ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.23(2017) No.3, pp.181-192

Numerical Solution of Fuzzy Fractional Differential Equations by Predictor-Corrector Method

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(Received 26 October 2015, accepted 19 January 2017)

Abstract: In this paper we study a numerical method for fuzzy fractional differential equations using Caupto fractional derivative by an Predictor-Corrector method. In addition, this method is illustrated by solving some numerical examples.
Keywords: Fuzzy system; Fuzzy Fractional Differential Equations; Predictor-Corrector Method

1 Introduction

Agarwal et, al. [1] have taken an initiative to introduce the concept of solution for Fuzzy Fractional Differential Equations(FFDEs). This contribution has motivated several authors to establish some results on the existence and uniqueness of solution [2]. Allahviranloo et, al. [3] derived the explicit solution of FFDEs using the Riemann-Liouville H-derivative. Recently, Salahshour et, al. [22] applied fuzzy Laplace transforms [4] to solve FFDEs. Basically, the proposed ideas are a generalization of the theory and solution of fuzzy differential equations [6, 7, 8, 14, 15, 23]. However, the authors considered FFDEs under the Riemann-Liouville H-derivative. Again, it requires a quantity of fractional H-derivative of an unknown solution at the fuzzy initial point. In particular Ahmad et, al. [5] have discussed numerical solution of FFDEs Euler method using Zadeh's extension principle. The theory and application of fractional differential equations under both types of fractional derivatives have been discussed by many authors [9, 11, 12, 13, 16, 17, 18, 19, 21, 24].

The structure of this paper is organized as follows. In section 2. we bring definitions to fuzzy valued functions. In section 3 we define fuzzy fractional differential systems. In sections 4 and 5, we present the solution of fuzzy fractional differential equations analytically and numerically using Predictor-Corrector method. The proposed algorithm is illustrated by solving some examples in section 6.

2 Preliminaries

In this section, some definitions and basic concepts which will be used in this paper.

Let $I = [0, 1] \subseteq R$ be as compact interval and let E^n denote the set of all $u : R^n \to I$ such that u satisfies the following conditions
(i) u is normal that is there exists an $x_0 \in R^n$ such that $u(x_0) = 1$,
(ii) u is fuzzy convex,
(iii) u is upper semicontinuous,
(iv) $[u]^0 = cl\{x \in R^n : u(x) > 0\}$ is compact. Then, from (i) - (iv), it follows that the $\alpha-$ level set $[u]^\alpha \in P_k(R)^n$ for all $0 \le \alpha \le 1$. If $g : R^n \times R^n \to R^n$ is a function, then using Zadeh's extension principle we can extend g to $E^n \times E^n \to E^n$ by the equation $\tilde{g}(u, v)(z) = \sup\limits_{z=g(x,y)} \min\Big\{u(x), u(y)\Big\}$ It is well known that $\tilde{g}\Big[(u, v)\Big]^\alpha = g\Big([u]^\alpha, [v]^\alpha\Big)$. For all $u, v \in E^n$, $0 \le \alpha \le 1$, and continuous function g. Further we have

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IJNS.2017.06.15/963

$$[u+v]^\alpha = \Big([u]^\alpha + [v]^\alpha\Big),$$
$$[ku]^\alpha = k[u]^\alpha.$$

where $k \in R$. The real numbers can be embedded in E^n by the rule $c \to \hat{c}(t)$
where

$$\hat{c}(t) = \begin{cases} 1\ ,\ \text{for t=c}, \\ 0\ ,\ \text{elsewhere}. \end{cases}$$

Definition 1 *A real function $x(t),\ t>0$, is said to be in the space $C_\mu,\ \mu \in R$, if there exist a real number $\rho > \mu$, such that $x(t) = t^p x_1(t)$, where $x_1(t) \in C(0,\infty)$ and it is said to be in the space C_μ^n if and only if $x^n \in C_\mu,\ n \in N$.*

Definition 2 *The Caputo fractional derivative of x of order $q>0$ with $a \geq 0$ is defined as*

$$^cD_a^q x(t) = \frac{1}{\Gamma(n-q)}\int_a^t (t-s)^{n-q-1}x^n(s)ds,$$

for $n-1 < q \leq n,\ n \in N,\ t \geq a,\ x \in C_{-1}^n$.

Two basic properties of the Caputo fractional derivative are as follows:
(i) Let $x \in C_{-1}^n,\ n \in N$. Then $^cD_a^q x,\ 0 \leq q \leq n$, is well defined and $^cD_a^q x \in C_{-1}$,
(ii) Let $n-1 < q \leq n,\ n \in N$, and $x \in C_\mu^n,\ \mu \geq -1$. Then

$$I_a^q(^cD_a^q)x(t) = x(t) - \sum_{k=0}^{n-1} x^{(k)}(a)\frac{(t-a)^k}{k!}.$$

The Laplace transform of the Caputo fractional derivative is given by

$$L\left\{^cD_a^q x(t)\right\} = s^q x(s) - \sum_{k=0}^{n-1} s^{q-k-1}x^k(0), \quad n-1 < q \leq n.$$

There exist a relation between the Riemann-Liouville fractional derivative and Caputo fractional derivative,

$$^cD_{a+}^q x(t) = D_{a+}^q x(t) - \sum_{k=0}^{n-1}\frac{x^k(a)}{\Gamma(k-q+1)}(t-a)^{k-q}.$$

