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Variational Inequality Theory for Elliptic Inequality Systems with Laplacian Type Operators and Related Population Models: An Overview and Recent Advances

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Abstract: Existence of nonzero positive solutions of elliptic inequality systems with Laplacian type operators and related population models of two or more species arising in mathematical biology is of great interest and challenging to study. In this paper, we deal with a survey of logistic models of biological populations and Laplacian elliptic equation, and discuss some related problems from variational equations to variational inequalities. Then, we present some works of studying elliptic variational inequalities with Laplacian type operators and related population models of one species arising in mathematical biology. Furthermore, we propose some theory of variational inequalities and the applications to study the existence of positive weak solutions for some elliptic variational inequalities and related population models. Finally, we display some remarks and open questions for researching in the further, which may help many present researchers for their numerical and computer realizations.

Keywords: Elliptic inequality system with Laplacian type operators; variational inequality theory; related population model; positive weak solution; further research question.

1 Introduction

As all we know, since the competition between species is of universal existence and importance [1], as a very popular and interesting topics, mathematicians as well as biologists have studied the interaction between species and their environment, which has long been and will continue to be one of the dominant themes in both ecology and mathematical ecology.

In 2008, Seo and Kot [2] analyzed a laissez-faire predator-prey model and a Leslie-type predator-prey model with type I functional responses. For the following Leslie-type model:

$$\begin{cases} \frac{du}{dt} = r_1 u \left(1 - \frac{u}{K}\right) - \phi(u)v, \\ \frac{dv}{dt} = r_2 v \left(1 - \frac{v}{hu}\right), \end{cases} \tag{1}$$

where r_1, r_2, K, h are positive constants, u and v represent the prey and predator populations at time t, and ϕ is Hollings type I functional response for positive constants a and b as $\phi(t) = \frac{b}{2a}t$ if $t < 2a$, otherwise, $\phi(t) = b$, the authors used a generalized Jacobian to determine how eigenvalues jump at the corner of the functional response and pointed out "the Leslie-type model may also exhibit super-critical and discontinuous Hopf bifurcationsand the existing analyse of predator-prey model systems is widely scattered and hard to find". We note that the system (1) includes the Leslie-Gower model [3] as special case, in where the prey grows logistically and involves a linear functional response while the predator population grows logistically with a carrying capacity proportional to the number of prey, and the biological meanings are can be shown clearly. Further, Heggerud and Lan [4] investigated the ratio-dependent predator-prey models with nonconstant predator harvesting rates as follows:

$$\begin{cases} \frac{du}{dt} = ru(1 - \frac{u}{K}) - \frac{c_1 uv}{u+mv}, \\ \frac{dv}{dt} = v(-d + \frac{c_2 u}{u+mv}) - \hbar v, \end{cases} \tag{2}$$

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where $\hat{h} > 0$ denotes the constant harvesting rate of prey, $\hbar \geq 0$ denotes the predator harvesting rates, $u(t)$ and $v(t)$ denote population densities of prey and predators at time t, respectively. For the biological meanings of other parameters, more predator-prey models with different functional responses or (and) harvesting, one can refer to [2–4] and references therein.

However, ordinary differential equations often model one dimensional dynamical systems, and multidimensional systems are frequently formulated by partial differential equations. As Mitidieri and Pohozaev [5] pointed out "the study of nonlinear partial differential inequalities is based on a special choice of test functions associated with the considered nonlinear problem". Thus, for two species competitive system, it is an open and interesting question to investigate existence of nonzero positive weak solutions for elliptic inequality systems involving harvesting rates of the form:

$$\begin{cases} -\Delta_p u(x) \geq f(x,u(x),v(x)) - h & \text{for almost every (a.e.) } x \in \Omega, \\ -\Delta_q v(x) \geq g(x,v(x),u(x)) & \\ u(x) = v(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{3}$$

where h describes the effect of harvesting on other biological species. This type of harvesting have been proposed (see [9]) constant-yield harvesting, described by a constant independent of the size of the population under harvest. The system of semilinear second order elliptic inequality (3) and equations arise in the study of Newtonian fluids, and in predator-prey system with monotonic functional response when $n = 1$ (see [6]). Further, and the generalizations of (3) arise in the study of non-Newtonian fluids, non-Newtonian filtrations, subsonic motion of gases, plasma physical models, population dynamics and some chemical reactions. For more related works, see, for instance, [6, 7] and references therein, and the following related works.

In 1838, Verhulst and Pearl first considered the following logistic equation for describing human population growth:

$$\frac{du}{dt} = uf(u), \quad f(u) = r - su, \tag{4}$$

where r is interpreted as the growth rate in case of unlimited resources and $s > 0$ is a constant which lead to the relative growth rate $f(u) = 0$ when population size. In a more reasonable form, $f(u)$ can be defined as

$$f(u) = r\left(1 - \frac{u}{K}\right),$$

where K is the maximum attainable population size. Further, (4) can be rewrited as

$$\frac{du}{dt} = ru(1-u) \quad \text{or} \quad u_{n+1} = ru_n\big(1-u_n\big).$$

Problem (4) was remained ignored until 1920, when it was reinvented. Further, as an extremely fertile area of mathematical applications, modelling more general population growth with time-lag (delay), stochastic and so on have been studied by many researchers (see [3]). Especially, for purpose of making the population persist on every location $x \in \Omega$, a bounded and connected open set in $\mathbb{R}^n$ $(n > 2)$ with $\text{meas}(\Omega) > 0$, Conway et al. [8] studied the following Laplacian elliptic equation with constant logistic growth rates arising in mathematical biology:

