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An Approximate Solution of Inelastic Two-Soliton Collision for BBM Equation with General Nonlinearity

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Abstract: In this paper, we consider the collision of two solitons with different velocities for the Benjamin-Bona-Mahony equation with general nonlinearity. In this case, one solitary wave is smaller than the other one in $H^1(R)$ energy space.The main method is introduced by Martel and Merle. The main argument is the construction of an explicit approximate solution in the collision region.

Keywords: approximate solution; BBM equation; collision

1 Introduction

In this work, let $c_1 > 1$ and $\lambda = \frac{c_1-1}{c_1} \in (0,1)$. We consider the BBM equations with general nonlinearity on the $(t,x) \in \mathrm{R}_t \times \mathrm{R}_x$,

$$(1-\lambda\partial_x^2)\partial_t u + \partial_x(\partial_x^2 u - u + u^2 + u^3) = 0. \tag{1.1}$$

Here, $u = u(t,x)$is a real-valued function and $f : \mathrm{R}_t \to \mathrm{R}_x$ is a nonlinear function. Where $f(u) = u^2 + u^3$, $f \in \mathrm{C}^{P+2}$ with $\lim\limits_{u\to 0} \frac{|u|^3}{|u|^2} = 0$.

We denote that

$$F(u) = \frac{1}{3}u^3 + \frac{1}{4}u^4. \tag{1.2}$$

Let us review some relevant works in this direction. The BBM equation was introduced by Peregrine [1]. The gBBM equation was introduced by Peregrine [2] and Benjamin, Bona and Mahony [3]. The work of Lax [4] developed a mathematical framework to study these problems. About the problem of collision of two solitary waves, Martel and Merle have made a lot of contributions. Martel and Merle [5] investigated the generalized KdV equation. Martel and Merle [6] considered gKdV with a general nonlinearity f. Martel, Merle and Mizumachi [7] proved that the collision of the two solitons is inelastic but almost elastic. Dika and Martel [8] proved the stability of N solitary waves for the gBBM equations. The main result of this paper is the following construction of an approximate 2-soliton solution.

2 Construction of an approximate 2-soliton solution

2.1 Reduction of the problem

Lemma 1 *(Claim 2.1 [8])*
(i) Let$c > 1$.

$$\tilde{Q}_\sigma(x) = \sigma\theta_\sigma Q(\sqrt{\sigma}x), \tag{2.1}$$

$$\sigma = \frac{c-1}{c\lambda}, \quad \theta_\sigma = \frac{1-\lambda}{1-\lambda\sigma}, \quad \mu_\sigma = \frac{1-\sigma}{1-\lambda\sigma}, \quad y_\sigma = x + \mu_\sigma t. \tag{2.2}$$

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(ii) Moreover, $\tilde{Q}_\sigma$ satisfies the following

$$\theta_\sigma \tilde{Q}''_\sigma + Q^2_\sigma + Q^3_\sigma = \sigma\theta_\sigma \tilde{Q}_\sigma. \tag{2.3}$$

$$\tilde{Q}''_\sigma = \sigma\tilde{Q}_\sigma - \frac{1}{\theta_\sigma}(Q^2_\sigma + Q^3_\sigma), \quad \left(\tilde{Q}'_\sigma\right)^2 = \sigma\tilde{Q}^2_\sigma - \frac{2}{\theta_\sigma}F(\tilde{Q}_\sigma). \tag{2.4}$$

Especially if $c = c_1$, then $\mu_\sigma = 0$ and $\sigma = 1$.

$$Q'' + Q^2 + Q^3 = Q \tag{2.5}$$

$$\left(Q'\right)^2 = Q^2 - 2F\left(Q\right) \quad , \quad \left(Q^k\right)'' = k^2 Q^k - 2k\left(k-1\right)Q^{k-2}F\left(Q\right) \; - kf\left(Q\right)Q^{k-1}. \tag{2.6}$$

2.2 Decomposition of the approximate solution

We first introduce a set of indices, depending on the cases we deal with. Let

$$\sum\nolimits_0 = \{(k,l) = (1,0)\,, (1,1)\,, (2,0)\,, (2,1)\,, (1,2)\,, (3,0)\}$$

We set two variables denoting the position of each soliton.

$$y_\sigma = x + \mu_\sigma t, \quad y = x - \alpha(y_\sigma),$$

where for $\left(a_{k,l}\right)_{(k,l)} \in \sum_0$

$$\alpha(s) = \int_0^s \beta(r)dr, \qquad \beta(s) = \sum_{(k,l)\in\sum_0} a_{k,l}\sigma^l Q^k_\sigma(s), \tag{2.7}$$