Theorem 1 *Let $\tilde{f}(x) \in C^F[0,a] \cap L^F[0,a]$, be a fuzzy valued function. The Riemann-Liouville integral of the $\tilde{f}(x)$, based on its $\alpha-$ cut representation can be expressed as follows:*

$$\Big[J^q\tilde{f}(x)\Big]^\alpha = \Big[J^q\underline{f}^\alpha(x), J^q\overline{f}^\alpha(x)\Big],\ \ 0 \leq \alpha \leq 1,$$

where

$$J^q\underline{f}^\alpha(x) = \frac{1}{\Gamma(q)}\int_0^x \frac{\underline{f}^\alpha(t)}{(x-t)^{1-q}}dt\ \ x,q \in R_+,$$
$$J^q\overline{f}^\alpha(x) = \frac{1}{\Gamma(q)}\int_0^x \frac{\overline{f}^\alpha(t)}{(x-t)^{1-q}}dt\ \ x,q \in R_+.$$

3 Fuzzy Cauchy Problem

Consider Fuzzy Fractional Initial Value Problem (FFIVP)

$$\begin{cases} ^cD_a^q\tilde{x}(t) = f(t,\tilde{x}(t)),\ \ 0<q\leq 1,\ t>a, \\ \tilde{x}(t_0) = \tilde{x}_0. \end{cases} \tag{1}$$

where $\tilde{x}(t)$ is a fuzzy function of t, $f(t, \tilde{x}(t))$ is a fuzzy function of the crisp variable t and the fuzzy variable $\tilde{x}(t)$, ${}^{c}D_{a}^{q}\tilde{x}(t)$ is the fuzzy Caupto fractional derivative of $\tilde{x}(t)$ and $\tilde{x}(t_0) = \tilde{x}_0$ is a triangular or a triangular shaped fuzzy number. Therefore we have a fuzzy Cauchy problem.

We denote the fuzzy function $\tilde{x}(t)$ by $\tilde{x}(t) = [\underline{x}(t;\alpha), \overline{x}(t;\alpha)]$. It means that the α-level set of $x(t)$ for $t \in [t_0, T]$ is

$$\Big[\tilde{x}(t_0)\Big]_{\alpha} = \Big[\underline{x}(t_0;\alpha),\ \overline{x}(t_0;\alpha)\Big],\ \Big[\tilde{x}(t)\Big]_{\alpha} = \Big[\underline{x}(t;\alpha),\ \overline{x}(t;\alpha)\Big], \quad \alpha \in (0,1].$$

By using the extension principle of Zadeh's we have the membership function

$$f\Big(t, \tilde{x}(t)\Big)(s) = \sup\Big\{x(t)(\tau) | s = f(t,\tau)\Big\}, \quad s \in R, \tag{2}$$

so $f\Big(t, \tilde{x}(t)\Big)$ is a fuzzy number. From this it follows that

$$\Big[\tilde{f}\Big(t, x(t)\Big)\Big]_{\alpha} = \Big[\underline{f}\Big(t, x(t;\alpha\Big),\ \overline{f}\Big(t, x(t;\alpha)\Big)\Big], \quad \alpha \in (0,1], \tag{3}$$

where

$$\begin{aligned} \underline{f}\Big(t, x(t);\alpha\Big) &= \min\Big\{\underline{f}(t,u) | u \in [\underline{x}(t;\alpha),\ \overline{x}(t;\alpha)]\Big\}, \\ \overline{f}\Big(t, x(t);\alpha\Big) &= \max\Big\{\overline{f}(t,u) | u \in [\underline{x}(t;\alpha),\ \overline{x}(t;\alpha)]\Big\}. \end{aligned} \tag{4}$$

4 Analytical Solution of Fuzzy Fractional Differential Equations

Consider the following fractional differential equations

$${}^{c}D_{a}^{q}x(t) = f(t, x(t)),\ x(t_0) = x_0, \tag{5}$$

where $f : [t_0, x(t)] \times R \to R$ is a real valued function, $x_0 \in R$, and $q \in (0,1]$. If $q = 1$, then (5) becomes an ordinary differential equation.

Assume that the initial value is replaced by a fuzzy number, then we have the following fuzzy fractional differential equation

$${}^{c}D_{a}^{q}\tilde{x}(t) = f(t, \tilde{x}(t)),\ \tilde{x}(t_0) = \tilde{x}_0, \tag{6}$$

where $x_0 \in F(R)$. If $q \in (0,1]$. If $q = 1$, then (6) becomes an fuzzy differential equation.

In order to find the solution of (6), we first find the solution of (5). Taking Laplace transform on both sides of (5), we get

$$Ł\left[{}^{c}D_{a}^{q}x(t)\right] = L\left[f(t, x(t))\right]. \tag{7}$$

It follows that

$$s^{q}L\left\{x(t)\right\} - x(t_0)s^{q-1} = L\left[f(t, x(t))\right],\ L\left[x(t)\right] = m(s). \tag{8}$$

Then by taking the inverse Laplace transform to (8), we have

$$x(t) = L^{-1}\left[m(s)\right] = g(t, q, x_0), \tag{9}$$

for $t \in [t_0, T]$ and $x_0 \in R$. In order to find the solution of (6), we fuzzify (9) using Zadeh's extension principle. Hence we have

$$\tilde{x}(t) = \tilde{g}(t, q, u) | u \in (\underline{x}_0, \overline{x}_0). \tag{10}$$

which is the solution of (6).

