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# On Multi-Soliton Solution of Degasperis-Procesi Equation with Self-Consistent Sources

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**Abstract:** Degasperis-Procesi equation with self-consistent sources is an integrable extension of the original system. We obtain the multi-soliton solutions of the Degasperis-Procesi equation with self-consistent sources by using the reciprocal transformation and the Darboux transformation.

**Keywords:** Degasperis-Procesi equation with self-consistent sources; reciprocal transformation; soliton

## 1 Introduction

The Camassa-Holm (CH) equation, which was implicitly contained in the class of multihamiltonian systems introduced by Fuchssteiner and Fokas in [1] and explicitly derived as a shallow water wave equation by Camassa and Holm in [2], has the form

$$u_t + 2\omega u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx}. \tag{1}$$

Since the works of Camassa and Holm, the CH equation has been studied from many different points of view[3]-[9].

In a recent study[10], considerable attention has been paid to the following equation:

$$u_t + \alpha u_x - u_{xxt} + (\beta + 1)uu_x = \beta u_x u_{xx} + uu_{xxx}. \tag{2}$$

Here $u = u(x,t)$, $\alpha$ and $\beta$ are constants. It is shown that the above equation is completely integrable when $\beta = 2$ or $\beta = 3$. In the case of $\beta = 2$, we get the CH equation, while in the case of $\beta = 3$, we have the equation called the Degasperis-Procesi (DP) equation. DP equation describes the unidirectional propagation of nonlinear shallow-water waves. It has a third order Lax pair and a bi-Hamiltonian structure [10]. The multi-peakon solution of the DP equation is discussed in [11]. In [12], the multi-soliton solution of the DP equation is studied.

In the present paper, we consider the DP equation:

$$u_t + 3\omega^3 u_x - u_{xxt} + 4uu_x = 3u_x u_{xx} + uu_{xxx} \tag{3}$$

where $\omega$ is a positive parameter. Let $m = u - u_{xx} + \omega^3$, the above equation could be written as

$$m_t + 3u_x m + um_x = 0. \tag{4}$$

Soliton equations with self-consistent sources (SESCS) have attracted much attention in recent years [13]-[19]. They are important integrable models in many fields of physics, such as hydrodynamics, state physics, plasma physics, etc. For example, the KdV equation with self-consistent sources describes the interaction of long and short capillary-gravity waves. The nonlinear Schrödinger equation with self-consistent sources represents the nonlinear interaction of an electrostatic high-frequency wave with the ion acoustic wave in a two component homogeneous plasma. The KP equation with self-consistent sources describes the interaction of a long wave with a short wave packet propagating on the $x$-$y$ plane at some angle to each other. The SESCS were first studied by Melnikov. A systematic way to construct the soliton equations with self-consistent sources and their zero-curvature representations was proposed.

In [19], we have derived the Degasperis-Procesi equation with self-consistent sources (DPESCS). The Lax representation of the DPESCS is presented. The conservation laws for DPESCS are constructed. The peakon solution of DPESCS is obtained by using the method of variation of constants.

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The present paper falls in that line of research, aiming at finding the multi-soliton solutions of DPESCS. We firstly relates DP equation to an alternative of associated Degasperis-Procesi (ADP) equation by considering the reciprocal transformation, which in a similar way, we relates the DPESCS with the associated DPESCS (ADPESCS). By using the Darboux transformation, we can find the N-soliton solution of alternative ADP equation. Then by means of the method of variation of constants, we can obtain the N-soliton solution for ADPESCS. Finally, using the inverse reciprocal transformation, we obtain the N-soliton solution of DPESCS.

This paper is organized as follows. In section 2, we briefly recall the DPESCS and its Lax representation. In section 3, we present the reciprocal transformation between DPESCS and ADPESCS. In section 4, we construct the N-soliton solution for DPESCS. In section 5, we give conclusion.