$$\begin{cases} -\Delta u(x) = ru(x)\big(1-u(x)\big) & \text{for a.e. } x \in \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $u(x)$ denotes the population density of one species at location x, $\Delta u(x) = \sum_{i=1}^{n} \frac{\partial^2 u(x)}{\partial x_i^2}$ is the Laplacian operator, r is the intrinsic growth rate of the species. Problem (5) includes problem (4) at critical point to variable t, that is, time independent item u is only respect to x_i at t for $i = 1, 2, \cdots, n$. If $n = 1$, Conway et al. ([8, Lemma 1.1]) proved that problem (5) has a unique nonzero positive solution in $C(\Omega)$ if $r \in (\mu_1, \infty)$, and has no nonzero positive solutions in $C(\Omega)$ if $r \in (0, \mu_1]$, where μ_1 is a constant with respect to $u(x)$. On the study of the Laplace equations related to problem (5), one can refer to [5, 9] and references therein, and the following Dirichlet problem for the Laplacian with homogeneous boundary conditions on a bounded domain Ω:

$$\begin{aligned} -\Delta u = f \quad & \text{in} \quad \Omega, \\ u = 0 \quad & \text{on} \quad \partial\Omega, \end{aligned} \tag{6}$$

where $f \in L^2(\Omega)$.

We note that the weak formulation of elliptic boundary value problem (6) is to finding $u \in H_0^1(\Omega) = W_0^{1,2}(\Omega)$ such that

$$\int_\Omega \nabla u \cdot \nabla v dx = \int_\Omega f v dx, \quad \forall v \in H_0^1(\Omega), \tag{7}$$

where $\nabla u(x) = \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \cdots, \frac{\partial u(x)}{\partial x_n}\right)$. By using the Lax-Milgram lemma, it is well know that problem (7) has a unique solution. Moreover, problem (7) is equivalent to a minimization problem of finding $u \in H_0^1(\Omega)$ such that

$$E(u) = \inf_{v \in H_0^1(\Omega)} E(v), \tag{8}$$

where

$$E(v) = \int_\Omega \left(\frac{1}{2}|v|^2 - fv\right) dx. \tag{9}$$

Remark 1 (i) *A minimization problem of the form* (8) *is a linear problem, owing to the properties that $E(\cdot)$ is a quadratic functional, and the set $H_0^1(\Omega)$, over which the infimum is sought, is a linear space. Problem* (8) *becomes nonlinear if the energy functional $E(v)$ is no longer quadratic (in particular, if $E(v)$ contains a non-differentiable term), or the energy functional is minimized over a general (convex) set instead of a linear space, or both.*

(ii) *Moreover, by minimizing the quadratic energy functional defined by* (9) *over a convex set, the following two concrete examples can be used to show that the variational equations can be more general forms, i.e., variational inequalities.*

Example 2 ([10]) *In an obstacle problem, determine the equilibrium position of an elastic membrane which*

(i) *passes through a closed curve $\partial\Omega$, the boundary of a planar domain Ω;*

(ii) *lies above an obstacle of height ψ;*

(iii) *is subject to the action of a vertical force of density τf, where τ is the elastic tension of the membrane, and f is a given function.*

Denote by u the vertical displacement component of the membrane. Since the membrane is fixed along the boundary $\partial\Omega$, we have the boundary condition $u = 0$ on $\partial\Omega$. In order to make the problem meaningful, we suppose that the obstacle function satisfies the condition $\psi \leq 0$ on $\partial\Omega$. Here, assume $\psi \in H^1(\Omega) = W^{1,2}(\Omega)$ and $f \in L^2(\Omega)$. Then the set of admissible displacements is

$$K = \{v \in H_0^1(\Omega) | \, v \geq \psi \quad \text{a.e. in} \quad \Omega\}.$$

The principle of minimal energy from mechanics asserts that the displacement u is a minimizer of the (scaled) total energy:

$$E(u) = \inf_{v \in K} E(v), \tag{10}$$

where $E(v)$ is the same as in (9). It is easy to show that the solution of (10) is also characterized by the variational inequality (see [10])

$$\int_\Omega \nabla u \cdot \nabla(v - u) dx \geq \int_\Omega f(v - u) dx, \quad \forall v \in K. \tag{11}$$

It is possible to derive a classical form of pointwise relations for the variational inequality (11). For this reason, suppose additionally that

$$f \in C(\Omega), \quad \psi \in C(\Omega), \quad u \in C^2(\Omega) \cap C(\bar{\Omega}).$$

Letting $v = u + \phi$ in (11) with $\phi \in C_0^\infty(\Omega)$ and $\phi \geq 0$ in Ω, then we get

$$\int_\Omega (\nabla u \cdot \nabla \phi - f\phi) dx \geq 0,$$

which is performed an integration by parts. Thus,

$$\int_\Omega (-\Delta u - f)\phi dx \geq 0, \quad \forall \phi \in C_0^\infty(\Omega), \, \phi \geq 0 \quad \text{in} \quad \Omega,$$

that is, u must satisfy the differential inequality

$$-\Delta u \geq f \quad \text{in} \quad \Omega, \tag{12}$$

which is a variational inequality from the variational equation in (6).

Example 3 ([10]) *Consider a problem of the form* (8) *with* $V = H^1(\Omega)$ *and*

$$E(v) = \int_\Omega \left[\frac{1}{2}\left(|\nabla v|^2 + v^2\right) - fv\right] + \alpha \int_{\partial\Omega} |v| dx,$$

where $\alpha > 0$ *and* $f \in L^2(\Omega)$ *are given, and*

$$|\nabla v| = \left[\sum_{k=1}^{n}\left(\frac{\partial v}{\partial x_k}\right)^2\right]^{1/2}.$$

It is clear to see that a minimization problem of finding $u \in V$ such that

$$E(u) = \inf_{v \in V} E(v)$$

is equivalent to the variational inequality

$$\int_\Omega [\nabla u \cdot \nabla(v-u) + u(v-u)]dx + \alpha \int_{\partial\Omega} (|v| - |u|)dx \geq \int_\Omega f(v-u)dx \tag{13}$$

for all $v \in V$, and the corresponding pointwise formulation of (13) is

$$\begin{aligned} &-\Delta u + u = f && \text{in} \quad \Omega, \\ &\left|\frac{\partial u}{\partial \nu}\right| \leq \alpha, \quad \frac{\partial u}{\partial \nu} u + \alpha |u| = 0 && \text{on} \quad \partial\Omega, \end{aligned}$$

where ν is the outward normal vector, and $\frac{\partial}{\partial \nu}$ is the outward normal derivative.

