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Explicit Solutions of Bogoyavlenskii Equation by the Exp-function Method and Ansatsz Method

E. Aksoy[1], A. C. Cevikel[1] *, A. Bekir[2], O. Guner[3]
[1]Yildiz Technical University, Education Faculty, Department of Mathematics Education, Istanbul-Turkey
[2] Eskisehir Osmangazi University, Art-Science Faculty, Department of Mathematics and Computer Science, Eskisehir-Turkey
[3] Cankırı Karatekin University, Faculty of Economics and Administrative Sciences, Department of International Trade, Cankiri-Turkey
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Abstract: In this paper, we employ the Exp-function method to obtain several families of exact solutions for Bogoyavlenskii equation. The solitary wave ansatsz method is used to retrieve the soliton solutions. It can be seen that the advantage of the Exp-function method and ansatsz method are found when selecting a variety of arbitrary values for the parameters.

Keywords: exp-function method; ansatz method; Bogoyavlenskii equation; exact solutions

1 Introduction

Nonlinear phenomena play a crucial role in applied mathematics and physics. Explicit solutions to nonlinear equations are of fundamental importance. It is known in literature that exact traveling wave solutions of nonlinear evolution equations have an important role in the studies related with nonlinear physical phenomena. The wave phenomena are observed in fluid dynamics, plasma, elastic media, optical fibers, etc. In the past several decades, both mathematicians and physicists have made significant progress in this direction. It was stated in literature that the nonlinear wave phenomena of dissipation, diffusion, dispersion, reaction, and convection have important roles in nonlinear wave equations [1]. New exact solutions might be used to find new phenomena in this field, such as the tanh-sech method [2–4], extended tanh method [5, 6], sine-cosine method [7–9], homogeneous balance method [10, 11], Jacobi elliptic function method [12, 13], extended mapping method [14–16], F-expansion method [17, 18], and Hirota method [19, 20] were used to develop nonlinear dispersive and dissipative problems.

A straightforward and concise method called as "Exp-function method" was proposed by He and Wu in order to get exact solutions of nonlinear evolution equations (NLEEs). The method, by the help of Matlab or Mathematica, has been successfully applied to many kinds of NLEEs. There are two main purposes in this paper. First one is to implement the Exp-function method to stress its power in handling nonlinear equations, so that one can apply it to models of various types of nonlinearity. Second one is to determine exact travelling wave solutions for the Bogoyavlenskii equation.

In this paper, the exp-function method and solitary wave ansatsz method will be used to carry out the integration of the Bogoyavlenskii equation. All the physical parameters in the soliton solutions (free parameters and velocity) are obtained. It is very interesting to note that the used solitary wave ansatsz method has been applied succesfully to different equations with constant and t-dependent coefficients.

2 The exp-funtion method

The Exp-function method has gained much popularity because it helps one to obtain exact and explicit solutions for NEEs in a concise manner. He and Wu were the first scientist to study the exp-function method in 2006 [21]. This method

* **Corresponding author**. *E-mail address*: acevikel@yildiz.edu.tr

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has been applied to various kinds of nonlinear problems arising in the applied sciences, and lately more attention is paid to its adaptation, generalization, and extension; just to mention a few, multi-dimensional equations [22, 23], differential-difference equations [24], coupled NEEs [25], NEEs with variable coefficients [26], stochastic equations [27], n-soliton solutions [28], rational solutions [29], double-wave solutions [30].

We consider a general nonlinear PDE in the form

$$P(u, u_t, u_x, u_{xx}, u_{tt}, ...) = 0. \tag{1}$$

Using a transformation

$$u(x,t) = u(\xi), \quad \xi = kx + wt + \zeta, \tag{2}$$

where k, w and ζ are constants, we can rewrite Eq.(1) in the following nonlinear ODE:

$$Q(u, u', u'', u''', ...) = 0, \tag{3}$$

where the prime denotes the derivation with respect to ξ.