The form of $\tilde{u}\left(t,x\right)$ is

$$\tilde{u}\left(t,x\right) := Q\left(y\right) + \tilde{Q}_\sigma\left(y_\sigma\right) + W\left(t,x\right), \tag{2.8}$$

$$W\left(t,x\right) := \sum_{(k,l)\in\sum_0} \sigma^l \left(Q^k_\sigma\left(y_\sigma\right)A_{k,l}\left(y\right) + \left(Q^k_\sigma\right)'\left(y_\sigma\right)B_{k,l}\left(y\right)\right), \tag{2.9}$$

where $a_{k,l}, A_{k,l}, B_{k,l}$ are unknown, and they will be computed. In order to measure the size of the error produced by inserting $\tilde{u}$ as defined in (2.8)-(2.9) in the equation (1.1). For this, let

$$S[\tilde{u}](t,x) := (1-\lambda\partial^2_x)\partial_t\tilde{u} + \partial_x(\partial^2_x\tilde{u} - \tilde{u} + f(\tilde{u})), \tag{2.10}$$

where $f(\tilde{u}) = \tilde{u}^2 + \tilde{u}^3$.

Proposition 2 *(Decomposition of $S[\tilde{u}](t,x)$). Define,*

$$Lw = -w'' + w - (2Q + 3Q^2)w, \tag{2.11}$$

Then,

$$\begin{aligned} S[\tilde{u}](t,x) \;&=\; \sum_{(k,l)\in\Sigma} \sigma^l \tilde{Q}^k_\sigma(y_\sigma)\left(a_{k,l}((\lambda-3)Q'' - 2Q^2 - 3Q^3)' - (LA_{k,l})' + F_{k,l}\right)(y) \\ &+\; \sum_{(k,l)\in\Sigma} \sigma^l (\tilde{Q}^k_\sigma)'(y_\sigma)\left((3-\lambda)A''_{k,l} + (2Q+3Q^2)A_{k,l} + a_{k,l}(2\lambda-3)Q'' - (LB_{k,l})' + G_{k,l}\right)(y) \\ &+\; \varepsilon(t,x), \end{aligned}$$

$$F_{1,0} = (2Q + 3Q^2)', \quad G_{1,0} = 2Q + 3Q^2$$

where the following holds:
(i) For all $(k,l) \in \sum_0$ such that $k + l = 3$, $F_{k,l}$, $G_{k,l}$ depend on $A_{k',l'}, B_{k',l'}$ for $1 \le k' + l' \le 2$. Moreover, if $A_{k',l'}$ are even and $B_{k',l'}$ are odd then $F_{k,l}$ are odd and $G_{k,l}$ are even.
(ii) If the functions $A_{k',l'}, B_{k',l'}$ are bounded then the rest term $\varepsilon(t,x)$ satisfies

$$\varepsilon(t,x) \le k\sigma^3 O(\tilde{Q}_\sigma), \tag{2.12}$$

Lemma 3 *(See Claim B.1 [9]) Let $h(t,x) = g(y) = g(x - \alpha(y_\sigma))$, where g is a C^3 function. Then we can obtain:*

$$\partial_t h = -\mu_\sigma \beta(y_\sigma) g'(y), \quad \partial_x h = (1-\beta(y_\sigma))g'(y), \quad \partial_x^2 h = (1-\beta(y_\sigma))^2 g''(y) - \beta(y_\sigma)g'(y),$$

$$\partial_x\partial_t h = -\mu_\sigma(1-\beta(y_\sigma))\beta(y_\sigma)g''(y) - \mu_\sigma\beta'(y_\sigma)g'(y),$$

$$\partial_x^3 h = (1-\beta(y_\sigma))^3 g'''(y) - 3(1-\beta(y_\sigma))\beta'(y_\sigma)g''(y) - \beta''(y_\sigma)g'(y),$$

$$\partial_x^2\partial_t h = \mu_\sigma \left\{ -(1-\beta(y_\sigma))^2\beta(y_\sigma)g'''(y) + 3\beta(y_\sigma)\beta'(y_\sigma)g''(y) \; -2\beta'(y_\sigma)g''(y) - \beta''(y_\sigma)g'(y) \right\}.$$

We follow the notation introduced in (2.8)-(2.11) and we also set

$$S(\tilde{u}) := (1-\lambda\partial_x^2)\partial_t\tilde{u} + \partial_x(\partial_x^2\tilde{u} - \tilde{u} + \tilde{u}^2 + \tilde{u}^3) = S_{mKdV}(\tilde{u}) + S_{BBM}(\tilde{u}),$$

$$S_{mKdV}(\tilde{u}) := \partial_t\tilde{u} + \partial_x(\partial_x^2\tilde{u} - \tilde{u} + \tilde{u}^2 + \tilde{u}^3), \qquad S_{BBM}(\tilde{u}) \; := -\lambda\partial_t\partial_x^2\tilde{u}.$$

$$\delta S_{mKdV}(W) := \partial_t W - \partial_x LW$$

$$L = -\partial_x^2 + 1 - (2Q + 3Q^2), \quad \tilde{u}(t,x) = Q(y) + \tilde{Q}_\sigma(y_\sigma) + W(t,x)$$