Theorem 2 *Let G be an open set in R and $[\tilde{x}_0]^\alpha \in F(R) \subset G$. Suppose that f is continuous and that for each $q \in (0,1)$ and each $x_0 \in G$ there exist a unique solution $g(t,q,x_0)$ of the problem (5) and that $g(t,q,x_0)$ is continuous in G for each $t \in [t_0,T]$ fixed. Then, there exist a unique fuzzy solution $\tilde{x}(t) = \tilde{g}(t,q,u)|u \in (\underline{x}_0, \overline{x}_0)$ of the problem (6).*

Theorem 3 *If $X : [t_0,T] \to F(R)$ is a fuzzy solution of (6) and denoting $[\tilde{x}(t)]^\alpha = [\underline{x}^\alpha(t;\alpha), \overline{x}^\alpha(t;\alpha)]$ for $\alpha \in [0,1]$ then*
(i) $[\tilde{x}(t)]^\alpha$ is compact subset of R,
(ii) $[\tilde{x}(t)]^{\alpha_2} \subseteq [\tilde{x}(t)]^{\alpha_1}$ for $0 \le \alpha_1 \le \alpha_2 \le 1$,
(iii) $[\tilde{x}(t)]^\alpha = \bigcap_{n=1}^{\infty}[\tilde{x}(t)]^{\alpha_n}$ for any nondecreasing sequence $\alpha_n \to \alpha$ in $[0,1]$.

Theorem 4 *If $x(t) = g(t,q,x_0)$ is obtained by using Theorem 2 and $[\tilde{x}(t)]^\alpha = [\underline{x}(t;\alpha), \overline{x}(t;\alpha)]$ for $\alpha \in [0,1]$, then $\underline{x}(t;\alpha)$ and $\overline{x}(t;\alpha)$ do not interchange at all $t \in [t_0,\infty)$.*

Proof. We know that $x(t)$ is obtained by Zadeh's extension principle through Theorem 2, then its membership function has the following form:

$$\tilde{x}(t)(y) = \begin{cases} \sup\limits_{x \in g^{-1}(t,q,y)} \tilde{x}_0(t), & \text{if y} \in \text{range(g)}, \\ 0, & \text{if y} \notin \text{range(g)}. \end{cases}$$

It follows that

$$\begin{aligned} \underline{x}(t;\alpha) &= \min\{g(t,q,u)|\ u \in [\underline{x}(t;\alpha), \overline{x}(t;\alpha)]\}, \\ \overline{x}(t;\alpha) &= \max\{g(t,q,u)|\ u \in [\underline{x}(t;\alpha), \overline{x}(t;\alpha)]\}, \end{aligned} \tag{11}$$

for $\alpha \in [0,1]$. It is obvious that $\underline{x}(t;\alpha) \le \overline{x}(t;\alpha)$. This holds for all $t \in [t_0,\infty)$. This Completes the proof.

■

5 The Predictor-Corrector Algorithm for Fuzzy Fractional Differential Equations

In this section, we show the Predictor-Corrector algorithm of the following FFIVP

$$\begin{cases} {}^cD_a^q\tilde{x}(t) = f(t,\tilde{x}(t)), 0 <\le 1, t > a, \\ \tilde{x}(0) = \tilde{x}_0. \end{cases} \tag{12}$$

where $\tilde{x}(t)$ is a fuzzy function of t, $f(t,\tilde{x}(t))$ is a fuzzy function of the crisp variable t and the fuzzy variable $\tilde{x}(t),^c D_a^q\tilde{x}(t)$ is the fuzzy Caupto derivative $\tilde{x}(t)$ and $\tilde{x}(t_0) = \tilde{x_0}$ is a triangular or a triangular shaped fuzzy fuzzy number. Using the Laplace transformation formula for the Caupto fractional derivative.

$$L\{{}^cD_a^q\} = s^q\tilde{x}(s) - \sum_{k=0}^{n-1} s^{q-k-1}\tilde{x}^(k)(0), n-1 < q < n. \tag{13}$$

from (12), we have

$$\begin{cases} s^q\tilde{x}(s) - \sum_{k=0}^{n-1} s^{q-k-1}\underline{x}^(k)(0) = F(s,\underline{x}(s),\overline{x}(s)), \\ s^q\overline{x}(s) - \sum_{k=0}^{n-1} s^{q-k-1}\tilde{x}^(k)(0) = G(s,\underline{x}(s),\overline{x}(s)), \end{cases} \tag{14}$$

or

$$\begin{cases} \underline{x}(s) = s^{-q}F(s,\underline{x}(s),\overline{x}(s)) + \sum_{k=0}^{n-1} a^{-k-1}\underline{x}^(k)(0), \\ \overline{x}(s) = s^{-q}F(s,\overline{x}(s),\overline{x}(s)) + \sum_{k=0}^{n-1} a^{-k-1}\overline{x}^(k)(0), \end{cases} \tag{15}$$

applying the inverse Laplace transform gives

$$\underline{x}(t) = \sum_{k=0}^{\lceil q \rceil -1} \underline{x}_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau,\underline{x}(\tau),\overline{x}(\tau))d\tau,$$

$$\overline{x}(t) = \sum_{k=0}^{\lceil q \rceil -1} \overline{x}_0^k \frac{t^k}{k!} + \frac{1}{\Gamma(q)} \int_0^t (t-\tau)^{q-1} f(\tau,\underline{x}(\tau),\overline{x}(\tau))d\tau, \tag{16}$$

where the fact

$$L^c D_a^q \tilde{x}(t) = L\left\{ \frac{1}{\Gamma(\mu)} \int_0^t \frac{\tilde{x}(\tau)}{(t-\tau)^{1-\mu}} d\tau \right\} = L\left\{ \frac{t^{\mu-1}}{\Gamma(\mu)} * \tilde{x}(t) \right\} = s^{-\mu}\tilde{x}(s),$$

and

$$L\{t^{\mu-1}\} = s^{\mu}\Gamma(\mu).$$

are used. The approximation is based on the equivalent from of thr Volterra integral equation (16). A fractional Adams Predictor-Corrector approach was firstly developed by [4] to numerically solve the problem (16). Using the standard quadrature techniques for the integral in (16), denote $g(\tau) = f(\tau, x(\tau))$, the integral is replaced by the trapezoidel quadrrature formula at the point t_{n+1}