# 2 The DPESCS and its Lax pair

In order to make our paper self-contained, we introduce the DPESCS briefly. It is known that the Lax pair for the DP equation is

$$f_{xxx} = f_x + m\lambda f, \tag{1a}$$
$$f_t = -\frac{1}{\lambda}f_{xx} - uf_x + (u_x + \frac{2}{3\lambda})f. \tag{1b}$$

Consider the following equations obtained from the spectral problem and its formal adjoint problem for $n$ distinct real $\lambda_j$:

$$q_{j,xxx} = q_{j,x} + m\lambda_j q_j, \quad j = 1, \cdots, n, \tag{2a}$$
$$r_{j,xxx} = r_{j,x} - m\lambda_j r_j, \quad j = 1, \cdots, n. \tag{2b}$$

It is not difficult to find that

$$\frac{\delta\lambda_j}{\delta m} = -\lambda_j q_j r_j, \quad j = 1, \cdots, n. \tag{3}$$

It is known that the DP equation possesses a hamiltonian structure, namely:

$$m_t = B_1 \frac{\delta H_1}{\delta m}, \tag{4}$$

where $B_1 = \partial_x(1 - \partial_x^2)(4 - \partial_x^2)$,$H_1 = -\frac{1}{6}\int u^3 dx$.

Then according to the approach proposed in [19], the DPESCS consists of the following equation:

$$m_t = B_1(\frac{\delta H_1}{\delta m} - \sum_{j=1}^{n} \alpha_j \frac{\delta\lambda_j}{\delta m})$$

and the equations (2.2), which by using (2.3) and taking $\alpha_j = -\frac{1}{6}$ leads to the DPESCS

$$m_t = -um_x - 3u_x m - \frac{1}{6}\sum_{j=1}^{n} \partial(1 - \partial^2)(4 - \partial^2)(\lambda_j q_j r_j), \tag{5a}$$
$$q_{j,xxx} = q_{j,x} + m\lambda_j q_j, \tag{5b}$$
$$r_{j,xxx} = r_{j,x} - m\lambda_j r_j, \quad j = 1, \cdots, n. \tag{5c}$$

We may assume that the Lax representation of the DPESCS has the form

$$\psi_{xxx} = \psi_x - m\lambda\psi, \tag{6a}$$
$$\psi_t = -\frac{1}{\lambda}\psi_{xx} - u\psi_x + (u_x + \frac{2}{3\lambda})\psi + a\psi_{xx} + b\psi_x + c\psi, \tag{6b}$$

where $a$, $b$ and $c$ are some functions of $q_j$ and $r_j$ to be determined. After some calculations we find that:

$$a = \sum_{j=1}^{n} \frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2} q_j r_j, \tag{7a}$$

$$b = \sum_{j=1}^{n} -\frac{1}{2}\frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2}(\lambda(q_j r_{j,x} - q_{j,x} r_j) + \lambda_j (q_j r_j)_x), \tag{7b}$$

$$c = \sum_{j=1}^{n} \frac{1}{6}\frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2}(3\lambda(q_j r_{j,xx} - q_{j,xx} r_j) - 4\lambda_j q_j r_j - 2\lambda_j (q_j r_j)_{xx}). \tag{7c}$$

which leads us to the Lax representation of the DPESCS

$$\psi_{xxx} = \psi_x - m\lambda\psi, \tag{8a}$$

$$\begin{aligned} \psi_t = {} & -\frac{1}{\lambda}\psi_{xx} - u\psi_x + (u_x + \frac{2}{3\lambda})\psi + (\sum_{j=1}^{n} \frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2} q_j r_j)\psi_{xx} \\ & +(\sum_{j=1}^{n} -\frac{1}{2}\frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2}(\lambda(q_j r_{j,x} - q_{j,x} r_j) + \lambda_j (q_j r_j)_x))\psi_x \\ & +(\sum_{j=1}^{n} \frac{1}{6}\frac{\lambda\lambda_j^2}{\lambda_j^2-\lambda^2}(3\lambda(q_j r_{j,xx} - q_{j,xx} r_j) - 4\lambda_j q_j r_j - 2\lambda_j (q_j r_j)_{xx}))\psi, \end{aligned} \tag{8b}$$

which means that DPESCS is Lax integrable.