$$S[\tilde{u}](t,x) : \; = I + II + III + IV + V$$

$$I = S(Q(y)) = (1-\lambda\partial_x^2)\partial_t(Q(y)) + \partial_x(\partial_x^2(Q(y)) - Q(y) + (Q^2(y) + Q^3(y)))$$

$$II = (((Q(y) + \tilde{Q}_\sigma(y))^2 + (Q(y) + \tilde{Q}_\sigma(y))^3) - (Q^2(y) + Q^3(y)) - (\tilde{Q}_\sigma^2(y_\sigma) + \tilde{Q}_\sigma^3(y_\sigma)))_x$$

$$III = \delta S_{mKdV} = W_t - (LW)_x, \qquad IV = S_{BBM} = -\lambda\partial_x^2\partial_t W$$

$$V = ((Q(y) + \tilde{Q}_\sigma(y) + W)^2 + (Q(y) + \tilde{Q}_\sigma(y) + W)^3) - ((Q(y) + \tilde{Q}_\sigma(y))^2 + (Q(y) + \tilde{Q}_\sigma(y))^3) - (2Q + 3Q^2)W)_x$$

Lemma 4

$$\mu_\sigma = \frac{1-\sigma}{1-\lambda\sigma} = 1 + (\lambda-1)\sigma\sum_{j=0}^{\infty}(\lambda\sigma)^j, \;\; \frac{1}{\theta_\sigma} = \frac{1}{1-\lambda} - \frac{\lambda}{1-\lambda}\sigma, \; \theta_\sigma = \frac{1-\lambda}{1-\lambda\sigma} = (1-\lambda)\sum_{j=0}^{\infty}(\lambda\sigma)^j.$$

Lemma 5 *Let*

$$\beta = a_{1,0}\tilde{Q}_\sigma + a_{1,1}\sigma\tilde{Q}_\sigma + a_{2,0}\tilde{Q}_\sigma^2 + a_{3,0}\tilde{Q}_\sigma^3 + a_{2,1}\sigma\tilde{Q}_\sigma^2 + a_{1,2}\sigma^2\tilde{Q}_\sigma.$$

Then,

$$\beta' = a_{1,0}(\tilde{Q}_\sigma)' + a_{1,1}\sigma(\tilde{Q}_\sigma)' + a_{2,0}(\tilde{Q}_\sigma^2)' + a_{3,0}(\tilde{Q}_\sigma^3)' + a_{2,1}\sigma(\tilde{Q}_\sigma^2)' + a_{1,2}\sigma^2(\tilde{Q}_\sigma)',$$

$$\beta'' = \sigma\tilde{Q}_\sigma a_{1,0} + \tilde{Q}_\sigma^2(-\frac{a_{1,0}}{1-\lambda}) + \sigma^2 a_{1,1}\tilde{Q}_\sigma + \sigma\tilde{Q}_\sigma^2(\frac{\lambda a_{1,0}}{1-\lambda} - \frac{\sigma a_{1,1}}{1-\lambda} + 4a_{2,0}) + \tilde{Q}_\sigma^3(-\frac{10a_{2,0} + 3a_{1,0}}{3(1-\lambda)}) + \sigma^3 O(\tilde{Q}_\sigma),$$

$$\beta^2 = a_{1,0}^2\tilde{Q}_\sigma^2 + a_{1,1}^2\sigma^2\tilde{Q}_\sigma^2 + 2a_{1,0}a_{1,1}\sigma\tilde{Q}_\sigma^2 + 2a_{1,0}a_{2,0}\tilde{Q}_\sigma^3 + \sigma^3 O(\tilde{Q}_\sigma),$$

$$(\beta^2)' = a_{1,0}^2(\tilde{Q}_\sigma^2)' + a_{1,1}^2\sigma^2(\tilde{Q}_\sigma^2)' + 2a_{1,0}a_{1,1}\sigma(\tilde{Q}_\sigma^2)' + 2a_{1,0}a_{2,0}(\tilde{Q}_\sigma^3)' + \sigma^3 O(\tilde{Q}_\sigma).$$

Lemma 6 *Let A and q be C^3-functions. Then*

$$\begin{aligned}
\delta S_{mKdV}(A(y)q(y_\sigma)) \;\; = \;\; & q(y_\sigma)[-(LA)'(y) + \beta(y_\sigma)(-3A'' - (2Q^2 + 3Q^3)A + (1-\mu_\sigma)A)'(y) \\
- \;\; & \beta'(y_\sigma)(3A'')(y) + \beta^2(y_\sigma)(3A''')(y) + (\beta^2)'(y_\sigma)(3A''/2)(y) - \beta''(y_\sigma)A'(y) - \beta^3(y_\sigma)A'''(y)] \\
+ \;\; & q'(y_\sigma)[3A''(y) + (2Q + 3Q^2)A + (\mu_\sigma - 1)A(y) - \beta(y_\sigma)(6A'')(y) \\
- \;\; & \beta'(y_\sigma)(3A')(y) + \beta^2(y_\sigma)(3A'')(y)] \\
+ \;\; & q''(y_\sigma)[3(1-\beta(y_\sigma))A'(y)] + q'''(y_\sigma)A(y).
\end{aligned}$$