Solution can be expressed in the form seen in the following part according to the Exp-function method [21].

$$u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)} = \frac{a_{-c}\exp[-c(\xi)] + ... + a_d \exp[d(\xi)]}{b_{-p}\exp[-p(\xi)] + ... + b_q \exp[q(\xi)]}, \tag{4}$$

where c, d, p and q are positive integer which could be freely chosen, a_n and b_m are unknown constants to be determined.

c and p the values are calculated by balance the linear term of lowest order in Eq.(3) with the lowest order nonlinear term. Similarly d and q the values are obtained by balance the linear term of highest order in Eq.(3) with the highest order nonlinear term.

Theorem 1 *Suppose that $u^{(r)}$ and u^s are respectively the highest order linear term and the highest order nonlinear term of a nonlinear ODE, where r and s are both positive integers. Then the balancing procedure using the Exp-function ansatz; $u(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}$ leads to $d = q$ and $c = p$, $\forall r \geq 1$, $\forall s \geq 2$. [31] it is suppoded that the solution of Eq.(3) can be expressed as*

$$u(\xi) = \sum_{i=1}^{n} a_i \phi^i, \tag{5}$$

where ϕ is the solution of the sub-equation $\phi' = \alpha + \beta\phi + \gamma\phi^2$. In a similar way, ϕ can be exressed in the form:

$$\phi(\xi) = \frac{\sum_{n=-c}^{d} a_n \exp(n\xi)}{\sum_{m=-p}^{q} b_m \exp(m\xi)}. \tag{6}$$

The different cases for Bogoyavlenskii equation system will be examined in order to show the efficiency of the method described in the previous part.

3 Applications of exp-function method

Let us second consider the Bogoyavlenskii equation [32]

$$4u_t + u_{xxy} - 4u^2 u_y - 4u_x v = 0, \tag{7}$$

$$uu_y = v_x, \tag{8}$$

describing the (2+1)-dimensional interaction of a Riemann wave propagating along the y axis with a long wave along the x axis. The $u = u(x, y, t)$ represents the physical field and $v = v(x, y, t)$ is some potential. The Lax pair and a nonisospectral condition of the spectral parameter was given by Bogoyavlenskii [32]. Peng has obtained periodic wave and travelling wave solutions [33]. Bogoyavlenskii equation was again derived as a member of (2+1) Schwarzian breaking soliton hierarchy and Kudryashov and Pickering has found rational solutions of this equation [34]. The periodic wave solutions of the equation were found by using first integral method in [35].

Using the transformation

$$v(x,y,t)=v(\xi),\qquad u(x,y,t)=u(\xi),\qquad \xi=x+y-kt, \tag{9}$$

and substituting the Eqs.(7) and (8) into Eqs.(9) yields

$$u'''(\xi)-4ku'(\xi)-4u^2(\xi)u'(\xi)-4u'(\xi)v(\xi) \tag{10}$$

$$u'(\xi)u(\xi)=v'(\xi) \tag{11}$$

where k is a real constant and the prime denotes the derivation with respect to ξ. Integrating Eq.(11) once and setting the constant of integration equal to zero we obtain

$$u^2(\xi)=2v(\xi), \tag{12}$$

Substituting Eq.(12) into Eq.(10) and then integrating Eq.(10) once and setting the constant of integration equal to zero we have

$$u''(\xi)-4ku(\xi)-2u^3(\xi)=0. \tag{13}$$

Eq. (4) can be re-written in an alternative form shown below;

$$u(x,t)=\frac{a_{-c}\exp[-c(\xi)]+...}{b_{-p}\exp[-p(\xi)]+...}. \tag{14}$$

By the same procedure which is shown in the section 2, the values of c and p can be determined by balancing u'' and u^3 in Eq.(13) and

$$u''=\frac{c_1\exp[-(c+3p)\xi]+...}{c_2\exp[-4p\xi]+...}, \tag{15}$$

and

$$u^3(\xi)=\frac{c_3\exp[-(3c)\xi]+...}{c_4\exp[-(3p)\xi]+...}=\frac{c_3\exp[-(3c+p)\xi]+...}{c_4\exp[-(4p)\xi]+...}, \tag{16}$$

where c_i are determined coefficients only for simplicity. Balancing highest order of Exp-function in Eqs.(15) and (16) we obtain