# 3 A reciprocal transformation for the DPESCS

Let $\rho^3 = m = u - u_{xx} + \omega^3$, by the reciprocal transformation

$$dy = \rho dx - \rho u ds, ds = dt,$$

and denoting $f = \rho^{-1}\phi$, the Lax pair of DP equation is transformed to

$$\phi_{yyy} + U\phi_y + \tfrac{1}{2}U_y\phi = \lambda\phi, \tag{1a}$$

$$\lambda\phi_s = \rho^2\phi_{yy} - \rho\rho_y\phi_y - (\tfrac{2}{3} + \rho\rho_{yy} - \rho_y^2)\phi, \tag{1b}$$

where

$$U = -2\frac{\rho_{yy}}{\rho} + \frac{\rho_y^2}{\rho^2} - \frac{1}{\rho^2}.$$

The compatibility condition of (3.1) leads to the associated DP (ADP) equation:

$$U_s + 6\rho\rho_y = 0, \tag{2a}$$

$$\tfrac{3}{2}(\rho\rho_y U_y)_y + (\rho\rho_y)_y U_y + 2U(\rho\rho_{yy})_y - (\rho_y\rho_{yy})_{yy} = 0 \tag{2b}$$

We now consider the reciprocal transformation for the DPESCS (2.5), (2.5a)gives

$$\rho_t + (\rho(u + \frac{1}{2}\sum_{j=1}^{N}(\lambda_j(q_{j,x} r_j - q_j r_{j,x}))))_x = 0$$

which shows that the 1-form

$$\rho dx - (\rho(u + \frac{1}{2}\sum_{j=1}^{N}(\lambda_j(q_{j,x} r_j - q_j r_{j,x}))))dt$$

is closed, so we can define a reciprocal transformation $(x, t) \to (y, s)$ by the relation

$$dy = \rho dx - (\rho(u + \frac{1}{2}\sum_{j=1}^{N}(\lambda_j(q_{j,x} r_j - q_j r_{j,x}))))dt, ds = dt.$$

Denoting $\psi = \rho^{-1}\varphi$, $q_j = \rho^{-1}\bar{q}_j$, $r_j = \rho^{-1}\bar{r}_j$ the Lax pair of DPESCS (2.5) is correspondingly rewritten as

$$\varphi_{yyy} + U\varphi_y + \tfrac{1}{2}U_y\varphi = \lambda\varphi, \tag{3a}$$

$$\lambda\varphi_s = \rho^2\varphi_{yy} - \rho\rho_y\varphi_y - (\tfrac{2}{3} + \rho\rho_{yy} - \rho_y^2)\varphi + \lambda(G\varphi_{yy} + H\varphi_y + I\varphi), \tag{3b}$$

where

$$G = \sum_{j=1}^{N}(-\frac{\lambda\lambda_j^2}{\lambda^2 - \lambda_j^2}\bar{q}_j\bar{r}_j),$$

$$H = \sum_{j=1}^{N}(\frac{1}{2}\frac{\lambda_j^3}{\lambda^2 - \lambda_j^2}W(\bar{q}_j, \bar{r}_j) + \frac{1}{2}\frac{\lambda\lambda_j^2}{\lambda^2 - \lambda_j^2}(\bar{q}_j\bar{r}_j)_y),$$

$$I = \sum_{j=1}^{N}(-\frac{1}{2}\frac{\lambda_j^3}{\lambda^2 - \lambda_j^2}W(\bar{q}_j, \bar{r}_j)_y - \frac{2}{3}\frac{\lambda\lambda_j^2}{\lambda^2 - \lambda_j^2}(\bar{q}_j\bar{r}_j) - \frac{1}{6}\frac{\lambda\lambda_j^2}{\lambda^2 - \lambda_j^2}(\bar{q}_j\bar{r}_j)_{yy}),$$

$W(f, g) = fg_y - f_y g$ is the Wronskian determinant.

