} \tag{14}$$

Compared with the map of integer order (1),the fractionalized one (13) has a discrete kernel function. In continuous dynamics system, time delay embedded in the variable is introduced into the dynamical equations to remember the previous signal, which is associated to the memory effect. However, in the map $x(n)$ depends on the past information $x(0), x(1), x(2), \cdots, x(n)$ and this is called discrete memory effect of the system. The memory effects of the discrete map means that their present state of evolution depends on all past states.

For $v = 0.8, b = 0.3, x(0) = y(0) = 0$, and $n = 100$, we can derive the numerical solutions $x(n), y(n)$ using the Matlab. In the following, the numerical solutions $x(n), y(n)$ for different a are plotted in Fig. 1,2 and 3. The results in Fig. 1,2,3 confirm that the fractional map can generate stable, periodical and chaotic state by selecting appropriate intrinsical parameter carefully. Furthermore, it is interesting to check the dependence of fractional order on map.

Now we consider more general chaos emergence of the map, the cases of the fractional difference order $v = 0.8, v = 0.6$ and $v = 1$. The phase portraits are plotted in Fig. 4,5 and 6. Indeed, the results in Fig. 4,5,6 confirm that chaos also emerges in the fractional order map by selecting different order for v. To discern the dependence of chaotic state on intrinsical parameter under different fractional order, the bifurcation diagram is also plotted to understand the transition from stable, periodical state to chaotic state.

Using the numerical formula (14), set the step size of a as 0.005 and the bifurcation diagrams are plotted in Fig. 7,8,9 and 10.

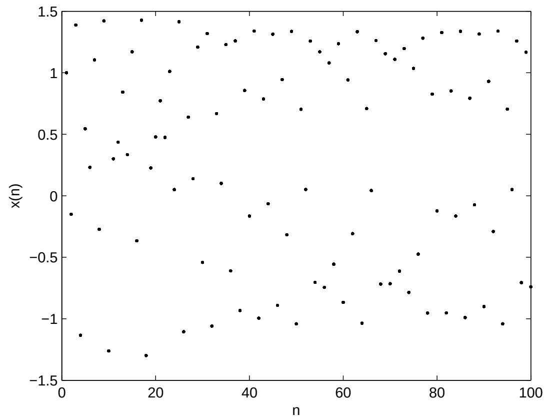

Figure 3: Chaos solution of the discrete fractional Hénon map for $a = 0.95, b = 0.3, v = 0.8$

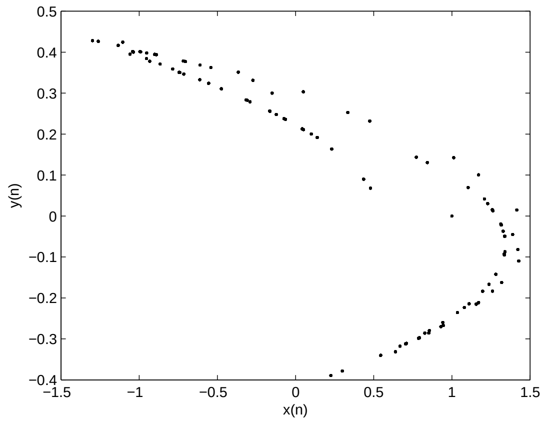

Figure 4: The phase portrait of the discrete fractional Hénon map for $a = 0.95, b = 0.3, v = 0.8$

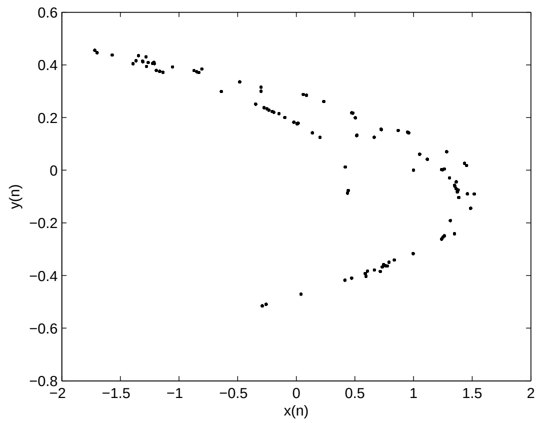

Figure 5: The phase portrait of the discrete fractional Hénon map for $a = 0.9, b = 0.3, v = 0.6$

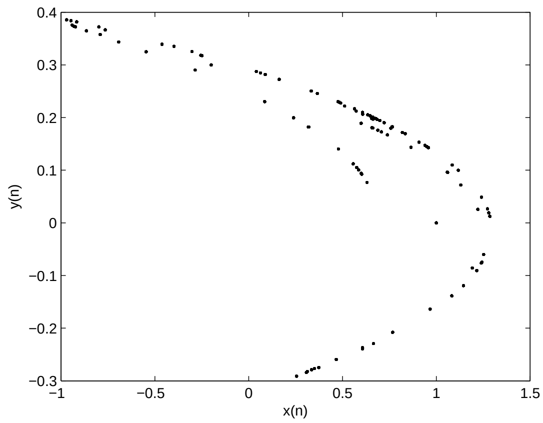

Figure 6: The phase portrait of the discrete fractional Hénon map for $a = 1.2, b = 0.3, v = 1$