Lemma 7 *Let A and q be C^3-functions. For simplify, we omit the variable y of $A(y)$ and y_σ of $q(y_\sigma), \beta(y_\sigma)$.Then*

$$\begin{aligned}
S_{BBM}(A(y)q(y_\sigma)) \;\; = \;\; & \lambda\mu_\sigma q[\beta A''' + \beta'(2A'')] \; + \lambda\mu_\sigma q[\beta^2(-2A''') + (\beta^2)'(-3A''/2) + \beta''A' + \beta^3 A'''] \\
+ \;\; & \lambda\mu_\sigma q'[-A'' + \beta(4A'') + \beta'(3A') + \beta^2(-3A'')] + \lambda\mu_\sigma q''[-2A' + \beta(3A')] + \lambda\mu_\sigma q'''(-A).
\end{aligned}$$

Proof. The proof of Lemma 2.3- Lemma 2.6 is similar to Claims B.1-B.5[9], so it is omitted. ■

Lemma 8

$$\begin{aligned} S(Q) &= \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}^k_\sigma(y_\sigma)a_{k,l}\left\{(\lambda-3)Q''-(2Q^2+3Q^3)\right\}'(y) + (\tilde{Q}^k_\sigma)'(y_\sigma)a_{k,l}(2\lambda-3)Q''(y)\right) \\ &+ \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}^k_\sigma(y_\sigma)F^I_{k,l}(y) + (\tilde{Q}^k_\sigma)'(y_\sigma)G^I_{k,l}(y)\right) + \sigma^3 O(\tilde{Q}_\sigma) \end{aligned}$$

We only explicitly compute to the order of $\sigma^l(\tilde{Q}^k_\sigma)'$ for $1 \le k+l \le 2$. From the $S(Q)$, we can know: $F^I_{1,0}=0,\ G^I_{1,0}=0$.

Proof. With $A(y)=Q(y)$ and $q=1, Q''+(Q^2+Q^3)=Q$,$1-\mu_\sigma=(1-\lambda)\sigma+\lambda(1-\lambda)\sigma^2+\sigma^3O(\varepsilon)$,by Lemma 2.5, we can obtain:

$$\begin{aligned} S_{mKdV}(Q(y)) &= \beta(y_\sigma)(-3Q''-(2Q^2+3Q^3))' - \beta'(y_\sigma)(3Q'') \\ &+ \beta^2(y_\sigma)(3Q''') + (\beta^2)'(y_\sigma)(3Q''/2) - \sigma\beta(y_\sigma)(\lambda-1)Q' \\ &- \beta''(y_\sigma)Q' - \beta^3(y_\sigma)Q''' - (\lambda-1)\lambda\sigma^2\beta(y_\sigma)Q' + \sigma^3O(\tilde{Q}_\sigma). \end{aligned}$$

Since,$\lambda\mu_\sigma = 1-\sigma\theta = \lambda-\lambda\sigma(1-\lambda)-\lambda^2\sigma^2(1-\lambda)$, using Lemma 2.6, we get:

$$\begin{aligned} S_{BBM}(Q(y)) &= \beta(y_\sigma)(\lambda Q''') + \sigma^2\beta(y_\sigma)\lambda^2(\lambda-1)Q''' + \sigma\beta(y_\sigma)\lambda(\lambda-1)Q''' \\ &+ \beta'(y_\sigma)(2\lambda Q'') + \sigma\beta'(y_\sigma)2\lambda(\lambda-1)Q'' + \sigma^2[\beta]'(y_\sigma)\lambda^2(\lambda-1)(2Q'') \\ &+ \beta^2(y_\sigma)(-2\lambda Q''') + \sigma\beta^2(y_\sigma)\lambda(\lambda-1)(-2Q''') + (\beta^2)'(y_\sigma)(-3\lambda Q''/2) \\ &+ \sigma(\beta^2)'(y_\sigma)\lambda(\lambda-1)(-3Q''/2) + \beta''(y_\sigma)(\lambda Q') + \sigma\beta''(y_\sigma)\lambda(\lambda-1)Q' + \beta^3(y_\sigma)(\lambda Q''') + \sigma^3(\tilde{Q}_\sigma). \\ S(Q) &= \tilde{Q}_\sigma(y_\sigma)a_{1,0}\left\{(\lambda-3)Q''-(2Q^2+3Q^3)\right\}' + \tilde{Q}_\sigma(y_\sigma)a_{1,0}(2\lambda-3)Q'' \\ &+ \tilde{Q}^2_\sigma(y_\sigma)(a_{2,0}\left\{(\lambda-3)Q''-(2Q^2+3Q^3)\right\}' + (3-2\lambda)a^2_{1,0}Q''' + a_{1,0}Q' \\ &+ (\tilde{Q}^2_\sigma)'(y_\sigma)(a_{2,0}(2\lambda-3)Q'' + \frac{3}{2}(1-\lambda)a^2_{1,0}Q'' \\ &+ \sigma\tilde{Q}_\sigma(y_\sigma)(a_{1,1}(\lambda-3)Q''-(2Q^2+3Q^3)' + \lambda(\lambda-1)a_{1,0}Q''' \\ &+ \sigma\tilde{Q}'_\sigma(y_\sigma)a_{1,1}(2\lambda-3)Q'' + 2\lambda(\lambda-1)a_{1,0}Q'' \\ &+ \sum_{k+l=3}\sigma^l\tilde{Q}^k_\sigma(y_\sigma)a_{k,l}\left\{(\lambda-3)Q''-(2Q^2+3Q^3)\right\}'(y) + (\tilde{Q}^k_\sigma)'(y_\sigma)a_{k,l}(2\lambda-3)Q''(y) \\ &+ \sum_{k+l=3}\sigma^l\tilde{Q}^k_\sigma(y_\sigma)F^I_{k,l} + (\tilde{Q}^k_\sigma)'(y_\sigma)G^I_{k,l} + \sigma^3(\tilde{Q}_\sigma), \end{aligned}$$