$$c+3p=3c+p, \tag{17}$$

which leads to the result

$$p=c. \tag{18}$$

d and q the values are determined by balance the linear term of the lowest order in Eq. (13) with the lowest order nonlinear term

$$u''=\frac{...+d_1\exp[(d+3q)\xi]}{...+d_2\exp[4q\xi]}, \tag{19}$$

and

$$u^3=\frac{...+d_3\exp[(3d)\xi]}{...+d_4\exp[3q\xi]}=\frac{...+d_3\exp[(3d+q)\xi]}{...+d_4\exp[4q\xi]}, \tag{20}$$

where d_i are determined coefficients only for simplicity. Balancing the lowest order of Exp-function in Eqs. (19) and (20), we have

$$d+3q=3d+q, \tag{21}$$

which gives

$$d=q. \tag{22}$$

Case 1: $p=c=1$, $d=q=1$

The values of c and d can be selected freely, but the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 1$ and $d = q = 1$, then Eq. (4) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi)}{b_1 \exp(\xi) + b_0 + b_{-1} \exp(-\xi)}. \tag{23}$$

Substituting Eq. (23) in Eq. (13) and equating to zero the coefficients of all powers of $\exp(n\xi)$ gives a set of algebraic equations for a_1, a_0, a_{-1}, b_0, b_{-1} and k. A set of non-trivial solutions of the system of algebraic equations using Maple are obtained as follows:

$$\left\{a_0 = a_0,\ a_1 = 0,\ a_{-1} = 0,\ b_0 = 0,\ b_1 = b_1,\ b_{-1} = -\frac{a_0^2}{4b_1},\ k = \frac{1}{4}\right\} \tag{24}$$

$$\left\{a_0 = \frac{b_0}{2},\ a_1 = 0,\ a_{-1} = -\frac{b_{-1}}{2},\ b_0 = b_0,\ b_1 = 0,\ b_{-1} = b_{-1},\ k = -\frac{1}{8}\right\} \tag{25}$$

$$\left\{a_0 = a_0,\ a_1 = \frac{4a_0^2 - b_0^2}{8b_{-1}},\ a_{-1} = \frac{b_{-1}}{2},\ b_0 = b_0,\ b_1 = -\frac{4a_0^2 - b_0^2}{4b_{-1}},\ b_{-1} = b_{-1},\ k = -\frac{1}{8}\right\} \tag{26}$$

Substituting Eqs. (24), (25) and (26) in Eq. (23), we obtain the following soliton solutions of Eq. (13)

$$u_1(\xi) = \frac{4b_1 a_0}{4b_1^2 \exp(\xi) - a_0^2 \exp(-\xi)}, \tag{27}$$

$$u_2(\xi) = \frac{1}{2} \frac{b_0 - b_{-1} \exp(-\xi)}{b_0 + b_{-1} \exp(-\xi)}, \tag{28}$$

$$u_3(\xi) = \frac{1}{2} \frac{(4a_0^2 - b_0^2) \exp(\xi) + 4b_{-1} a_0 + 4b_{-1}^2 \exp(-\xi)}{-(4a_0^2 - b_0^2) \exp(\xi) + 4b_{-1} b_0 + 4b_{-1}^2 \exp(-\xi)}. \tag{29}$$

Also, if we take $a_0 = \mp 2b_1$ and $k = \frac{1}{4}$ then Eq. (27) admits special solution

$$u_4(\xi) = \mp csch(\xi). \tag{30}$$

Substituting Eqs. (27)- (30) into Eq. (12), respectively, we gives

$$v_1(\xi) = \frac{1}{2} \left\{\frac{4b_1 a_0}{4b_1^2 \exp(\xi) - a_0^2 \exp(-\xi)}\right\}^2, \tag{31}$$

$$v_2(\xi) = \frac{1}{8} \left\{\frac{b_0 - b_{-1} \exp(-\xi)}{b_0 + b_{-1} \exp(-\xi)}\right\}^2, \tag{32}$$

$$v_3(\xi) = \frac{1}{8} \left\{\frac{(4a_0^2 - b_0^2) \exp(\xi) + 4b_{-1} a_0 + 4b_{-1}^2 \exp(-\xi)}{-(4a_0^2 - b_0^2) \exp(\xi) + 4b_{-1} b_0 + 4b_{-1}^2 \exp(-\xi)}\right\}^2, \tag{33}$$

and

$$v_4(\xi) = \frac{1}{2} csch^2(\xi), \tag{34}$$

where $\xi = x + y - \frac{1}{4}t$, and $\xi = x + y + \frac{1}{8}t$, respectively.