Driven by elliptic inequality systems (3) with Laplacian type operators and related population models, we shall present some works of studying elliptic variational inequalities with Laplacian type operators and related population models of one species arising in mathematical biology in Section 2. In Sections 3, we will propose some theory of variational inequalities and the applications to study the existence of positive weak solutions for some elliptic variational inequalities and related problems. We also display some remarks and open questions to be solved in further research in Section 4.

2 Elliptic variational inequalities with Laplacian type operators

In this section, we will provide some works on the generalizations of elliptic variational inequalities with Laplacian type operators and related population models of one species arising in mathematical biology.

In 1965, for purpose of studying the regularity problem for partial differential equations, Lions and Stampacchia [11] proved his generalization to the Lax-Milgram theorem in [12] and coined the name "variational inequality" for all the problems involving inequalities of this kind. Indeed, as we all know, a variational inequality is an inequality involving a functional, which has to be solved for all possible values of a given variable, belonging usually to a convex set. In particular, variational inequalities of monotone type operators have arisen in physics, mechanics, engineering, control, optimization, nonlinear potential theory and elliptic inequalities, and have been widely studied. See, for example, [13] and the references therein. Further, the following variational inequality forms an important family of nonlinear problems: Find $u \in K \subset \mathcal{X}$ such that

$$\langle Au, v-u\rangle \leq \langle Fu, v-u\rangle, \quad \forall v \in K, \tag{14}$$

where $\mathcal{X}$ is a real reflexive Banach space with its dual $\mathcal{X}^*$, $\langle \cdot, \cdot \rangle$ is the duality pairing between $\mathcal{X}^*$ and $\mathcal{X}$, K is a nonempty closed convex subset, and A and F are two operators from K to $\mathcal{X}^*$. And some of the more complex physical processes are described by inequality (14) and its generalized forms, which includes population models arising in mathematical biology. For related works, for example, see [3, 7, 14, 15] and the references therein.

On the research of physical problems with partial differential equations, there are many Laplace equations involving both space and time, i.e., parabolic or evolutionary type equations, such as frictional contact problem [16], heat conduction [17] and so on. If the time derivative of the unknown function u appears in the variational inequality form (14) (and, therefore, an initial condition for u is needed), then we refer to it as an evolutionary variational inequality. Otherwise, it is called an elliptic variational inequality. However, under some suitable conditions, the following example show that time-dependent form of partial differential equations can be rewritten as a time-independent form.

Example 4 *Consider the following Helmholtz equation:*

$$\frac{1}{C^2}\frac{\partial^2\hat{u}(x,t)}{\partial t^2} - \Delta_x\partial^2\hat{u}(x,t) = \hat{f}(x,t), \tag{15}$$

where C denotes the wave speed. Let $\hat{f}(x,t) = f(x)e^{i\omega t}$ and $\hat{u}(x,t) = u(x)e^{i\omega t}$, where ω is the angular frequency. Then (15) *is equivalent to*

$$-\Delta u(x) = f(x, u(x)),$$

where $f(x,u) = f(x) + k^2u$ and $k = \frac{\omega}{C}$ is the wave vector.

Evolutionary variational equations (15) is also called German physician, physicist and anatomist equation, and second order wave equation [16], represents an elliptic variational form of the wave equation, and is a special case of Dirichlet problem (6). Further, if both the data and the solution of a variational inequality depend on the time variable which plays the role of a parameter, then the corresponding inequality is called a time-dependent variational inequality (i.e., $f \in ([0,T];X)$ and $u : [0,T] \to X$, where $T > 0$ and $[0,T]$ denotes the time interval of interest). Finally, if an integral term containing the solution or its derivative appears in the formulation of a variational inequality, we refer to it as a history-dependent variational inequality. This classification is not strict and is intended to distinguish among the types of variational inequalities used in the mathematical theory of contact mechanics. See [16] and the references therein.

In this paper, we shall devote to review elliptic variational inequalities with Laplacian type operators and their applications to related population models. It is well known that the theory of elliptic variational inequalities has as its model the variational theory of boundary value problems for elliptic equations, but differs from this theory in that the competing functions belong to a convex set, rather than an affine space of functions. Recently, there are many researchers to bend themselves to study the theory of elliptic variational inequalities and applications due to several special problems of obstacle type. In [12], Stampacchia proved the existence of a solution for the following variational inequality problem: Find $u \in K$ such that

$$\int_\Omega D_j u D_j(v-u)dx \geq 0, \quad \forall v \in K, \tag{16}$$

where $D_j u = \partial/\partial x_j$ is the partial derivative of u with respect to x_j for $j = 1, 2, \cdots, n$,

$$K = K_\psi = \left\{\chi : \chi \text{ is Lipschitz in } \bar{\Omega}, \begin{array}{ll} \chi \geq \psi & \text{in } \Omega \\ \chi = 0 & \text{on } \partial\Omega \end{array}\right\}$$

is a convex set, $\Omega \subset \mathbb{R}^n$ is a convex domain with smooth boundary $\partial\Omega$ and $\psi \in C^2(\bar{\Omega})$ is an "obstacle", that is, $\max_\Omega \psi \geq 0$ and $\psi \leq 0$ on $\partial\Omega$.