$$\int_0^{t_{n+1}} (t_{n+1}-\tau)^{q-1}\tilde{g}(\tau)d\tau \approx \int_0^{t_{n+1}} (t_{n+1}-\tau)^{q-1}\tilde{g}_{n+1}(\tau)d\tau, \tag{17}$$

where $\tilde{g}_{n+1}$ is the piecewise linear interpolation of $\tilde{g}$ with nodes $t_j, j = 0, 1, 2, ..., n+1$. After some elementary calculations, the right hand side of (17) gives

$$\int_0^{t_{n+1}} (t_{n+1}-\tau)^{q-1}\tilde{g}_{n+1}(\tau)d\tau = \frac{h^q}{q(q+1)} \sum_{j=0}^{n+1} a_{j,n+1}\tilde{g}(t_j), \tag{18}$$

where the uniform mesh is used and h is the step size. And if we use the product rectangle rule, the right hand of (18) can be written as

$$\int_0^{t_{n+1}} (t_{n+1}-\tau)^{q-1}\tilde{g}_{n+1}(\tau)d\tau = \sum_{j=0}^{n+1} b_{j,n+1}\tilde{g}(t_j), \tag{19}$$

where

$$a_{j,n+1} = \begin{cases} n^{q+1} - (n-q)(n+1)^q, & if \;\; j=0 \\ (n-j-2)^{q+1} - 2(n-j+1)^{q+1} + (n-j)^{q+1}, & if \;\; 1 \le j \le n, \\ 1 & if \;\; j = n+1 \end{cases} \tag{20}$$

and

$$b_{j,n+1} = \frac{h^q}{q}[(n+1-j)^q - (n-j)^q, \qquad if \;\; 0 \le j \le n+1].$$

Then the predictor and corrector formula for solving (16) are given, respectively, by

$$\underline{x}_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q \rceil -1} \frac{t_{n+1}^k}{k!} x_0^(k) + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} F(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)),$$

$$\overline{x}_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q\rceil -1} \frac{t_{n+1}^k}{k!} x_0^(k) + \frac{1}{\Gamma(q)} \sum_{j=0}^{n} b_{j,n+1} G(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)), \tag{21}$$

and

$$\begin{cases} \underline{x}_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q\rceil -1} \frac{t_{n+1}^k}{k!} x_0^(k) + \frac{1}{\Gamma(q+2)} \sum_{j=0}^{n} F(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)) \\ \quad + \frac{1}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} F(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)) \end{cases} \tag{22}$$

$$\begin{cases} \overline{x}_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q\rceil -1} \frac{t_{n+1}^k}{k!} x_0^(k) + \frac{1}{\Gamma(q+2)} \sum_{j=0}^{n} G(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)) \\ \quad + \frac{1}{\Gamma(q+2)} \sum_{j=0}^{n} a_{j,n+1} F(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)) \end{cases} \tag{23}$$

The approximation accuracy of the scheme (21)-(22) is $O(h^{min[2,q+1]})$. Now we make some improvements for scheme (21)-(22). We modify the approximation of (17) as

$$\int_0^{t_{n+1}} (t_{n+1}-\tau)^(q-1)\tilde{g}(\tau)d\tau \approx \int_0^{t_n} (t_{n+1}-\tau)^(q-1)\tilde{g}_n(\tau)d\tau + \int_0^{t_{n+1}} (t_{n+1}-\tau)^(q-1)\tilde{g}_n(\tau)d\tau \tag{24}$$

where $\tilde{g}_n$ is the piecewise linear interpolation for $\tilde{g}$ with nodes and knots chosen at $t_j, j = 0, 1, 2, ..., n$. Then using the standard quadrature technique,the right hand of (24) can be written as

$$\int_0^{t_n} (t_{n+1}-\tau)^(q-1)\tilde{g}_n(\tau)d\tau + \int_0^{t_{n+1}} (t_{n+1}-\tau)^(q-1)\tilde{g}_n(\tau)d\tau = \frac{h^q}{q(q+1)} \sum_{j=0}^{n} \tilde{b}_{j,n+1}, \tilde{g}(t_j) \tag{25}$$

where

$$b_{j,n+1} = \begin{cases} \begin{cases} a_{j,n+1}, & if \quad 0 \le j \le n-1. \\ 2^{q+1}-1, & if \quad j = n \end{cases} & if \quad n > 0 \\ b_{0,1} = q+1, & if \quad n > 0 \end{cases}$$

Hence , this algorithm for the predictor step can be improved as

$$\underline{x}_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q\rceil -1} \frac{t_{n+1}^k}{k!} x_0^(k) + \frac{1}{\gamma(2-q)} \sum_{j=0}^{n} b_{j,n+1} F(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)),$$

$$\overline{x}_h^p(t_{n+1}) = \sum_{k=0}^{\lceil q\rceil -1} \frac{t_{n+1}^k}{k!} x_0^(k) + \frac{1}{\gamma(2-q)} \sum_{j=0}^{n} b_{j,n+1} G(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)), \tag{26}$$