where for all $k+l=3,\ F^I_{k,l}\in\ y$ and $G^I_{k,l}\in\ y$ are as in the statement of Lemma 2.7. ∎

Lemma 9 *In Lemma 2.8, $l=1, k=1, l+k=3$ are not needed.*

$$II = \sum_{(k,l)\in\Sigma_0}\sigma^l\left(\tilde{Q}^k_\sigma(y_\sigma)F^{II}_{k,l}(y) + (\tilde{Q}^k_\sigma)'(y_\sigma)G^{II}_{k,l}(y)\right) + \sigma^3O(\tilde{Q}_\sigma)$$

$$F^{II}_{1,0}(y) = (2Q+3Q^2)', \qquad G^{II}_{1,0}(y) = (2Q+3Q^2)$$

Proof. First, we define $II^* = ((Q(y)+\tilde{Q}_\sigma(y))^2 + (Q(y)+\tilde{Q}_\sigma(y))^3) - (Q^2(y)+Q^3(y)) - (\tilde{Q}^2_\sigma(y_\sigma)+\tilde{Q}^3_\sigma(y_\sigma))$ By Taylor expansion, we can get:

$$\begin{aligned} II=(II^*)_x &= (2Q+3Q^2)'(1-\beta)\tilde{Q}_\sigma + (2Q+3Q^2)\tilde{Q}'_\sigma + \frac{1}{2}(2Q+3Q^2)'(1-\beta)\tilde{Q}^2_\sigma + \frac{1}{2}(2+6Q)(\tilde{Q}^2_\sigma)' \\ &+ (1-\beta)\tilde{Q}^3_\sigma + (\tilde{Q}^3_\sigma)' - (\tilde{Q}^2_\sigma+\tilde{Q}^3_\sigma)' + \sigma^3O(\tilde{Q}_\sigma) \end{aligned}$$

∎

Using the Lemma 2.4, we can obtain:

$$\begin{aligned} II &= (2Q+3Q^2)'\tilde{Q}_\sigma + (2Q+3Q^2)\tilde{Q}' + \frac{1}{2}(((2+6Q)') - a_{1,0}(2Q+3Q^2)'\tilde{Q}^2_\sigma + \frac{1}{2}(1+6Q)(\tilde{Q}^2_\sigma)' \\ &- (2Q+3Q^2)'a_{1,1}\sigma\tilde{Q}_\sigma + (1-\frac{1}{2}a_{1,0}(2+6Q) - (2Q+3Q^2)a_{2,0})'\tilde{Q}^3_\sigma + \sigma^3O(\tilde{Q}_\sigma) \end{aligned}$$

Lemma 10

$$III = \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}^k_\sigma(y_\sigma)(-LA_{k,l})'(y) + (\tilde{Q}^k_\sigma)'(y_\sigma)((-LB_{k,l})' + 3A''_{k,l} + (2Q+3Q^2)A_{k,l})(y)\right) + \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}^k_\sigma(y_\sigma)F^{III}_{k,l}(y) + (\tilde{Q}^k_\sigma)'(y_\sigma)G^{III}_{k,l}(y)\right) + \sigma^3 O(\tilde{Q}_\sigma)$$

$$F^{III}_{1,0} = 0, \quad G^{III}_{1,0} = 0$$

And for all $(k,l)\in\Sigma_0$ such that $1 \le k'+l' \le 2, F^{III}_{k,l},\ G^{III}_{k,l}$, depend on $A_{k',l'}, B_{k',l'}$ for$1 \le k'+l' \le 2$. Moreover, if $A_{k',l'}$, are even and $B_{k',l'}$ are odd then $F^{III}_{k,l}$are odd and $G^{III}_{k,l}$ are even.