Moreover, under some suitable conditions, Brezis and Kinderlehrer [18] introduced and studied several questions about the smoothness of solutions to the following nonlinear variational inequalities with obstacles: Find $u \in K$ such that

$$\int_\Omega a_j(Du)D_j(v-u)dx \geq \int_\Omega f(v-u)dx, \quad \forall v \in K, \tag{17}$$

where K is the same as in (16), $Du = \nabla u$ is the gradient of u and $a_j(p)$ is the jth components of n dimension vector $a(p)$ for $p = (p_1, p_2, \cdots, p_n) \in \mathbb{R}^n$.

Further, Atkinson and Han [10] introduced and studied more elliptic variational inequalities of the forms (12) and (13) arising in mechanics, which can be equivalently expressed as convex minimization problems, and pointed out "convex minimization is a rich source for many elliptic variational inequalities". The authors also presented some results on existence, uniqueness and stability of solutions to the elliptic variational inequalities, and discussed numerical approximations of the elliptic variational inequalities because a general family of elliptic variational inequalities that do not necessarily relate to minimization problems, and considered some contact problems in elasticity that lead to elliptic variational inequalities. For more works, see for examples, [7, 10, 19–21] and the references therein In many applications, the operator F in (14) is linear and corresponds to a bilinear form on a real Hilbert space $\mathcal{H}$:

$$\langle Fu, v\rangle = a(u,v), \quad \forall u, v \in \mathcal{H}.$$

By this case, variational inequalities of the form (14) can be divided into two kinds of elliptic variational inequalities defined as follows.

Definition 1 *Let $\mathcal{H}$ denote a real Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and norm $\|\cdot\|$, and K is a nonempty closed convex subset of $\mathcal{H}$. Let $a:\mathcal{H}\times\mathcal{H}\to\mathbb{R}$ and $j:\mathcal{H}\to(-\infty,\infty]$. Then a problem of the form is called*

(i) *elliptic variational inequality of the first kind, if for each $f\in\mathcal{H}$, there exists an element $u\in K$ such that*

$$a(u,v-u)\geq\langle f,v-u\rangle,\quad\forall v\in K.\tag{18}$$

(ii) *elliptic variational inequality of the second kind, if for given $f\in\mathcal{H}$, there exists an element $u\in\mathcal{H}$ such that*

$$a(u,v-u)+j(v)-j(u)\geq\langle f,v-u\rangle,\quad\forall v\in\mathcal{H}.\tag{19}$$

Remark 5 (i) *From the historical point of view, elliptic variational inequality* (18) *was first studied and constituted a point of departure for introduction and study of the elliptic variational inequality* (19). *Many free boundary value problems, and a lot of constant mechanical and physical problems in partial differential equations can be ascribed to the variational inequality* (19). *Further, Sofonea and Matei* [16] *showed that the elliptic variational inequality* (19) *reduces to elliptic variational inequality* (18) *when j in* (19) *is the indicator function of a nonempty, convex, and closed subset $K\subset\mathcal{H}$, that is, $j=\psi_K:\mathcal{H}\to(-\infty,\infty]$ defined by*

$$\psi_K(v)=\begin{cases}0 & \text{if}\ \ v\in K,\\ \infty & \text{if}\ \ v\notin K.\end{cases}$$

(ii) *When the nondifferentiable convex functional j in* (19) *depends explicitly on u or on its time derivative $\dot{u}$, we refer to the corresponding variational inequality as a quasivariational inequality.*

Sofonea and Matei [16] proved a basic existence and uniqueness result for elliptic variational inequalities (19) and (18) when a is a bilinear symmetric form with continuity and $\mathcal{H}$-ellipticity (i.e., there exist positive constants M and m such that $|a(u,v)|\leq M\|u\|\|v\|$ and $a(u,u)\geq m\|u\|^2$ for any $u\in X$ and $v\in Y$, where Y is a normed space and j is a proper convex l.s.c. function. Further, Sofonea and Matei [16] provided convergence results and extended part of these results to the study of elliptic quasivariational and time-dependent variational and quasivariational inequalities, respectively. The results presented in Chapter 3 of [16] is used to study static antiplane frictional contact problems with elastic materials. They also are crucial tools in deriving existence results for evolutionary variational inequalities.

Example 6 ([16]) *If $a(u,v)=\langle u,v\rangle$ for all $u,v\in\mathcal{H}$, then, using the inequality characterized by the projection operator P_K onto K (i.e. $\langle P_Ku-u,v-P_Ku\rangle\geq 0$), it is easy to see that in this case the solution of* (18) *is the projection of the element f on K, i.e.,*

$$u=P_Kf.$$

Moreover, elliptic variational inequality (18) arises in the study of obstacle problems, fixed point problems and unilateral contact between a linearly elastic body and a rigid foundation, see [3, 10] and the references therein.

In 2015, under some suitable setting, using the developed theory to the variational inequality (14) and based on the equivalent elliptic variational inequality

$$\sum_{i=1}^{n}\int_\Omega\big(|\nabla u(x)|^{p-2}\frac{\partial u}{\partial x_i}\big)\frac{\partial v}{\partial x_i}\,dx\geq\int_\Omega f(x,u(x))v(x)\,dx\quad\text{for } v\in K\subset\mathcal{X},\tag{20}$$

Lan [7] studied existence of nonzero positive weak solutions for the following p-Laplacian elliptic inequalities, depending on a parameter p:

$$\begin{cases}-\Delta_p u(x)\geq f(x,u(x)) & \text{for a.e. } x\in\Omega,\\ u(x)=0 & \text{on } \partial\Omega,\end{cases}\tag{21}$$

where $2\leq n<p<\infty$, $\Delta_p u(x)=\sum_{i=1}^n\frac{\partial}{\partial x_i}\left(|\nabla u(x)|^{p-2}\frac{\partial u(x)}{\partial x_i}\right)$ is the p-Laplacian operator, which is changed the square of Laplacian operator to a p^{th} power, $|\cdot|$ stands for the Euclidean norm of $\mathbb{R}^{\mathbf{N}}$, Ω is a bounded and connected open set in $\mathbb{R}^n$.