The new predictor-corrector approach (26)and (22) has numerical accuracy $O(h^{min[2,2q+1]})$. Obiviously half of the computational cost can be reduced, for $0 < q \le 1$, if we reformulate (26) and (22) as

$$\underline{x}_h^p(t_{n+1}) = \begin{cases} \underline{x_0} + \frac{h^q}{\Gamma(q+1)} F(t_0, \underline{x}_h(t_0), \overline{x}_h(t_0)), & if \quad n = 0 \\ \underline{x_0} + \frac{h^q}{\Gamma(q+2)} (2^{q+1}-1) F(t_n, \underline{x}_h(t_n), \overline{x}_h(t_n)), & \\ + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n-1} a_{j,n+1} F(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)), & if \quad n \ge 1. \end{cases} \tag{27}$$

$$\underline{x}_h^p(t_{n+1}) = \begin{cases} \underline{x_0} + \frac{h^q}{\Gamma(q+1)} G(t_0, \underline{x}_h(t_0), \overline{x}_h(t_0)), & if \quad n = 0 \\ \underline{x_0} + \frac{h^q}{\Gamma(q+2)} (2^{q+1}-1) G(t_n, \underline{x}_h(t_n), \overline{x}_h(t_n)), & \\ + \frac{h^q}{\Gamma(q+2)} \sum_{j=0}^{n-1} a_{j,n+1} G(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)), & if \quad n \ge 1. \end{cases} \tag{28}$$

Table 1: The approximate solution bypredictor-corrector method to the FFIVP(31) - $\overline{x}(t;\alpha)$ for $q = 0.5$.

t	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	1.0813	1.1173	1.1534	1.1894	1.2255	1.2615	1.2975	1.3336	1.3696	1.4057	1.4417
0.2	1.3181	1.3620	1.4059	1.4499	1.4938	1.5377	1.5817	1.6256	1.6695	1.7135	1.7574
0.3	1.5471	1.5987	1.6503	1.7018	1.7534	1.805	1.8566	1.9081	1.9597	2.00113	2.0628
0.4	1.7849	1.8444	1.9039	1.9634	2.0229	2.0824	2.1419	2.2014	2.2609	2.3204	2.3798
0.5	2.0381	2.106	2.1739	2.2419	2.3098	2.3777	2.4457	2.5136	2.5815	2.6495	2.7174
0.6	2.3111	2.3881	2.4652	2.5422	2.6192	2.6963	2.7733	2.8503	2.9274	3.0044	3.0814
0.7	2.6077	2.6946	2.7815	2.8684	2.9553	3.0423	3.1292	3.2161	3.303	3.3899	3.4769
0.8	2.9313	3.029	3.1267	3.2244	3.3221	3.4198	3.5175	3.6153	3.713	3.8107	3.9084
0.9	3.2855	3.3951	3.5046	3.6141	3.7236	3.8331	3.9426	4.0522	4.1617	4.2712	4.3807
1.0	3.6741	3.7966	3.9191	4.0415	4.164	4.2865	4.4089	4.5314	4.6539	4.7764	4.8988

and

$$\underline{x}_h^p(t_{n+1}) = \begin{cases} \underline{x_0} + \frac{h^q}{\Gamma(q+2)}\left(F(t_1, \underline{x}_h^p(t_1), \overline{x}_h^p(t_1)) + qF(t_0, \underline{x}_h^p(t_0), \overline{x}_h^p(t_0))\right) \qquad if \quad n = 0 \\ +\underline{x_0} + \frac{h^q}{\Gamma(q+2)}\big(F(t_{n+1}, \underline{x}_h^p(t_{n+1})) + (2^{q+1}-2)F(t_n, \underline{x}_h(t_n), \overline{x}_h(t_n))\big) \\ +\frac{h^q}{\Gamma(q+2)}\sum_{j=o}^{n-1} a_{j,n+1}F(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)) \qquad if \quad n \geq 0 \end{cases} \tag{29}$$

$$\underline{x}_h^p(t_{n+1}) = \begin{cases} \underline{x_0} + \frac{h^q}{\Gamma(q+2)}\left(G(t_1, \underline{x}_h^p(t_1), \overline{x}_h^p(t_1)) + qG(t_0, \underline{x}_h^p(t_0), \overline{x}_h^p(t_0))\right) \qquad if \quad n = 0 \\ +\underline{x_0} + \frac{h^q}{\Gamma(q+2)}\big(G(t_{n+1}, \underline{x}_h^p(t_{n+1})) + (2^{q+1}-2)F(t_n, \underline{x}_h(t_n), \overline{x}_h(t_n))\big) \\ +\frac{h^q}{\Gamma(q+2)}\sum_{j=o}^{n-1} a_{j,n+1}G(t_j, \underline{x}_h(t_j), \overline{x}_h(t_j)) \qquad if \quad n \geq 0 \end{cases} \tag{30}$$

6 Numerical Examples

Example 5 *Consider the following FFIVP*

$$\begin{cases} {}^cD_0^q\tilde{x}(t) = \tilde{x}(t), \;\; t \in [0,1], \\ \tilde{x}(0) = (0.75 + 0.25\alpha, 1.125 - 0.125\alpha), \quad 0 < \alpha \leq 1. \end{cases} \tag{31}$$

where $q \in (0,1),\ t > 0$. By using (27)(28)(29) and (30) with N=10, we get the approximate solution as x(1;1)=4.8988. The exact solution is given by

$$\underline{x}(t;\alpha) = (0.75 + 0.25\alpha)E_q(t^q), \quad \overline{x}(t;\alpha) = (1.125 - 0.125\alpha)E_q(t^q),$$

where $E_q(t^q) = \sum_{k=0}^{\infty}\frac{(t^q)^k}{\Gamma(kq+1)} = \sum_{k=0}^{\infty}\frac{(t^q)^k}{(kq)!}.$

The approximate solution by predictor-corrector method are plotted at $t \in [0,1]$ and q=0.5. (see tables 1 - 2 and figure 1) The exact and the approximate solutions by predictor-corrector method are compared and plotted at $t = 1$ and q=0.5. (see tables 1 - 4 and figure 2)