Proof. First, we compute$\delta S_{mKdV}(A_{1,0}\tilde{Q}_\sigma)$, using Lemma 2.4 and Lemma 2.5, we can get:

$$\begin{aligned}\delta S_{mKdV}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma)) &= \tilde{Q}_\sigma(y_\sigma) - (LA_{1,0})' + a_{1,0}\tilde{Q}_\sigma(y_\sigma)(-3A''_{1,0} - (2Q+3Q^2)A_{1,0})' - a_{1,0}\tilde{Q}'_\sigma(y_\sigma)(3A''_{1,0})\} \\ &+ \tilde{Q}'_\sigma(y_\sigma)3A''_{1,0} + (2Q+3Q^2)A_{1,0} - a_{1,0}\tilde{Q}_\sigma(y_\sigma)(6A''_{1,0}) + \tilde{Q}''_\sigma(y_\sigma)(3A'_{1,0}) \\ &+ \tilde{Q}'''_\sigma(y_\sigma)A_{1,0} + \sigma\tilde{Q}'_\sigma(y_\sigma)(\lambda-1)A_{1,0} + \sigma^2 O(\tilde{Q}_\sigma(y_\sigma)).\end{aligned}$$

■

Note in particular that we have used $(\tilde{Q}'_\sigma)^2 = \sigma^2 O(\tilde{Q}_\sigma), f(\tilde{Q}_\sigma) = \tilde{Q}^2_\sigma + \sigma^2 O(\tilde{Q}_\sigma)$.

$$\begin{aligned}\tilde{Q}''_\sigma(y_\sigma)(3A'_{1,0}) + \tilde{Q}'''_\sigma(y_\sigma)A_{1,0} &= \sigma\tilde{Q}_\sigma(y_\sigma) - \frac{1}{1-\lambda}\tilde{Q}^2_\sigma(y_\sigma)(3A'_{1,0}) \\ &+ (\sigma\tilde{Q}'_\sigma(y_\sigma) - \frac{1}{1-\lambda}(\tilde{Q}^2_\sigma(y_\sigma))')A_{1,0} + \sigma^2 O(\tilde{Q}_\sigma).\end{aligned}$$

So

$$\begin{aligned}\delta S_{mKdV}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma)) &= \tilde{Q}_\sigma(y_\sigma)(-LA_{1,0})' + \tilde{Q}'_\sigma(y_\sigma)3A''_{1,0} + (2Q+3Q^2)A_{1,0} \\ &+ \tilde{Q}^2_\sigma(y_\sigma)(a_{1,0}(-3A''_{1,0} - (2Q+3Q^2)A_{1,0})' - \frac{3A'_{1,0}}{1-\lambda}) \\ &+ \sigma\tilde{Q}_\sigma(y_\sigma)(3A'_{1,0}) + \lambda A_{1,0}\sigma\tilde{Q}'_\sigma(y_\sigma) + \sigma^2 O(\tilde{Q}_\sigma).\end{aligned}$$

The compute of $\delta S_{mKdV}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma))$ is in a similar way:

$$\begin{aligned}\delta S_{mKdV}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma)) &= \tilde{Q}'_\sigma(y_\sigma)(-LB_{1,0})' + \tilde{Q}^2_\sigma(-\frac{1}{1-\lambda}(3B''_{1,0} + (2Q+3Q^2)B_{1,0})) \\ &+ (\tilde{Q}^2_\sigma)'(y_\sigma)a_{1,0}\frac{(-3B''_{1,0} - (2Q+3Q^2)B_{1,0})'}{2} - \frac{3B'_{1,0}}{1-\lambda} \\ &+ \sigma\tilde{Q}_\sigma(y_\sigma)(3B''_{1,0} + (2Q+3Q^2)B_{1,0}) + \sigma\tilde{Q}'_\sigma(y_\sigma)3B'_{1,0} + \sigma^2 O(\tilde{Q}_\sigma(y_\sigma)).\end{aligned}$$

Similarly, we have for all (k,l) with $2 \le k+l \le 3$, combining the above, we obtain Lemma 2.9.

Lemma 11

$$S_{BBM}(W) = \sum_{(k,l)\in\Sigma_0} \sigma^l (\tilde{Q}^k_\sigma)'(y_\sigma)(-\lambda A''_{k,l})(y) + \sum_{(k,l)\in\Sigma_0} \sigma^l \left(\tilde{Q}^k_\sigma(y_\sigma)F^{IV}_{k,l}(y)) + (\tilde{Q}^k_\sigma)(y_\sigma)G^{IV}_{k,l}(y))\right) + \sigma^3 O(\tilde{Q}_\sigma(y_\sigma)),$$

$$F^{IV}_{1,0} = 0, \quad G^{IV}_{1,0} = 0.$$

And for all $(k,l)\in\Sigma_0$ such that $1 \le k'+l' \le 2, F^{III}_{k,l},\ G^{III}_{k,l}$, depend on $A_{k',l'}, B_{k',l'}$ for$1 \le k'+l' \le 2$. Moreover, if $A_{k',l'}$, are even and $B_{k',l'}$ are odd then $F^{III}_{k,l}$ are odd and $G^{III}_{k,l}$ are even.