Case 2: $p = c = 1, d = q = 2$

As mentioned above the values of c and d can be freely chosen. Also there are some free parameters, we set $p = c = 1$, $b_1 = 1$ and $d = q = 2$, then Eq. (4) reduces to

$$u(\xi) = \frac{a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi) + a_{-2} \exp(-2\xi)}{\exp(\xi) + b_0 + b_{-1} \exp(-\xi) + b_{-2} \exp(-2\xi)}. \tag{35}$$

By the same calculation which is shown above, we have

$$\left\{a_0 = -\frac{4b_{-2}}{a_{-1}},\ a_1 = 0,\ a_{-1} = a_{-1},\ a_{-2} = 0,\ b_0 = -\frac{a_{-1}^2}{4b_{-2}},\ b_{-1} = -\frac{4b_{-2}^2}{a_{-1}^2}, b_{-2} = b_{-2},\ k = \frac{1}{4}\right\} \tag{36}$$

$$\left\{a_0 = 0,\ a_1 = 1,\ a_{-1} = -b_{-1},\ a_{-2} = 0,\ b_0 = 0,\ b_{-1} = b_{-1}, b_{-2} = 0,\ k = -\frac{1}{2}\right\} \tag{37}$$

Substituting Eqs. (36) and (37) in Eq. (35), we obtain the following soliton solutions of Eq. (13):

$$u_5(\xi) = \frac{4b_{-2}a_{-1}[-4b_{-2} + a_{-1}\exp(-\xi)]}{4b_{-2}a_{-1}^2\exp(\xi) - a_{-1}^4 - 16b_{-2}^3\exp(-\xi) + 4b_{-2}^2a_{-1}^2\exp(-2\xi)}, \tag{38}$$

$$u_6(\xi) = \frac{\exp(\xi) - b_{-1}\exp(-\xi)}{\exp(\xi) + b_{-1}\exp(-\xi)}. \tag{39}$$

If we set $b_{-1} = 1$ and $k = -\frac{1}{2}$, Eq. (39) becomes

$$u_7(\xi) = \tanh(\xi). \tag{40}$$

Substituting Eqs. (38)- (40) into Eq. (12), respectively, we have

$$v_5(\xi) = \frac{1}{2}\left\{\frac{4b_{-2}a_{-1}[-4b_{-2} + a_{-1}\exp(-\xi)]}{4b_{-2}a_{-1}^2\exp(\xi) - a_{-1}^4 - 16b_{-2}^3\exp(-\xi) + 4b_{-2}^2a_{-1}^2\exp(-2\xi)}\right\}^2, \tag{41}$$

$$v_6(\xi) = \frac{1}{2}\left\{\frac{\exp(\xi) - b_{-1}\exp(-\xi)}{\exp(\xi) + b_{-1}\exp(-\xi)}\right\}^2, \tag{42}$$

and

$$v_7(\xi) = \frac{1}{2}\tanh^2(\xi), \tag{43}$$

where $\xi = x + y - \frac{1}{4}t$ and $\xi = x + y + \frac{1}{2}t$, respectively.

Case 3: $p = c = 2, d = q = 1$

As mentioned above the values of c and d can be freely chosen, now we set $p = c = 2$, and $d = q = 1$. Also there are some free parameters, we set $b_2 = 1$, for simplicity, then Eq. (4) becomes

$$u(\xi) = \frac{a_2\exp(2\xi) + a_1\exp(\xi) + a_0 + a_{-1}\exp(-\xi)}{\exp(2\xi) + b_1\exp(\xi) + b_0 + b_{-1}\exp(-\xi)}. \tag{44}$$