The compatibility condition of the above Lax pair leads to ADPESCS

$$U_s + 6\rho\rho_y = 3\sum_{j=1}^{N}\lambda_j^2(\bar{q}_j\bar{r}_j)_y, \tag{4a}$$

From the reciprocal transformation, we have

$$\frac{\partial x}{\partial y} = \frac{1}{\rho}, \frac{\partial x}{\partial s} = u - \frac{1}{2}\sum_{j=1}^{N}(\lambda_j(q_{j,x}r_j - q_j r_{j,x})) \tag{5}$$

which leads us to find

$$x(y, s) = \int \frac{1}{\rho}dy.$$

The solution of the DPESCS with respect to the variable $(y, s)$ are given by

$$m = \rho^3(y, s),\ q_j(y, s) = \tfrac{\bar{q}_j}{\rho},\ r_j(y, s) = \tfrac{\bar{r}_j}{\rho}, \tag{6a}$$

$$u(y, s) = \rho^3 - \rho(\ln\rho)_{ys} + \rho(\rho(\tfrac{1}{2}\textstyle\sum_{j=1}^{N}(\lambda_j(q_{j,x}r_j - q_j r_{j,x})))_y)_y + \omega^3, \tag{6b}$$

$$x(y, s) = \textstyle\int \tfrac{1}{\rho}dy. \tag{6c}$$

# 4 The solutions for the DPESCS

We first review the Darboux transformation for the third order spectral problem $\varphi_{yyy} + U\varphi_y + \frac{1}{2}U_y\varphi = \lambda\varphi$, where $\varphi$ is the eigenfunction with the spectral parameter $\lambda$. $\psi$ is the eigenfunction of the spectral problem with the spectral parameter $\mu$. Then the Darboux transformation for the spectral problem is:

$$\tilde{\varphi} = (\lambda + \mu)\varphi - 2\mu\psi\delta(\varphi, \psi)/\delta(\psi, \psi), \tag{1a}$$

$$\tilde{U} = U + 3\ln_{yy}\delta(\psi, \psi), \tag{1b}$$

where

$$\delta(\varphi, \psi) = \varphi_{yy}\psi - \varphi_y\psi_y + \varphi\psi_{yy} + U\varphi\psi.$$

Based on this Darbour transformation, we have the generalized Darboux transformation for ADP equation. Let $\phi_0(y, s, \lambda)$, $\Psi_1(y, s, \lambda_1)$, $\cdots$, $\Psi_n(y, s, \lambda_n)$ be different solutions of (3.1) with $U = -\frac{1}{\omega^2}$, $\rho = \omega$ and the corresponding $\lambda$ and

$\lambda = \lambda_1, \cdots, \lambda_n$, respectively. We define two determinants from these functions.

$$W_N(\Psi_1,\cdots,\Psi_N) = \begin{vmatrix} \int\Psi_1^2dy & \int\Psi_1\Psi_2dy & \cdots & \int\Psi_1\Psi_Ndy \\ \int\Psi_2\Psi_1dy & \int\Psi_2^2dy & \cdots & \int\Psi_2\Psi_Ndy \\ \vdots & \vdots & \ddots & \vdots \\ \int\Psi_N\Psi_1dy & \int\Psi_N\Psi_2dy & \cdots & \int\Psi_N^2dy \end{vmatrix}, \tag{2a}$$

$$F_N(\Psi_1,\cdots,\Psi_N,\phi_0) = \begin{vmatrix} \int\Psi_1^2dy & \int\Psi_1\Psi_2dy & \cdots & \int\Psi_1\Psi_Ndy & \int\Psi_1\phi_0dy \\ \int\Psi_2\Psi_1dy & \int\Psi_2^2dy & \cdots & \int\Psi_2\Psi_Ndy & \int\Psi_2\phi_0dy \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \int\Psi_N\Psi_1dy & \int\Psi_N\Psi_2dy & \cdots & \int\Psi_N^2dy & \int\Psi_N\phi_0dy \\ \Psi_1 & \Psi_2 & \cdots & \Psi_N & \phi_0 \end{vmatrix}. \tag{2b}$$