Proof. First, we compute $S_{BBM}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma))$.

$$\begin{aligned} S_{BBM}(A_{1,0}(y)\tilde{Q}_\sigma(y_\sigma)) &= \tilde{Q}'_\sigma(y_\sigma)(-\lambda A''_{1,0}) + \sigma\tilde{Q}_\sigma(y_\sigma)(-2\lambda A'_{1,0}) \\ &+ \sigma\tilde{Q}'_\sigma(y_\sigma)(\lambda(1-\lambda)A''_{1,0} - \lambda A_{1,0}) \\ &+ Q^2_\sigma(y_\sigma)(\lambda a_{1,0}A'''_{1,0} + \frac{2\lambda}{1-\lambda}A'_{1,0}) \\ &+ (Q^2_\sigma)'(y_\sigma)(3\lambda a_{1,0}A''_{1,0} + \frac{\lambda}{1-\lambda}A_{1,0}) + \sigma^2 O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

Similarly, we obtain

$$\begin{aligned} S_{BBM}(B_{1,0}(y)\tilde{Q}'_\sigma(y_\sigma)) &= \sigma\tilde{Q}_\sigma(y_\sigma)(-\lambda B''_{1,0}) + \sigma\tilde{Q}'_\sigma(y_\sigma)(-2\lambda B'_{1,0}) + \tilde{Q}^2_\sigma\frac{\lambda}{1-\lambda}B''_{1,0} \\ &+ (\tilde{Q}^2_\sigma)'\left(\frac{\lambda a_{1,0}}{2}B'''_{1,0} + \frac{2\lambda}{1-\lambda}B'_{1,0}\right) + \sigma^2 O(\tilde{Q}_\sigma(y_\sigma)). \end{aligned}$$

Finally, we check that for (k,l) such that $2 \le k+l \le 3$,

$$S_{BBM}(\sigma^l\tilde{Q}^k_\sigma(y_\sigma)A_{k,l}(y)) = \sigma^l(\tilde{Q}^k_\sigma)'(y_\sigma)(-\lambda A''_{k,l}) + \sigma^2 O(\tilde{Q}_\sigma(y_\sigma)),$$

$$S_{BBM}(\sigma^l(\tilde{Q}^k_\sigma)'(y_\sigma)B_{k,l}(y)) = \sigma^2 O(\tilde{Q}_\sigma(y_\sigma)).$$

■

Lemma 12

$$V = \sum_{(k,l)\in\Sigma_0}\sigma\left(\tilde{Q}^k_\sigma(y_\sigma)F^V_{k,l}(y)) + (\tilde{Q}^k_\sigma)(y_\sigma)G^V_{k,l}(y))\right) + \sigma^3 O(\tilde{Q}_\sigma(y_\sigma)),$$

$$F^V_{1,0} = 0, \quad G^V_{1,0} = 0$$

and for all $(k,l) \in \Sigma_0$,$F^V_{k,l}$, $G^V_{k,l}$, depend on $A_{k',l'}$, $B_{k',l'}$ for $1 \le k'+l' \le 2$. Moreover, if $A_{k',l'}$, are even and $B_{k',l'}$are odd then $F^V_{k,l}$ are odd and $G^V_{k,l}$ are even.

Proof. Define:

$$\begin{aligned} V^* &= f(Q(y) + \tilde{Q}_\sigma(y_\sigma) + W) - f(Q(y) + \tilde{Q}_\sigma(y_\sigma)) - f'(Q(y))W \\ &= (Q(y) + \tilde{Q}_\sigma(y) + W)^2 + (Q(y) + \tilde{Q}_\sigma(y) + W)^3) \\ &- ((Q(y) + \tilde{Q}_\sigma(y))^2 + (Q(y) + \tilde{Q}_\sigma(y))^3) - (2Q + 3Q^2)W \end{aligned}$$

Using Taylor expansion, we can obtain:

$$\begin{aligned} V^* &= f''(Q)(\tilde{Q}_\sigma W + \frac{1}{2}W^2) + \frac{1}{2}f^{(3)}(Q)(\tilde{Q}^2_\sigma W + \tilde{Q}_\sigma W^2 + \frac{1}{3}W^3) + \sigma^3 O(\tilde{Q}_\sigma) \\ &= V^*_2 + V^*_3 + \sigma^3 O(\tilde{Q}_\sigma) \end{aligned}$$