Remark 7 *The elliptic inequalities of the form* (21) *and equations arise in the study of Newtonian fluids* ($p=2$) *and their generalizations arise in the study of non-Newtonian fluids* ($p\neq 2$) *such as dilatant fluids* ($p>2$) *and pseudoplastic fluids* ($1<p<2$), *non-Newtonian filtrations, subsonic motion of gases, plasma physical models, population dynamics, some chemical reaction-diffusion, nonlinear elasticity (for instance, torsional creep), glaciology and so on. See, for example,* [7] *and the references therein.*

As illustrations, Lan [7] considered a population model of one species containing the functional response of general Holling type III as follows

$$\begin{cases} -\Delta_p u(x) \geq r u^{p-1}(x) \left(1 - \frac{u(x)}{K}\right) - \frac{a u^{\alpha}(x)}{b + u^{\gamma}(x)} & \text{for a.e. } x \in \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{22}$$

arising in mathematical biology, where Δ_p and Ω are the same as in (21), $u(x)$ denotes the population density of one species at location x, r is the intrinsic growth rate of the species, $K > 0$ is the carrying capacity of the species, the term $u^{p-1}(x)(1 - u(x))$ represents the logistic growth rate of order p, and the parameters $a \geq 0, b, \alpha, \gamma > 0$. Further, if $p = 2$ and $a = 0$, then the population model (22) reduces to the following Laplacian elliptic inequality:

$$\begin{cases} -\Delta u(x) \geq r u(x) \left(1 - \frac{u(x)}{K}\right) & \text{for a.e. } x \in \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{23}$$

which includes (5) as a special case. Thus, Lan [7] pointed out "it is interesting to know whether (22) has nonzero positive solutions in $W_0^{1,p}$ even when $a = 0$.

The Laplacian elliptic inequality (21) with $p = 2$, i.e.,

$$\begin{cases} -\Delta u(x) \geq f(x, u(x)) & \text{for a.e.} \quad x \in \Omega, \\ u(x) = 0, & \text{on} \quad \partial\Omega \end{cases} \tag{24}$$

was studied in [19], and if $2 \leq p < n$, by using the critical point theory, existence of positive or nonzero positive weak solutions of Laplacian elliptic inequality (21) have been discussed by many researchers. See, for example, [22] and the references therein. Further, Lan [19] considered the existence of positive weak solutions for the upper or lower growth problem of replacing the right end of the first inequality in (23) by $-a_1(x) - b_1(x)u^{\sigma} + a(x) + b(x)u^{\delta}$, which biological meaning can also be found in [20]. Some more complex related works, we refer to [15, 21].

If for all $x \in \Omega$, $|\nabla u(x)|^{p-2}$ in (21) is replaced by a matrix field $a : \mathbb{R}^n \to M^{n\times n}(\mathbb{R})$ defined by $a(x) = \big(a_{ij}(x)\big)_{n\times n}$, then we have the following second-order uniformly elliptic inequality:

$$\begin{cases} -\nabla \cdot \big(a(x)\nabla u(x)\big) \geq f(x, u(x)) & \text{for a.e.} \quad x \in \Omega, \\ u(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{25}$$

which simple necessary and sufficient conditions of the existence for a nonzero positive weak solution of (25) is worth exploring.

Furthermore, if $h(x)$ is nonnegative and $\hat{f}(x, u(x)) := f(x, u(x)) + h(x)$ for a.e. $x \in \Omega$, then the elliptic inequality (25) can be rewritten as

$$\begin{cases} -\nabla \cdot \big(a(x)\nabla u(x)\big) = \hat{f}(x, u(x)) & \text{for a.e. } x \in \Omega, \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases} \tag{26}$$

which is a stationary state of general periodic reaction-diffusion equations with Dirichlet smooth boundary conditions.

In the last three decades, this type of inequality (25) and equations (26) has been widely used to model spatial propagation or spreading of biological species (bacteria, epidemiological agents, insects, plants, etc), so far as to model the spread of human cultures. See, for example, [23] and the reference therein. Furthermore, Şahiner and Zafer [24] considered and studied the oscillation of solutions for the following elliptic inequalities with $p(x)$-Laplacian of the form

$$\begin{aligned} u\Big[\nabla \cdot \Big(a(x)|\nabla u|^{p(x)-2}\nabla u\Big) &- \log|u||\nabla u|^{p(x)-2}(a(x)\nabla u) \cdot \nabla p(x) \\ &+ \langle b(x), |\nabla u|^{p(x)-2}\nabla u\rangle + c(x)|u|^{\beta(x)-2}u + f(x)\Big] \leq 0 \end{aligned} \tag{27}$$

for all $x \in \Upsilon$, where Υ is an exterior domain in $\mathbb{R}^N$ containing the set $\Upsilon(r_0) := \{x \in \mathbb{R}^N |, |x| \geq r_0\}$ for a fixed positive real number r_0, $\beta(x) \geq p(x) > 1$ for all $x \in \Upsilon$, the matrix $a = (a_{ij})_{N\times N}$ is a real symmetric positive definite matrix with $a_{ij} \in C^1(\Upsilon, \mathbb{R})$, $b = (b_i)_{N\times 1}$ is a real vector with $b_i \in C(\Upsilon, \mathbb{R})$, $c, f \in C(\Upsilon, \mathbb{R})$ and $p \in C^1(\Upsilon, (1, \infty))$.