We find the Darboux transformation for (3.1)

$$U[N] = -\tfrac{1}{\omega^2} + 3\ln_{yy} W_N(\Psi_1,\cdots,\Psi_N), \tag{3a}$$

$$\rho[N]^2 = \omega^2 - \ln_{yy} W_N(\Psi_1,\cdots,\Psi_N), \tag{3b}$$

$$\phi[N](y,s,\lambda) = F_N(\Psi_1,\cdots,\Psi_N,\phi_0)/W_N(\Psi_1,\cdots,\Psi_N), \tag{3c}$$

namely $U[N]$, $\rho[N]$ and $\phi[N]$ satisfies (3.1).

Let $\alpha_j$, $\beta_j$ and $-(\alpha_j+\beta_j)$ be the solutions of $x^3 - 1/\omega^2 x = \lambda_j$ and $\alpha_j - \beta_j = k_j$. Then we have

$$\alpha_j = \frac{1}{2}(k_j + \frac{\sqrt{4-k_j^2\omega^2}}{\sqrt{3}\omega}), \tag{4a}$$

$$\beta_j = \frac{1}{2}(-k_j + \frac{\sqrt{4-k_j^2\omega^2}}{\sqrt{3}\omega}). \tag{4b}$$

Take

$$\Psi_j = \sqrt{\tfrac{\alpha_j}{\beta_j}}e^{\theta_j} + e^{\sigma_j} = \exp(\tfrac{\theta_j+\sigma_j}{2})(\sqrt{\tfrac{\alpha_j}{\beta_j}}e^{\xi_j} + e^{-\xi_j}), \tag{5a}$$

$$q_j = \sqrt{\tfrac{\alpha_j}{\beta_j}}e^{\theta_j} - e^{\sigma_j} = \exp(\tfrac{\theta_j+\sigma_j}{2})(\sqrt{\tfrac{\alpha_j}{\beta_j}}e^{\xi_j} - e^{-\xi_j}), \tag{5b}$$

$$r_j = 0, \int\Psi_i r_j dy = \delta_{ij}, j = 1,\cdots,N \tag{5c}$$

to be the solution of (3.1) with $U = -\frac{1}{\omega^2}$, $\rho = \omega$ and $\lambda_j = -\alpha_j\beta_j(\alpha_j+\beta_j)$, where

$$\theta_j = \alpha_j y + \frac{1}{\lambda_j}(\omega^2\alpha_j^2 - \frac{2}{3})s + C_j,$$

$$\sigma_j = \beta_j y + \frac{1}{\lambda_j}(\omega^2\beta_j^2 - \frac{2}{3})s - C_j,$$

$$\xi_j = \frac{\theta_j - \sigma_j}{2} = \frac{1}{2}(k_j y + \frac{3k_j\omega^4}{k_j^2\omega^2 - 1}s) + C_j$$

and $C_j$ are arbitrary constants. By using the Darboux transformation (4.3), we obtain the N-soliton solution of the ADP equation from these seed solutions. Furthermore, we can use the method of variation of constants to find the N-soliton solution of the ADPESCS. And Finally, by taking the reciprocal transformation (3.6) we get the N-soliton solution of the DPESCS. Here we take the one soliton solution as an example.

When $N = 1$, (4.3) gives rise to one soliton solution and the corresponding eigenfunction and the adjoint eigenfunction for ADP equation with $\lambda_1 = -\alpha_1\beta_1(\alpha_1+\beta_1)$

$$U = -\tfrac{1}{\omega^2} + 3\ln_{yy}(\cosh(2\xi_1) + a_1), \tag{6a}$$

$$\rho = \omega\frac{\cosh(2\xi_1) - a_1 + \frac{2}{a_1}}{\cosh(2\xi_1) + a_1}, \tag{6b}$$

$$q_1 = \frac{k_1}{\alpha_1+\beta_1}\frac{e^{\theta_1} + \sqrt{\frac{\alpha_1}{\beta_1}}e^{\sigma_1}}{\cosh(2\xi_1) + a_1}, \tag{6c}$$

$$r_1 = -\beta_1\frac{e^{\theta_1} + \sqrt{\frac{\alpha_1}{\beta_1}}e^{\sigma_1}}{\cosh(2\xi_1) + a_1}, \tag{6d}$$

where $a_1 = 2\frac{\sqrt{\alpha_1\beta_1}}{\alpha_1+\beta_1}$.