By the same calculation as illustrated above, we obtain

$$\left\{a_0 = -\frac{4b_{-1}}{a_1},\ a_1 = a_1,\ a_{-1} = 0,\ a_2 = 0,\ b_0 = -\frac{a_1^2}{4},\ b_1 = -\frac{4b_{-1}}{a_1^2}, b_{-1} = b_{-1},\ k = \frac{1}{4}\right\} \tag{45}$$

$$\left\{a_0 = \frac{a_1^2}{2} - \frac{b_1}{8},\ a_1 = a_1,\ a_{-1} = 0,\ a_2 = \frac{1}{2},\ b_0 = -a_1^2 + \frac{b_1}{4},\ b_1 = b_1, b_{-1} = 0,\ k = -\frac{1}{8}\right\} \tag{46}$$

Substituting (45) and (46) into (44) yields

$$u_8(\xi) = \frac{4a_1[a_1^2\exp(\xi) - 4b_{-1}]}{4a_1^2\exp(2\xi) - 16b_{-1}\exp(\xi) + a_1^4 + 4a_1^2b_{-1}\exp(-\xi)}, \tag{47}$$

$$u_9(\xi) = \frac{4\exp(2\xi) + 8a_1\exp(\xi) + 4a_1^2 - b_1^2}{2[4\exp(2\xi) + 4b_1\exp(\xi) - 4a_1^2 + b_1^2]}, \tag{48}$$

Now, respectively, substituting Eqs. (47)- (48) into Eq. (12), we have

$$v_8(\xi) = \frac{1}{2}\left\{\frac{4a_1[a_1^2\exp(\xi) - 4b_{-1}]}{4a_1^2\exp(2\xi) - 16b_{-1}\exp(\xi) + a_1^4 + 4a_1^2b_{-1}\exp(-\xi)}\right\}^2, \tag{49}$$

and

$$v_9(\xi) = \frac{1}{8}\left\{\frac{4\exp(2\xi) + 8a_1\exp(\xi) + 4a_1^2 - b_1^2}{4\exp(2\xi) + 4b_1\exp(\xi) - 4a_1^2 + b_1^2}\right\}^2, \tag{50}$$

where $\xi = x + y - \frac{1}{4}t$ and $\xi = x + y + \frac{1}{8}t$,respectively.

Case 4: $p = c = 2, d = q = 2$

The values of c and d can be selected freely, but the final solution does not strongly depend upon the choice of values of c and d. For simplicity, we set $p = c = 2$, $b_2 = 1$, $b_1 = b_{-1} = 0$ and $d = q = 1$, then Eq. (4) reduces to

$$u(\xi) = \frac{a_2 \exp(2\xi) + a_1 \exp(\xi) + a_0 + a_{-1} \exp(-\xi) + a_{-2} \exp(-2\xi)}{\exp(2\xi) + b_0 + b_{-2} \exp(-2\xi)}, \tag{51}$$

Substituting Eq. (51) in Eq. (13) and using Maple, equating to zero the coefficients of all powers of $exp(n\xi)$ gives a set of algebraic equations for $a_2, a_1, a_0, a_{-1}, a_{-1}, b_0, b_{-2}$, and k. Solving the systems of algebraic equations using Maple gives the following sets of nontrivial solutions:

$$\left\{ \begin{array}{c} a_0 = a_0,\ a_1 = -\frac{4b_{-2}}{a_{-1}},\ a_{-1} = a_{-1},\ a_2 = 0,\ a_{-2} = 0, \\ b_0 = -\frac{16b_{-2}^3 + a_{-1}^4}{4a_{-1}^2 b_{-2}},\ b_1 = -\frac{a_0 a_{-1}}{4b_{-2}}, b_{-1} = \frac{b_{-2} a_0}{a_{-1}}, b_{-2} = b_{-2},\ k = \frac{1}{4} \end{array} \right\} \tag{52}$$

$$\left\{ a_0 = a_0,\ a_1 = 0,\ a_{-1} = 0,\ a_2 = 0,\ b_0 = 0,\ b_1 = 0, b_{-1} = 0, b_{-2} = -\frac{a_0^2}{16},\ k = 1 \right\} \tag{53}$$