3 Some existing variational inequality theory

In this section, we will propose some existing variational inequality theorems, especially new theory for variational inequalities due to Lan [7, 19] and their applications to study the existence of positive weak solutions for some elliptic inequalities.

Since the theory of variational inequalities plays an important role in the study of both the qualitative and numerical analysis of nonlinear boundary value problems arising in mechanics, physics and engineering science, the mathematical literature dedicated to this field is extensive, and the progress made in the over past four decades is impressive. Further, an increasing number of authors and researchers have been studied the theory of variational inequalities and applications to some problems arising in economics, mechanics, differential equations, and engineering science. See, for example, [7, 15, 19] and the references therein. In particular, Atkinson and Han [10] pointed out that it is more difficult to study solution regularity for variational inequalities than for ordinary boundary value problems of partial differential equations.

Based on the classic fixed point index for condensing self-maps developed by Nussbaum [25], Lan and Webb [26] studied a fixed point equation $u = (rA)u$, which is equivalent to u is a solution of (14) with $F = I$, that is,

$$\langle u - Au, u - v\rangle \leq 0, \forall v \in K. \tag{28}$$

He also obtained a fixed point index for generalized inward maps of condensing type defined on cones in Banach spaces. Recently, Lan and Lin [21] established a variational inequality index for γ-condensing maps in $\mathcal{H}$ and proved some new results on existence of nonzero positive solutions of the variational inequality (28) for such maps by using the theory of variational inequality index. As applications of such a theory, the authors presented the existence of nonzero positive weak solutions for the elliptic inequality (24). Moreover, Lan and Lin [21] pointed out that the previous results of variational inequalities for S-contractive maps cannot be applied and few results on existence of nonzero positive solutions of the variational inequalities are derived in [15].

On the other hand, it is well known that variational methods were often employed to study the functionals defined on a closed and convex subset of a Banach space, and a critical point theory for closed convex sets is required because of having to encounter problems with inequality constraints (such as variational inequalities). Indeed, if $2 \leq p < n$, by using the critical point theory, many researchers studied variational inequalities of the form due to elliptic inequality (21). See, for example, [22] and the references therein.

However, in the 1920s, in order to distinguish between different kinds of critical points of a functional defined on $\mathbb{R}^{\mathbf{N}}$, Morse [14] first told the difficulties convincing the experts at the time to distinguish between critical points, which are and are not extrema or absolute minimizers of the associated functionals, a subject now called Morse theory, which has been extended to compact, smooth, finite-dimensional manifolds. See, for example, [27] and the references therein. In fact, as we all know, one can apply this distinction with great mastery and connect Morse's study of critical points with the topology of both finite-dimensional and infinite-dimensional manifolds. That is to say, Morse theory enables one to analyze the differential topology of a manifold by studying differentiable functions on that manifold, and can be used to explore some nonlinear elliptic boundary value problems with resonance at zero. For more related works, one can refer to [22, 27] and the references therein.

In the sequel, we shall review literatures on a new theory for variational inequalities of the form (14) and its applications to study the existence of positive weak solutions for some elliptic inequalities.

In [19], Lan pointed out "there are little applications of Granas topological transversality to the study of variational inequalities for nonlinear maps" and first introduced the so-called essential maps for (28) in the class of demicontinuous S-contractive maps A in $\mathcal{H}$. Further, Lan [19] proved existence property, normalization and homotopy property of the variational inequality of the map A, which are similar to those of fixed point theory for demicontinuous weakly inward S-contractive maps [28] and to those of fixed point index for maps of condensing type. Then, using these properties and adding some new conditions, Lan [19] proved that maps which satisfy the Leray-Schauder type conditions related to variational inequalities are essential and that maps which satisfy the conditions similar to those implying that the fixed point index is zero are not essential. Combining such results, the author obtained some new results on the existence of nonzero positive solutions and eigenvalues for elliptic variational inequality (24) with nonlinearities which satisfy suitable lower or upper bound growth conditions involving the critical Sobolev exponent.

Further, in order to improve main results of [19], by using the developed new principles of variational inequality (28) for demicontinuous pseudo-contractive maps in Hilbert spaces, Lan [20] emerged the existence, uniqueness and convergence of approximants of positive weak solutions for (24), which nonlinearities satisfy lower bound conditions containing the critical Sobolev exponents and suitable upper or monotonicity conditions. Similar illustrations discussed in [19], Lan [20] considered the existence of positive weak solutions of the following generalized inequality:

$$\begin{cases} -\Delta u(x) \geq -a_1(x) - b_1(x)u^\sigma(x) + g(x)h(u) & \text{for a.e. } x \in \Omega, \\ u(x) = 0, & \text{on } \partial\Omega, \end{cases}$$

where g and h are two functions satisfying some suitable conditions. For more developments on this theory, for example, one can refer to [29] and the references therein.

Moreover, as Lan [20] pointed out, in order to consider (28) with p-Laplacian operators or more general elliptic differential operators, it would be interesting to generalize the theory of variational inequalities from Hilbert spaces to uniformly convex Banach spaces or reflexive Banach spaces. Therefore, in order to study existence of nonzero positive weak solutions for the p-Laplacian elliptic inequality (22), Lan [7] developed the theory for variational inequalities of the form (14) with $F \equiv J$, the duality map with a gauge function, in reflexive smooth Banach spaces by employing the method used in [19], where the variational inequality theory for demicontinuous S-contractive map in Hilbert spaces is established. The main ideas originate from the Granas topological transversality which was developed in order to study existence of fixed points for nonlinear maps [28]. Firstly, based on the results from [13], Lan [7] let us know that if K is a wedge in $\mathcal{X}$, then $x \in D \subset \mathcal{X}$ is a solution of the variational inequality

$$\langle Ju - Au, u - v\rangle \leq 0 \quad \text{for } v \in K, \tag{29}$$

which is a special case of (14), where $J : \mathcal{X} \to \mathcal{X}^*$ is a duality map and $A : D \subset \mathcal{X} \to \mathcal{X}^*$ is an S-contractive map on D, if and only if $x \in D$ is a solution of the following complementary problem for A:

$$\langle Ju - Au, u\rangle = 0 \quad \text{and} \quad \langle Ju - Au, v\rangle \geq 0 \quad \text{for } v \in K. \tag{30}$$

Further, if K is a subspace of $\mathcal{X}^*$, then $x \in D$ is a solution of (29) if and only if $\langle Ju - Au, v\rangle = 0$ for all $v \in K$, that is, $Ju - Au$ is orthogonal to K.