We can use the method of variation of constants on one soliton solution of the ADP equation to get the one soliton solution of ADPESCS. Taking $C_1$ in $\xi_1$ to be time-dependent function $C_1(s)$ and requiring that

$$\bar{U} = -\frac{1}{\omega^2} + 3\ln_{yy}(\cosh(2\bar{\xi}_1) + a_1), \tag{7a}$$

$$\bar{\rho} = \omega\frac{\cosh(2\bar{\xi}_1) - a_1 + \frac{2}{a_1}}{\cosh(2\bar{\xi}_1) + a_1}, \tag{7b}$$

$$\bar{q}_1 = \gamma_1(s)\frac{k_1}{\alpha_1+\beta_1}\frac{e^{\bar{\theta}_1} + \sqrt{\frac{\alpha_1}{\beta_1}}e^{\bar{\sigma}_1}}{\cosh(2\bar{\xi}_1) + a_1}, \tag{7c}$$

$$\bar{r}_1 = -\gamma_1(s)\beta_1\frac{e^{\bar{\theta}_1} + \sqrt{\frac{\alpha_1}{\beta_1}}e^{\bar{\sigma}_1}}{\cosh(2\bar{\xi}_1) + a_1}, \tag{7d}$$

satisfy the ADPESCS (3.3) for $N = 1$. We find that $C_1(s)$ can be an arbitrary function of $s$ and

$$\gamma_1(s) = \sqrt{\frac{2C_1(s)}{\lambda_1^2}}.$$

So we can get the one soliton solution for DPESCS with $N = 1$ parametrically

$$\bar{\rho} = \omega\frac{\cosh(2\bar{\xi}_1) - a_1 + \frac{2}{a_1}}{\cosh(2\bar{\xi}_1) + a_1}, \tag{8a}$$

$$u = \frac{8\omega^3 a_1(\frac{1}{a_1^2} - 1)(\frac{1}{a_1^2} - \frac{1}{4})}{\cosh(2\bar{\xi}_1) - a_1 + \frac{2}{a_1}}, \tag{8b}$$

$$\bar{q}_1 = \sqrt{\frac{2C_1(s)}{\lambda_1^2}}\frac{k_1}{\alpha_1+\beta_1}\frac{e^{\bar{\theta}_1} + \sqrt{\frac{\alpha_1}{\beta_1}}e^{\bar{\sigma}_1}}{\cosh(2\bar{\xi}_1) + a_1}, \tag{8c}$$

$$\bar{r}_1 = -\sqrt{\frac{2C_1(s)}{\lambda_1^2}}\beta_1\frac{e^{\bar{\theta}_1} + \sqrt{\frac{\alpha_1}{\beta_1}}e^{\bar{\sigma}_1}}{\cosh(2\bar{\xi}_1) + a_1}, \tag{8d}$$

$$x = \frac{y}{\omega} + \frac{2a_1 - \frac{2}{a_1}}{\omega}\int_{-\infty}^{y}\frac{1}{\cosh(2\bar{\xi}_1) + \frac{2}{a_1} - a_1}dy. \tag{8e}$$

We have an arbitrary function of $s$ in the soliton solution, this may cause the variation of the speed of soliton depending on what kind of $C_1(s)$ we choose.

Taking this method repeatedly, we can find the N-soliton solution of the DPESCS parametrically.

# 5 Conclusion

In this paper, by considering the reciprocal transformation, we present the links between the DPESCS and ADPESCS. The multi-soliton solution of the DPESCS is derived. Comparing with the soliton solution of the original Degasperis-Procesi equation, this new solution has an arbitrary function of $s$ that its dynamics may have more freedom. We may also find some other types of solutions of DPESCS in the future.

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