Substituting Eqs. (52) and (53) in Eq. (51), we obtain multiple soliton solutions of Eq. (13) as follows:

$$u_{10}(\xi) = \frac{4a_{-1}b_{-2}[-4b_{-2}\exp(\xi) - a_0 a_{-1} + a_{-1}^2 \exp(-\xi)]}{4a_{-1}^2 b_{-2} \exp(2\xi) - a_0 a_{-1}^3 \exp(\xi) - (16b_{-2}^3 + a_{-1}^4) + 4a_0 a_{-1} b_{-2}^2 \exp(-\xi) + 4a_{-1}^2 b_{-2}^2 \exp(-2\xi)}, \tag{54}$$

and

$$u_{11}(\xi) = \frac{a_0}{16 \exp(2\xi) - a_0^2 \exp(-2\xi)}. \tag{55}$$

Also, if we take $a_0 = \mp 4$ then Eq. (55) admits special solution

$$u_{12}(\xi) = \mp \frac{1}{8} csch 2\xi. \tag{56}$$

Therefore, respectively, substituting Eqs. (54)- (56) into Eq. (12), we get

$$v_{10}(\xi) = \frac{1}{2} \left\{ \frac{4a_{-1}b_{-2}[-4b_{-2}\exp(\xi) - a_0 a_{-1} + a_{-1}^2 \exp(-\xi)]}{4a_{-1}^2 b_{-2} \exp(2\xi) - a_0 a_{-1}^3 \exp(\xi) - (16b_{-2}^3 + a_{-1}^4) + 4a_0 a_{-1} b_{-2}^2 \exp(-\xi) + 4a_{-1}^2 b_{-2}^2 \exp(-2\xi)} \right\}^2, \tag{57}$$

$$v_{11}(\xi) = \frac{1}{2} \left\{ \frac{a_0}{16 \exp(2\xi) - a_0^2 \exp(-2\xi)} \right\}^2, \tag{58}$$

and

$$v_{12}(\xi) = \frac{1}{128} csch^2(2\xi), \tag{59}$$

where $\xi = x + y - \frac{1}{4}t$ and $\xi = x + y - t$,respectively.

Remark 2 *As a result, we find hyperbolic solutions of the Bogoyavlenskii system different from the solutions which are found in [33, 35].*

Remark 3 *With the aid of Maple, we have verified all solutions we obtained in Section 3, by putting them back into the original Eqs. (7)-(8).*

Remark 4 *Comparing other methods and the Exp-function method shows that the latter gives an abundant variety of solutions compared to the other methods. This can be easily obtained by selecting a variety of arbitrary values for the parameters p, c, d, and q provided that $p = c$, $d = q$.*

4 The solitary wave ansatz method

The solitary wave ansatz method proposed by Biswas [36] and Triki et al. [37] is particularly notable in its power and applicability in solving nonlinear problems, and this method has been successfully applied to many kinds of nonlinear partial differential equations [38–44]

5 Topological soliton solution

In order the start off with the solution hypothesis, the following ansatsz is assumed

$$u(x,y,t) = A_1 \tanh^p (B_1 x + B_2 y - ct) \tag{60}$$

and

$$v(x,y,t) = A_2 \tanh^r (B_1 x + B_2 y - ct) \tag{61}$$

Here in (60) and (61), A_1, A_2, B_1 and B_2 are the free parameters of the solitons and c is the velocity of the soliton.

Now, from (60) and (61) it is possible to obtain

$$u_t = A_1 pc\{\tanh^{p+1}\tau - \tanh^{p-1}\tau\}, \tag{62}$$

$$\begin{aligned} u_{xxy} \;=\;& p(p-1)(p-2)A_1 B_1^2 B_2 \tanh^{p-3}\tau - \\ & \left\{p(p-1)(p-2) + 2p^3\right\} A_1 B_1^2 B_2 \tanh^{p-1}\tau \\ & + \left\{p(p+1)(p+2) + 2p^3\right\} A_1 B_1^2 B_2 \tanh^{p+1}\tau \\ & -p(p+1)(p+2)A_1 B_1^2 B_2 \tanh^{p+3}\tau, \end{aligned} \tag{63}$$

$$u^2 u_y = A_1^3 B_2 p \left\{\tanh^{3p-1}\tau - \tanh^{3p+1}\tau\right\}, \tag{64}$$

$$u_x = A_1 B_1 p \left\{\tanh^{p-1}\tau - \tanh^{p+1}\tau\right\}, \tag{65}$$

$$v_x = A_2 B_1 r \left\{\tanh^{r-1}\tau - \tanh^{r+1}\tau\right\}, \tag{66}$$

where $\tau = B_1 x + B_2 y - ct$.