Based on important properties of essential maps and under general conditions of ensuring that (29) has nonzero positive solutions, the following two results on the existence of solutions and eigenvalues for (29) was obtained in [7], respectively.

Theorem *LA* ([7, Theorem 3.3, p. 75]) *Let K be a wedge in a reflexive smooth Banach space $\mathcal{X}$ and let D^1, D be bounded open sets in $\mathcal{X}$ such that $0 \in D^1$ and $\overline{D^1_K} \subset D_K$. Assume that J is of S^+-type and strictly monotone. Assume that $A : \overline{D}_K \to \mathcal{X}^*$ is a bounded demicontinuous S-contractive map satisfying the following conditions:*

(i) *(LS) There exists $x_0 \in D_K$ such that the variational inequality of $tA + (1-t)\hat{Jx_0}$ has no solutions on ∂D_K for each $t \in (0, 1)$.*

(ii) *There exists $e \in K$ with $\|e\| = 1$ such that the variational inequality of $A + \beta\hat{Je}$ has no solutions on ∂D^1_K for each $\beta > 0$.*

Then (30) *has a solution on $\overline{D}_K \setminus D^1_K$.*

Theorem *LB* ([7, Theorem 3.5, p. 78]) *Let K be a wedge in X with $J(K) \cap K^* \neq \{0\}$ and D a bounded open set in X such that $\partial D_K \neq \emptyset$, where*

$$K^* = \{u \in \mathcal{X}^* : (u, v) \geq 0 \quad \textit{for } v \in K\}.$$

Suppose J is of S^+-type, $A : \overline{D}_K \to \mathcal{X}^*$ is a bounded demicontinuous S-contractive map and $B : \overline{D}_K \to \mathcal{X}^*$ is a compact map. Suppose that the following conditions hold:

(h_1) *A satisfies $\underline{(LS)}$ of Theorem LA on ∂D_K.*

(h_2) *Either $\overline{B(\partial D_K)} \cap (-K^*) = \emptyset$ or the following conditions hold:*

(a) $\inf\{\|Bx\| : x \in \partial D_K\} > 0$; (b) $\overline{B(\partial D_K)} \cap ((-K^*) \setminus J(K)) = \emptyset$.

Then there exists $\lambda \geq 0$ such that the variational inequality of $A + \lambda B$

$$(Ju - (A + \lambda B)u, u - v) \leq 0 \quad \textit{for } v \in K$$

has a solution on ∂D_K.

Since the Sobolev space $W^{1,p}_0 := W^{1,p}_0(\Omega)$ with the standard norm

$$\|u\|_{W^{1,p}_0} = \Big(\int_\Omega |\nabla u(x)|^p\, dx\Big)^{1/p},$$

is a uniformly convex and smooth Banach space (see [30]), where Ω is a bounded and connected open set in $\mathbb{R}^n$ ($n \geq 1$). It is known that $W_0^{1,p}$ is a reflexive, strictly convex and smooth Banach space with property (P) (or Kadee-Klee property, see [31]), i.e., $y_n \rightharpoonup y$ and $\|y_n\| \to \|y\|$ together imply $y_n \to y$. The dual space of $W_0^{1,p}$ is denoted by $W^{-1,q}(\Omega)$, where $1/p + 1/q = 1$. Let

$$P = \left\{ u \in W_0^{1,p} |\, u(x) \geq 0 \quad \text{a.e. on } \Omega \right\}.$$

Then P is a standard positive cone in $W_0^{1,p}$, and it follows from (29) and (30) that $u \in P$ is a solution of (29) with $K = P$ if and only if $u \in P$ is a solution of the complementary problem

$$\langle Ju, u\rangle = \langle Au, u\rangle$$

and

$$\langle Ju, v\rangle \geq \langle Au, v\rangle \quad \text{for } v \in P. \tag{31}$$

Thus, finding a positive weak solution $u \in W_0^{1,p}$ of (21) is equivalent to solving the inequality (31), that is (see [7, Definition 4.1, p. 80]), finding $u \in P$ such that (20) with $K = P$ holds. Lan [7] respectively defined operators $J : W_0^{1,p} \to W^{-1,q}$ and $A : P \to W^{-1,q}$ in (31) by

$$Ju(x) = -\Delta_p u(x), \quad (Au, v) = \int_\Omega f(x, u(x))v(x)\, dx.$$

By the property of $W_0^{1,p}$, and Example 2.110 and Lemma 2.111 of [32], one can know that J is of S^+-type and is strictly monotone. Further, Lan [7] prove that A is compact and so demicontinuous S-contractive map under some hypotheses([7, Lemma 4.2]. It is nevertheless the case that conditions and properties remain to be considered for concerning nonlinear operator F in (14). Besides the improvement of existing variational inequality theory and derivation of proper existence results, our feeling is that some promising approach would be to develop variational inequality theory similar to those in [20] allowing to generalize from a Hilbert space to a Banach space.