Table 2: The approximate solution by predictor-corrector method to the FFIVP(31) - $\overline{x}(t;\alpha)$ for $q = 0.5$.

t	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	1.6219	1.6039	1.5859	1.5679	1.5498	1.5318	1.5138	1.4958	1.4778	1.4597	1.4417
0.2	1.9771	1.9551	1.9332	1.9112	1.8892	1.8673	1.8453	1.8233	1.8014	1.7794	1.7574
0.3	2.3207	2.2949	2.2691	2.2433	2.2176	2.1918	2.166	2.1402	2.1144	2.0886	2.0628
0.4	2.6773	2.6476	2.6178	2.5881	2.5583	2.5286	2.4988	2.4691	2.4393	2.4096	2.3798
0.5	3.0571	3.0231	2.9892	2.9552	2.9212	2.8873	2.8533	2.8193	2.7853	2.7514	2.7174
0.6	3.4666	3.4281	3.3896	3.3511	3.3125	3.274	3.2355	3.197	3.1585	3.12	3.0814
0.7	3.9115	3.868	3.8246	3.7811	3.7376	3.6942	3.6507	3.6073	3.5683	3.5201	3.3.4769
0.8	4.3969	4.3481	4.2991	4.2504	4.2015	4.1527	4.1038	4.055	4.0061	3.9572	3.9084
0.9	4.9283	4.8735	4.8188	4.764	4.7093	4.6545	4.5998	4.545	4.4902	4.4355	4.3807
1.0	5.5112	5.4499	5.3887	5.3275	5.2662	5.205	5.1438	5.0825	5.0213	4.9601	4.8988

Table 3: The exact solution to the FFIVP(31) - $\underline{x}(t;\alpha)$ for $q = 0.5$.

t	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	1.6726	1.654	1.6354	1.6169	1.5983	1.5797	1.5611	1.5425	1.5239	1.5053	1.4868
0.2	2.0239	2.0014	1.9789	1.9564	1.9339	1.9115	1.889	1.8665	1.844	1.8215	1.7990
0.3	2.3712	2.3448	2.3185	2.2921	2.2658	2.2394	2.2131	2.1867	2.1604	2.134	2.1077
0.4	2.7338	2.7034	2.673	2.6426	2.6123	2.5819	2.5515	2.5211	2.4908	2.4604	2.4300
0.5	3.121	3.0863	3.0516	3.0169	2.9822	2.9476	2.9129	2.8782	2.8435	2.8089	2.7742
0.6	3.5392	3.4998	3.4605	3.4212	3.3819	3.3425	3.3032	3.2639	3.2246	3.1852	3.1459
0.7	3.9938	3.9495	3.9051	3.8607	3.8163	3.7720	3.7276	3.6832	3.6388	3.5945	3.5501
0.8	4.4902	4.4403	4.3904	4.3405	4.2906	4.2407	4.1908	4.1409	4.091	4.0412	3.9913
0.9	5.0334	4.9774	4.9215	4.8656	4.8096	4.7537	4.6978	4.6419	4.5859	4.5300	4.4741
1.0	5.6351	5.5725	5.5099	5.4473	5.3847	5.3220	5.2594	5.1968	5.1342	5.0716	5.0033

Table 4: The exact solution to the FFIVP(31) - $\overline{x}(t;\alpha)$ for $q = 0.5$.

t	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	1.1151	1.1522	1.1894	1.2266	1.2637	1.3009	1.3381	1.3753	1.4124	1.4496	1.4868
0.2	1.3493	1.3942	1.4392	1.4842	1.5292	1.5741	1.6191	1.6641	1.7091	1.7540	1.7990
0.3	1.5808	1.6335	1.6862	1.7388	1.7915	1.8442	1.8969	1.9496	2.0023	2.0550	2.1077
0.4	1.8225	1.8833	1.9440	2.0048	2.0655	2.1263	2.1870	2.2478	2.3085	2.3693	2.4300
0.5	2.0806	2.1500	2.2193	2.2887	2.3581	2.4274	2.4968	2.5661	2.6355	2.7048	2.7742
0.6	2.3594	2.4381	2.5167	2.5954	2.6740	2.7527	2.8313	2.9100	2.9886	3.0673	3.1459
0.7	2.6626	2.7513	2.8401	2.9288	3.0176	3.1063	3.1951	3.2838	3.3726	3.4613	3.5501
0.8	2.9935	3.0932	3.193	3.2928	3.3926	3.4924	3.5921	3.6919	3.7917	3.8915	3.9913
0.9	3.3556	3.4674	3.5793	3.6911	3.803	3.9148	4.0267	4.1385	4.2504	4.3622	4.4741
1.0	3.7567	3.8820	4.0072	4.1324	4.2576	4.3829	4.5081	4.6333	4.7585	4.8838	5.0033