Substituting (62)-(66) into (7)-(8) yields

$$\begin{aligned} & 4A_1 pc\{\tanh^{p+1}\tau - \tanh^{p-1}\tau\} + p(p-1)(p-2)A_1 B_1^2 B_2 \tanh^{p-3}\tau \\ & - \left\{p(p-1)(p-2) + 2p^3\right\} A_1 B_1^2 B_2 \tanh^{p-1}\tau \\ & + \left\{p(p+1)(p+2) + 2p^3\right\} A_1 B_1^2 B_2 \tanh^{p+1}\tau \\ & -p(p+1)(p+2)A_1 B_1^2 B_2 \tanh^{p+3}\tau - 4A_1^3 B_2 p \left\{\tanh^{3p-1}\tau - \tanh^{3p+1}\tau\right\} \\ & -4pA_1 A_2 B_1 \left\{\tanh^{p+r-1}\tau - \tanh^{p+r+1}\tau\right\} \\ =\;& 0, \end{aligned} \tag{67}$$

and

$$A_1^2 B_2 p \left\{\tanh^{2p-1}\tau - \tanh^{2p+1}\tau\right\} - A_2 B_1 r \left\{\tanh^{r-1}\tau - \tanh^{r+1}\tau\right\} = 0. \tag{68}$$

Now from (68), equating the coefficients of $2p+1$ and $r+1$ gives

$$2p + 1 = r + 1, \tag{69}$$

so that

$$2p = r. \tag{70}$$

It need to be noted that the same result is yielded when the exponent $2p-1$ and $r-1$ is equated the other.

Again from (67) equating the coefficients of $3p+1$ and $p+3$ gives

$$3p+1=p+3, \tag{71}$$

that leads to

$$p=1, \tag{72}$$

so that from (70),

$$r=2 \tag{73}$$

Setting the coefficients of $\tanh^{2p+j}\tau$ and $\tanh^{r+j}\tau$ in (68), to zero where $j=-1,1$ gives

$$A_1^2B_2p-A_2B_1r=0, \tag{74}$$

which gives after using (72) and (73):

$$A_2=\frac{A_1^2B_2}{2B_1}. \tag{75}$$

Again from (67), setting the coefficients of $\tanh^{3p+1}\tau$, $\tanh^{p+3}\tau$,and $\tanh^{p+r+1}\tau$ terms to zero, one obtains:

$$4A_1^3B_2p-p(p+1)(p+2)A_1B_1^2B_2+4pA_1A_2B_1=0, \tag{76}$$

this leads to after using (72), (73) and (75):

$$A_1=\pm B_1. \tag{77}$$

Finally, setting the coefficients of $\tanh^{p+1}\tau$, $\tanh^{3p-1}\tau$ and $\tanh^{p+r-1}\tau$ in (67), to zero gives

$$4A_1pc+\left\{p(p+1)(p+2)+2p^3\right\}A_1B_1^2B_2-4A_1^3B_2p-4pA_1A_2B_1=0, \tag{78}$$

the latter gives after using (72), (73) and (77):

$$c=-\frac{B_1^2B_2}{2}. \tag{79}$$

By inserting (77) in (75), it is possible to recover

$$A_2=\frac{B_1B_2}{2}. \tag{80}$$

Lastly, the dark soliton solution for the Bogoyavlenskii equation system (6)-(7) is given by

$$u(x,y,t)=\pm B_1\tanh\left(B_1x+B_2y+\left(\frac{B_1^2B_2}{2}\right)t\right) \tag{81}$$

and

$$v(x,y,t)=\frac{B_1B_2}{2}\tanh^2\left(B_1x+B_2y+\left(\frac{B_1^2B_2}{2}\right)t\right) \tag{82}$$

We see from (77) and (80) that the free parameters A_1 and A_2 are dependent on the other free parameters B_1 and B_2. Also in (79) the velocity of the soliton c is dependent on the other free parameters B_1 and B_2, too.