Next, by using Theorem *LA* and the results in [33], Lan [7] proved the following two results on the existence of nonzero positive weak solutions of (21) and (22) involving the functional response of general Holling type III (see Theorems 4.1 and 4.2 in [7]), respectively.

Further, in Theorem 4.3 of [7], the following eigenvalue result on variational inequality (21):

$$\begin{cases} -\|u\|^{2-p}_{W_0^{1,p}} \Delta_p u(x) \geq f(x, u(x)) + \lambda g(x, u(x)) & \text{for a.e. } x \in \Omega, \\ u(x) = 0 \quad \text{on } \partial\Omega. \end{cases} \tag{32}$$

was obtained via applying Theorem *LB* and the corresponding results in [34].

4 Concluding remarks and challenges

In this section, we list some remarks and some challenging questions on elliptic variational inequalities with Laplacian type operators and applications to related population models, which are worth being solved in further research.

In this paper, we provided some background with Logistic models of biological populations and research on elliptic equations and inequalities, and presented some related problems from variational equations to variational inequality. Then, we gave some works of studying elliptic inequalities with Laplacian type operators and related population models of one species arising in mathematical biology. Moreover, based on fixed point index, critical point theory and Granas topological transversality ideas, we proposed some theory of variational inequalities, especially new theory for variational inequalities due to Lan [7, 19] in Hilbert spaces or Banach spaces, and their applications to study the existence of positive weak solutions for some elliptic inequalities.

However, the fixed point index theories cannot be used to treat variational inequalities of the form (14) when A is not continuous and rA is not a map of condensing type, where r is the metric (generalized) projection from $\mathcal{X}^*$ to a closed convex set K of $\mathcal{X}$ (see [7, 19]). Further, the topological degree is used to investigate the linking properties related to critical point theory, also in bifurcation theory, in handling different nonlinear elliptic equations. As Lan [19] pointed out, there are little applications of Granas topological transversality to the study of variational inequalities for nonlinear maps. In particular, concerning nonlinear operator $F \neq J$, the duality map with the gauge function, how to study the theory of variational inequality (14). Therefore, it is the case that challenges remain to be tackled. That is to say, there are still many questions on elliptic variational inequalities and applications to related population models to be solved in further research.

Based on the above perspectives and discussions, now we list four challenging questions for further research as follows:

Question 1 *Can other variational inequality theory, such as fixed point index theory* [15]*, critical point theory* [22]*or Morse theory* [14]*be used to study existence of nonzero positive weak solutions for the generalizations of Laplacian elliptic inequalities* (21)*,* (25) *and* (27) *, and population models* (22) *and* (23)*?*

Question 2 *On the one hand, it is important to obtain existence of positive solutions of* (14) *by studying convergence of suitable approximants since such an approach not only shows the existence of positive solutions but also provides approximations and numerical schemes. Hence, it is necessary and important to develop a variational inequality approximation principle for nonlinear maps on closed convex subsets of reflexive smooth Banach spaces. On the other hand, if* $2 \le p < n$ *or* $1 < p < 2$*, whether does there exist nonzero positive weak solutions for* p*-Laplacian elliptic inequality* (21) *and population model* (22)*? The reader is referred to* [15, 20] *for further details.*

Question 3 *When the operator* F *in* (14) *is more general elliptic differential operators or any nonlinear operators, whether the results of Theorems LA and LB hold? For instance, if* $Fu(x) = -\nabla \cdot \big(a(x)\nabla u(x)\big)$ *for all* $x \in \Omega$*, then one can clearly to say that* F *is not a duality map with the gauge function* $\Phi(t) = t^{p-1}$ *for* $t \in \mathbb{R}^+$ *(see* [35]*). Thus, the conditions for* F *and* A *in* (14) *should be explored to develop a theory for variational inequalities of the form* (14)*, which can be employed to investigate existence of nonzero positive weak solutions for elliptic inequality* (25) *and related population models of one species arising in mathematical biology.*

As the end of this paper, we will emerge the fourth question in relation to problem (3). In fact, problem (3) not only is in view of ordinary differential equations (1) and (2), but also can be come from the open question proposed by Lions in [36], that is, whether the following system of Laplace equations

$$\begin{cases} -\Delta z_i(x) = f_i(\mathbf{z}(x)) & \text{for a.e.} \quad x \in \Omega, \quad i = 1, 2, \cdots, n, \\ z_i(x) = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{33}$$

where $\mathbf{z}(x) = (z_1(x), \cdots, z_n(x))$, Ω is a bounded open set in $\mathbb{R}^m (m \ge 2)$ with smooth boundary $\partial\Omega$, has a nonzero positive solution under sublinear or superlinear conditions which involve the principal eigenvalues of the corresponding linear systems (see [36, question (c) in Section 4.2]).

On the other hand, by using the coincidence degree theory due to Mawhin [37], Zhu [38] introduced and studied the following system of variational inequalities involving the linear operators: Find $(u, v) \in K \times K$ such that

$$\begin{cases} \langle Au, y - u\rangle \ge \langle g(v), y - u\rangle, \quad \forall y \in X, \\ \langle Bu, y - u\rangle \ge \langle h(v), y - u\rangle, \quad \forall y \in K, \end{cases} \tag{34}$$

where K is a nonempty closed convex subset of reflexive Banach space X with its dual X^*, and $A, B : X \to X^*$ are two mappings, but g, h from K to X^*. Further, the author proved some existence results of positive solutions for this variational inequality systems and gave an example as an application of the results.

Motivate and inspired the above works, now we propose the following question of research for the future:

Question 4 *How to develop the theory of variational inequality system* (34) *for studying the existence of nonzero positive weak solutions to the elliptic inequality system* (3) *and its associated with problems* (33) *and the corresponding population model of more than one species arising in mathematical biology?*

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