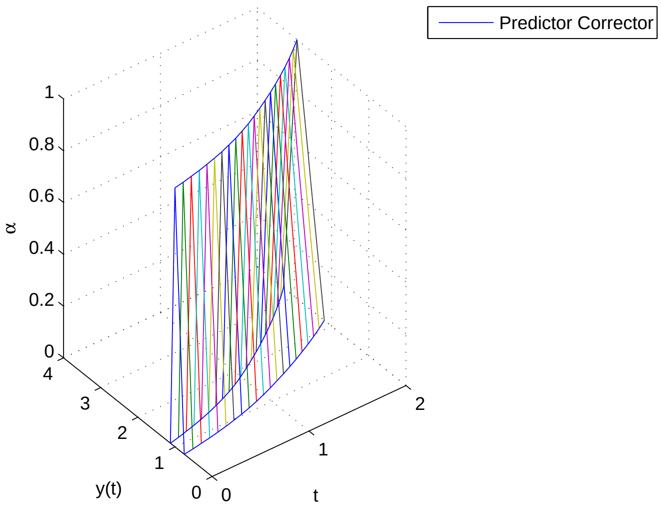

Figure 1: For h=0.1

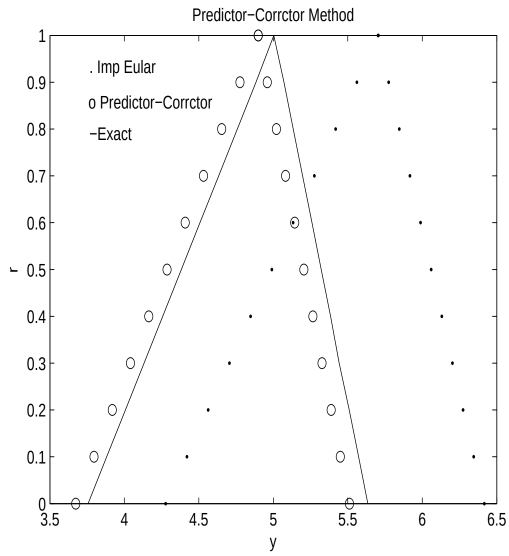

Figure 2: For h=0.1

Example 6 *Consider the following FFIVP*

$$\begin{cases} {}^{c}D_0^q\tilde{x}(t) = -\tilde{x}(t), \;\; t \in [0,1], \\ \tilde{x}(0) = (0.75 + 0.25\alpha, 1.125 - 0.125\alpha), \quad 0 < \alpha \leq 1. \end{cases} \tag{32}$$

where $q \in (0,1),\; t > 0.$
By using (29) and (30) with N=10, we get the approximate solution as x(1;1)=0.3924.
The exact solution is given by

$$\underline{x}(t;\alpha) = (0.75 + 0.25\alpha)E_q(-t^q), \quad \overline{x}(t;\alpha) = (1.125 - 0.125\alpha)E_q(-t^q),$$

where $E_q(-t^q) \;= \sum_{k=0}^{\infty} \frac{(-t^q)^k}{\Gamma(kq+1)} \;= \sum_{k=0}^{\infty} \frac{(-t^q)^k}{(kq)!}.$

The approximate solution by predictor-corrector method are plotted at $t \in [0,1]$ and q=0.75. (see tables 5-6 and figure 3) The exact and the approximate solutions by predictor-corrector are compared and plotted at $t = 1$ and q=0.75.(see tables 5 - 8 and figure 4)

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Table 5: The approximate solution by predictor-corrector method to the FFIVP(32) - $\underline{x}(t;\alpha)$ for $q = 0.75$.

t	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.9313	0.9210	0.9107	0.9003	0.8900	0.8796	0.8693	0.8589	0.8486	0.8385	0.8279
0.2	0.8229	0.8134	0.8047	0.7955	0.7864	0.7772	0.7681	0.7589	0.7498	0.7406	0.7315
0.3	0.7464	0.7334	0.7251	0.7169	0.7086	0.7004	0.6921	0.6839	0.6757	0.6674	0.6592
0.4	0.6761	0.6686	0.6611	0.6536	0.6461	0.6386	0.6310	0.6235	0.6160	0.6085	0.6010
0.5	0.6216	0.6147	0.6078	0.6009	0.5940	0.5871	0.5802	0.5732	0.5663	0.5594	0.5525
0.6	0.5752	0.5688	0.5624	0.5560	0.5496	0.5432	0.5368	0.5304	0.5240	0.5177	0.5113
0.7	0.5350	0.5291	0.5232	0.5172	0.5113	0.5053	0.4994	0.4934	0.4875	0.4815	0.4756
0.8	0.4999	0.4944	0.4888	0.4833	0.4777	0.4722	0.4666	0.4611	0.4555	0.4499	0.4444
0.9	0.4689	0.4637	0.4585	0.4533	0.4481	0.4429	0.4377	0.4325	0.4273	0.4220	0.4168
1.0	0.4414	0.4365	0.4316	0.4267	0.4218	0.4169	0.4120	0.4071	0.4022	0.3973	0.3924

Table 6: The approximate solution by improved Euler method to the FFIVP (22) in Example 7.2 - $\overline{x}(t;\alpha)$ for $q = 0.75$.

t	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.6209	0.6416	0.6623	0.6830	0.7037	0.7244	0.7451	0.7658	0.7865	0.8072	0.8279
0.2	0.5486	0.5669	0.5852	0.6035	0.6218	0.6401	0.6583	0.6766	0.949	0.7132	0.7315
0.3	0.4944	0.5109	0.5273	0.5438	0.5603	0.5768	0.5933	0.6097	0.6262	0.6427	0.6592
0.4	0.4507	0.4658	0.4808	0.4958	0.5108	0.5259	0.5409	0.5559	0.5709	0.5860	0.6010
0.5	0.4144	0.4282	0.4420	0.4558	0.4696	0.4835	0.4973	0.5111	0.5249	0.5387	0.5525
0.6	0.3834	0.3962	0.4090	0.4218	0.4346	0.4473	0.4601	0.4729	0.4857	0.4985	0.5113
0.7	0.3567	0.3686	0.3805	0.3924	0.4042	0.4161	0.4280	0.4399	0.4518	0.4637	0.4756
0.8	0.3333	0.3444	0.3555	0.3666	0.3777	0.3888	0.3999	0.4111	0.4222	0.4333	0.4444
0.9	0.3126	0.3230	0.3335	0.3439	0.3543	0.3647	0.3751	0.3856	0.3960	0.4064	0.4168
1.0	0.2943	0.3041	0.3139	0.3237	0.3335	0.3433	0.3531	0.3629	0.3727	0.3825	0.3924