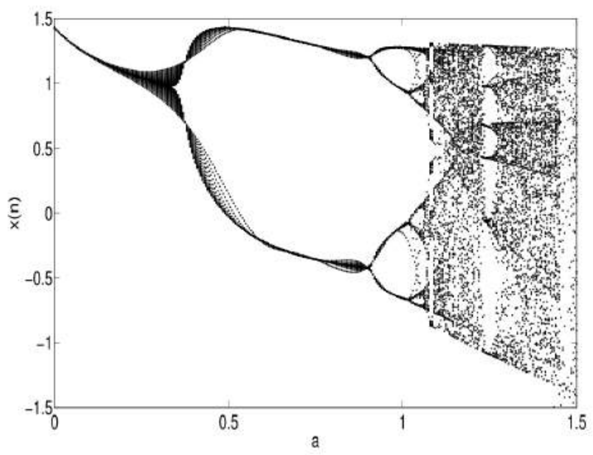

Figure 7: The bifurcation diagram of the discrete fractional Hénon map for $b = 0.3, v = 1$

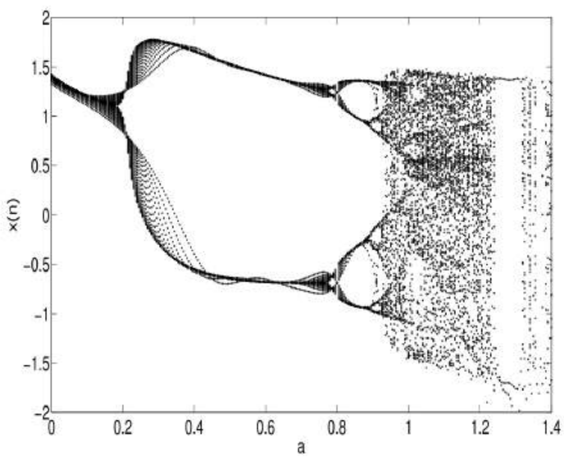

Figure 8: The bifurcation diagram of the discrete fractional Hénon map for $b = 0.3, v = 0.8$

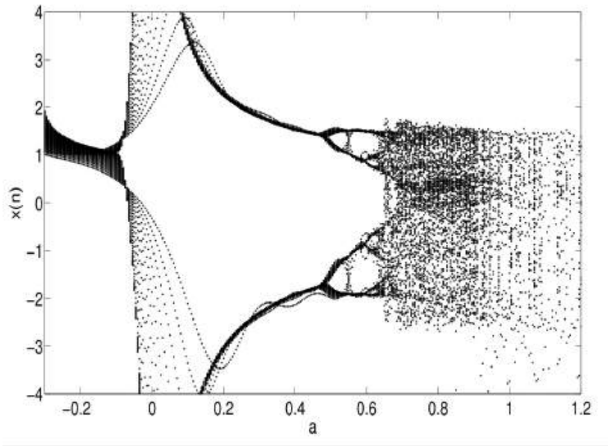

Figure 9: The bifurcation diagram of the discrete fractional Hénon map for $b = 0.3, v = 0.2$

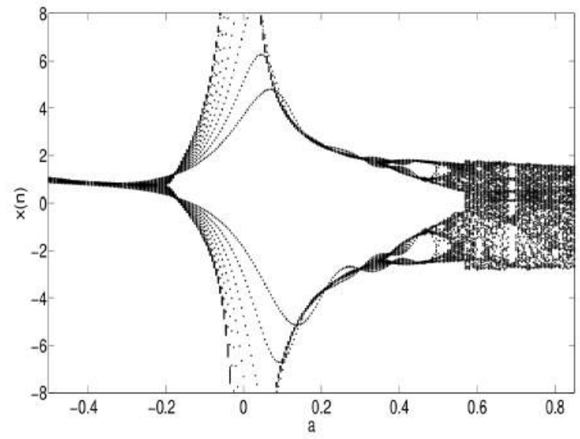

Figure 10: The bifurcation diagram of the discrete fractional Hénon map for $b = 0.3, v = 0.05$

4 Conclusion

The discrete dynamics of complex systems are frequently encountered in the interdisciplinary fields, many systems can be described by discrete fractional equations. In order to describe the long range interaction traits, this study applies the discrete fractional calculus to nonlinear maps. Fractional Hénon map is then proposed. Compared with the one of the integer order, the fractional map has a discrete memory and a fractional order v. The map is given in iteration formulae which are fractional generalizations of the classical ones. The bifurcation diagrams and the phase portraits are given to confirm the emergence of chaos by changing the control bifurcation parameter and order, respectively. It is interesting to point out that the chaotic zones not only depends on the coefficients a, b but the difference order v. There results show that the discrete fractional calculus is an efficient tool for fractional generations of the discrete maps.

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