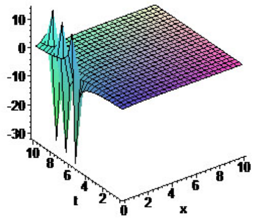

Figure 1: Exact solutions $u_1(x, y, t)$ for Eqs. (7-8) with $a_0 = -1$, $b_1 = 2$ and $y = 0$.

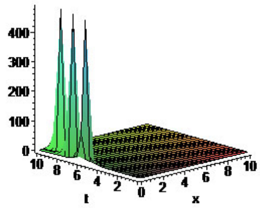

Figure 2: Exact solutions $v_1(x, y, t)$ for Eqs. (7-8) with $a_0 = -1$, $b_1 = 2$ and $y = 0$.

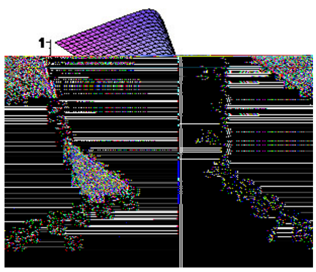

Figure 3: Exact solutions $u_6(x, y, t)$ for Eqs. (7-8) with $b_{-1} = 1$ and $y = 0$.

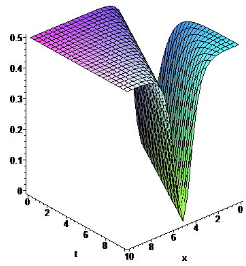

Figure 4: Exact solutions $v_6(x, y, t)$ for Eqs. (7-8) with $b_{-1} = 1$ and $y = 0$.

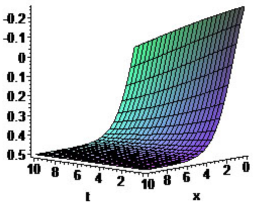

Figure 5: Exact solutions $u_9(x, y, t)$ for Eqs. (7-8) with $a_1 = -1$, $b_1 = 4$ and $y = 0$.

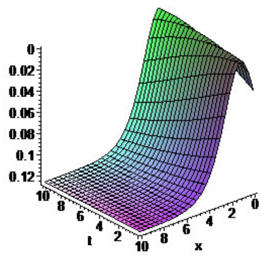

Figure 6: Exact solutions $v_9(x, y, t)$ for Eqs. (7-8) with $a_1 = -1$, $b_1 = 4$ and $y = 0$.

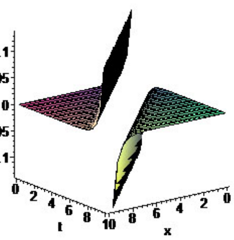

Figure 7: Exact solutions $u_{11}(x, y, t)$ for Eqs. (7-8) with $a_0 = 4$ and $y = 0$.

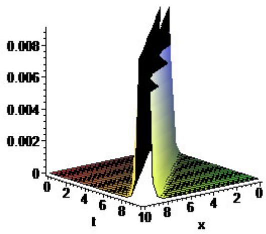

Figure 8: Exact solutions $v_{11}(x, y, t)$ for Eqs. (7-8) with $a_0 = 4$ and $y = 0$.

6 Conclusions

In this study, the Exp-function method and ansatz method with a computerized symbolic computation system Maple is used for finding the generalized solitary solutions and topological (dark) solutions to Bogoyavlenskii equation arising in mathematical physics. New exact solutions, not obtained by the previously available methods, are also found. It can be seen that the Exp-function method yields more general solutions in comparison with the other method. The solitary ansatz wave method has initiated an explosive reaction in the scientific community to find supposedly new exact solutions of nonlinear evolution equations. It is very likely that the flow of papers describing the application of this method will continue in the future. The Exp-function and ansatsz methods show effectiveness and reliability in handling nonlinear evolution equations. Solitonary wave solutions were succesfully obtained using these methods. It is well known in the literature that nonlinear wave equations were handled by these methods to show the new solutions compared to the solutions obtained in the literature [33, 35].

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