Table 7: The exact solution to the FFIVP(22) in Example 7.2 - $\underline{x}(t;\alpha)$ for $q = 0.75$.

t	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.9317	0.9214	0.9110	0.9007	0.8903	0.8800	0.8696	0.8593	0.8489	0.8386	0.8282
0.2	0.8241	0.8150	0.8058	0.7966	0.7875	0.7783	0.7692	0.7600	0.7509	0.7417	0.7325
0.3	0.7428	0.7346	0.7263	0.7181	0.7098	0.7016	0.6933	0.6851	0.6768	0.6685	0.6603
0.4	0.6773	0.6698	0.6623	0.6548	0.6472	0.6397	0.6322	0.6247	0.6171	0.6096	0.6021
0.5	0.6228	0.6158	0.6089	0.6020	0.5951	0.5882	0.5812	0.5743	0.5674	0.5605	0.5536
0.6	0.5763	0.5699	0.5635	0.5571	0.5507	0.5443	0.5379	0.5315	0.5250	0.5186	0.5122
0.7	0.5361	0.5301	0.5242	0.5182	0.5123	0.5063	0.5003	0.4944	0.4884	0.4825	0.4765
0.8	0.5009	0.4953	0.4898	0.4842	0.4786	0.4731	0.4675	0.4619	0.4564	0.4508	0.4452
0.9	0.4698	0.4646	0.4594	0.4542	0.4490	0.4437	0.4385	0.4333	0.4281	0.4229	0.4176
1.0	0.4422	0.4373	0.4324	0.4275	0.4225	0.4176	0.4127	0.4078	0.4029	0.3980	0.3931

Table 8: The exact solution to the FFIVP(22) in Example 7.2 - $\overline{x}(t;\alpha)$ for $q = 0.75$.

t	α										
	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1
0.1	0.6212	0.6418	0.6626	0.6833	0.7040	0.7247	0.7454	0.7661	0.7868	0.8075	0.8282
0.2	0.5494	0.5677	0.5860	0.6043	0.6227	0.6410	0.6593	0.6776	0.6959	0.7142	0.7325
0.3	0.4952	0.5117	0.5282	0.5447	0.5612	0.5777	0.5943	0.6108	0.6273	0.6438	0.6603
0.4	0.4515	0.4666	0.4817	0.4967	0.5118	0.5268	0.5419	0.5569	0.5720	0.5870	0.6021
0.5	0.4152	0.4290	0.4428	0.4567	0.4705	0.4844	0.4982	0.5120	0.5259	0.5397	0.5536
0.6	0.3842	0.3970	0.4098	0.4226	0.4354	0.4482	0.4610	0.4738	0.4866	0.4994	0.5122
0.7	0.3574	0.3693	0.3812	0.3931	0.4050	0.4169	0.4289	0.4408	0.4527	0.4646	0.4765
0.8	0.3339	0.3451	0.3562	0.3677	0.3785	0.3896	0.4007	0.4119	0.4230	0.4341	0.4452
0.9	0.3132	0.3237	0.3341	0.3445	0.3550	0.3654	0.3759	0.3863	0.3968	0.4072	0.4176
1.0	0.2948	0.3046	0.3144	0.3243	0.3341	0.3439	0.3538	0.3636	0.3734	0.3832	0.3931

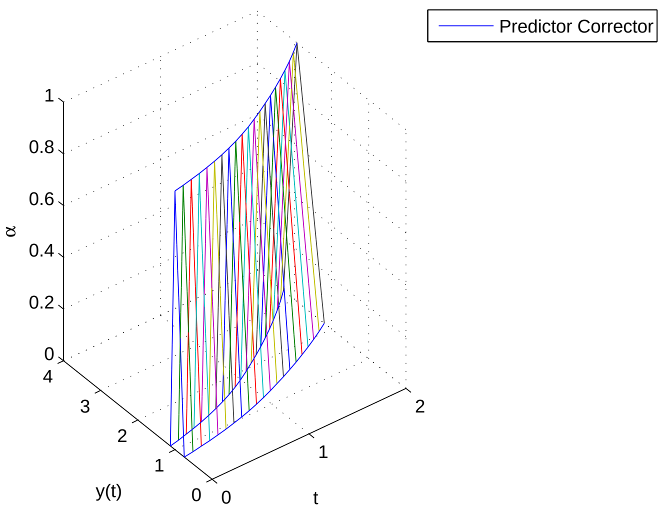

Figure 3: For h=0.1

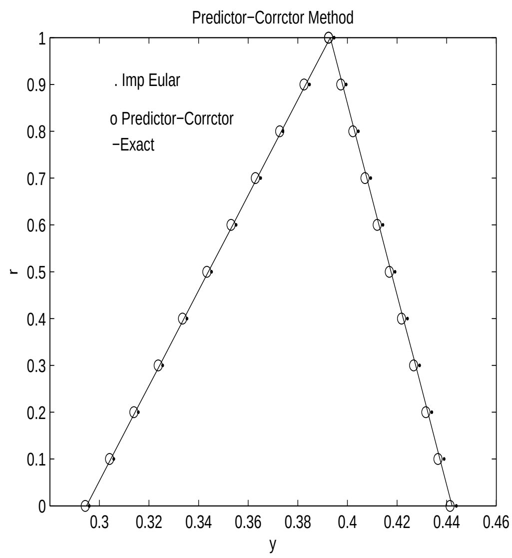

Figure 4: For h=0.1

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