First of all, let us consider the third order term V^*_3.A quick computation gives us,

$$\tilde{Q}^2_\sigma W = A_{1,0}\tilde{Q}^3_\sigma + \frac{1}{3}B_{1,0}(\tilde{Q}_\sigma)' + \sigma^3 O(\tilde{Q}_\sigma), \ \ W^2\tilde{Q}_\sigma = A^2_{1,0}\tilde{Q}^3_\sigma + \frac{2}{3}A_{1,0}B_{1,0}(\tilde{Q}^3_\sigma)' + \sigma^3 O(\tilde{Q}_\sigma)$$

$$W^3 = A^3_{1,0}\tilde{Q}^3_\sigma + A_{1,0}B_{1,0}(\tilde{Q}^3_\sigma)' + \sigma^3 O(\tilde{Q}_\sigma)$$

If we suppose $A_{1,0} \in y$and $B_{1,0}$bounded (this is actually the case), we will obtain:

$$V^*_3 = (2A_{1,0} + 2A^2_{1,0} + A^3_{1,0})\tilde{Q}^3_\sigma + B_{1,0}(\tilde{Q}^3_\sigma)' + 3A_{1,0}B_{1,0}(\tilde{Q}^3_\sigma)' + \sigma^3 O(\tilde{Q}_\sigma).$$

Now, let us compute V^*_2, it is easy to check that, up to third order,

$$\tilde{Q}_\sigma W = A_{1,0}\tilde{Q}^2_\sigma + \frac{1}{2}B_{1,0}(\tilde{Q}^2_\sigma)' + A_{1,1}\sigma\tilde{Q}^2_\sigma + \frac{1}{2}B_{1,1}\sigma(\tilde{Q}^2_\sigma)' + A_{2,0}\tilde{Q}^3_\sigma + \frac{2}{3}B_{2,0}(\tilde{Q}^3_\sigma)' + \sigma^3 O(\tilde{Q}_\sigma).$$

And

$$\begin{aligned} W^2 &= A_{1,0}^2\tilde{Q}_\sigma^2 + A_{1,0}B_{1,0}(\tilde{Q}_\sigma^2)' + (2A_{1,0}A_{1,1} + B_{1,0}^2)\sigma\tilde{Q}_\sigma^2 + (A_{1,0}B_{1,1} + B_{1,0}A_{1,1})\sigma(\tilde{Q}_\sigma^2)' \\ &+ (2A_{1,0}A_{2,0} - \frac{2}{3}B_{1,0}^2)\tilde{Q}_\sigma^3 + \frac{2}{3}(A_{1,0}B_{2,0} + B_{1,0}A_{2,0})(\tilde{Q}_\sigma^3)' + \sigma^3O(\tilde{Q}_\sigma). \end{aligned}$$

Then, we obtain:

$$\begin{aligned} V_2^* &= (1+3Q)(2A_{1,0} + A_{1,0}^2)\tilde{Q}_\sigma^2 + (1+3Q)(B_{1,0} + A_{1,0}B_{1,0})(\tilde{Q}_\sigma^2)' + (1+3Q)(2A_{1,1} + 2A_{1,0}A_{1,1} + B_{1,0}^2)\sigma\tilde{Q}_\sigma^2 \\ &+ (1+3Q)(A_{1,0}B_{1,1} + B_{1,1} + A_{1,1}B_{1,0})\sigma(\tilde{Q}_\sigma^2)' + (1+3Q)(2A_{1,0}A_{2,0} + 2A_{2,0} - \frac{2}{3}B_{1,0}^2)\tilde{Q}_\sigma^3 \\ &+ (1+3Q)(\frac{2}{3}A_{1,0}B_{2,0} + \frac{4}{3}B_{2,0} + \frac{2}{3}A_{2,0}B_{1,0})(\tilde{Q}_\sigma^3)' + \sigma^3O(\tilde{Q}_\sigma) \end{aligned}$$

Combining V_2^* and V_3^*, we can have:

$$\begin{aligned} V &= (1+3Q)(2A_{1,0} + A_{1,0}^2))'\tilde{Q}_\sigma^2 \\ &+ (1+3Q)(2A_{1,0} + A_{1,0}^2) + ((B_{1,0} + A_{1,0}B_{1,0}))'(\tilde{Q}_\sigma^2)' \\ &+ F_{2,1}^V\sigma\tilde{Q}_\sigma^2 + G_{2,1}^V(\sigma\tilde{Q}_\sigma^2)' + F_{3,0}^V\sigma\tilde{Q}_\sigma^3 + G_{3,0}^V(\sigma\tilde{Q}_\sigma^3)' + \sigma^3O(\tilde{Q}_\sigma) \end{aligned}$$

■

Putting together Lemmas 2.7-2.11, we obtain Proposition 2.1, in particular, the explicit expressions of $F_{k,l}$ and $G_{k,l}$ for $1 \leq k+l \leq 2$.

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