**ISSN** 1749-3889 (print), 1749-3897 (online)
**International Journal of Nonlinear Science**
Vol.23(2017) No.3, pp.142-150

# Hybrid Projective Synchronization Between two Identical New 4-D Hyper-chaotic Systems via Active Control Method

Ayub Khan, Arti Tyagi *
Department of Mathematics, Jamia Millia Islamia, New Delhi-110 025, India
*(Received 15 September 2016, accepted 23 February 2017)*

**Abstract:** In this manuscript the Active Control method has been employed to discuss the hybrid projective synchronization between the two identical new 4-d hyper-chaotic systems with different initial conditions. Hybrid projective synchronization is one of the important type of synchronization as it also includes complete synchronization, anti-synchronization and projective synchronization as its special cases. Linear feedback control method is adopted to achieve hybrid projective synchronization based on Lyapunov stability theory. Moreover, numerical simulations are given to illustrate and verify the analytical results.

**Keywords:** New 4-D hyper-chaotic system; Hybrid projective synchronization ; Active control method; Lyapunov stability theorem

## 1 Introduction

Many complex and interesting phenomena in nature are due to nonlinear phenomena. The theory of nonlinear dynamical systems, also termed as 'chaos theory'. A nonlinear dynamical system is said to be chaotic if it is deterministic and has long-term unpredictable behaviors. A well known inherent properties of chaotic systems is the sensitive dependence on initial conditions i.e two nearby points in state space will separate rapidly as they evolve in time. Whereas, chaotic behavior is a common feature of nonlinear dynamics. In 1963, Lorenz found the first chaotic attractor in a three-dimensional (3D) autonomous system when he studied the atmospheric convection [1]. After that in 1976, Rössler [2] introduced a special three-dimensional chaotic system with only one nonlinear term, then in 1983, Chua [3] constructed the notable Chua circuit which can exhibit two scroll chaotic behavior. Later on Chen [4], Lü [5] and many other researchers tried to introduce new chaotic systems, such as Qi, Yang, Liu, Wang and so on [6]-[8]. Generally, chaotic systems are three or four dimensional, having one positive Lyapunov exponent. Having more than one positive Lyapunov exponent cause the system to show very complex and irregular behaviors. These systems are known as hyper-chaotic systems with complex dynamical behavior. The dynamics of a hyper-chaotic attractor are expanding in more than one direction, giving rise to more complex chaotic dynamics. By now, the usual technique to get a new hyper-chaotic system is to add one more state variable to the well known three-dimensional chaotic systems, but this is [9]. Yet,it is not easy task to create a new chaotic system or hyper-chaotic system with a more topological structure, with more wings, with more scrolls, which also fulfills the agreement between lyapunov exponent and dissipation. The first classical hyper-chaotic system is the well known hyper-chaotic Rössler system [10]. After that, many hyper-chaotic systems have been developed and the applications of these models have been enhanced recently. Over the past decades, many other hyper-chaotic systems have been introduced,such as hyper-chaotic Chen system [11] , hyper-chaotic Chen system [12] , hyper-chaotic Lü system [13], hyper-chaotic Nikolov system [14], hyper-chaotic Lorenz system [15], and hyper-chaotic Lorenz system[16].

Since the pioneering work of Pecora and Corroll [17], chaos synchronization has become an active research area in the field of non linear dynamical systems for its many potential applications in physics, secure communication, chemical reactor, biological networks, neural networks, etc. The main idea of synchronization is to use the output of the master system to control the slave system so that the output of the latter follows the output of the master system asymptotically. So far,

*Corresponding author. *E-mail address*: artityagi28@gmail.com


IJNS.2017.06.15/959

there exist many types of synchronization such as complete synchronization , anti synchronization, hybrid synchronization , phase synchronization, partial synchronization , generalized synchronization , lag synchronization , Q-S synchronization etc [18]-[31]. For attaining these types of synchronization various methods are proposed by the researchers such as active control method , adaptive control method , optimal control method, feedback control method, back-stepping control method, sliding mode method and so on [32]-[42] .

As Hybrid projective synchronization includes complete synchronization, anti-synchronization and projective synchronization as its special cases . So,in our paper we present the hybrid projective synchronization between two identical new 4-d hyper-chaotic systems with different initial conditions via active control method.The paper is organised as : Section 2 contains the description of a new 4-D Hyper-chaotic system. Section 3 presents the hybrid projective synchronization of new 4-d hyper-chaotic system. In Section 4, numerical simulations are used to verify the effectiveness of the proposed scheme. Finally, in the last section conclusions are drawn.

# 2 Model description

A new 4-D hyper-chaotic system [43] is defined by:

$$\begin{aligned} \dot{x} &= ax - byz, \\ \dot{y} &= -cy + xz, \\ \dot{z} &= kx - dz + xy, \\ \dot{w} &= hw + xy. \end{aligned} \tag{1}$$

where $(x, y, z, w)^T \in R^4$ is the state vector, and a, b, c, d, k and h are positive constant parameters of the system.

## 2.1 Phase portraits and time responses

For the parameters a=4.55, b=1.532, c=10.1, d=5.5 , k=3.5 and h=.04, the system shows hyper-chaos. The new system is Hyper chaotic having the Lyapunov exponents $LE_1 = 1.5278, LE_2 = .041041, LE_3 = 0.0023108$ and $LE_4 = -12.5454$. It is clear that $LE_3 = 0.0023108$ is close to 0. Thus for the values of chosen parameters, the system exhibits hyper-chaos. The corresponding phase portraits ,time series and lyapunov exponents are displayed in Figure 1, Figure 2 and Figure 3 respectively.

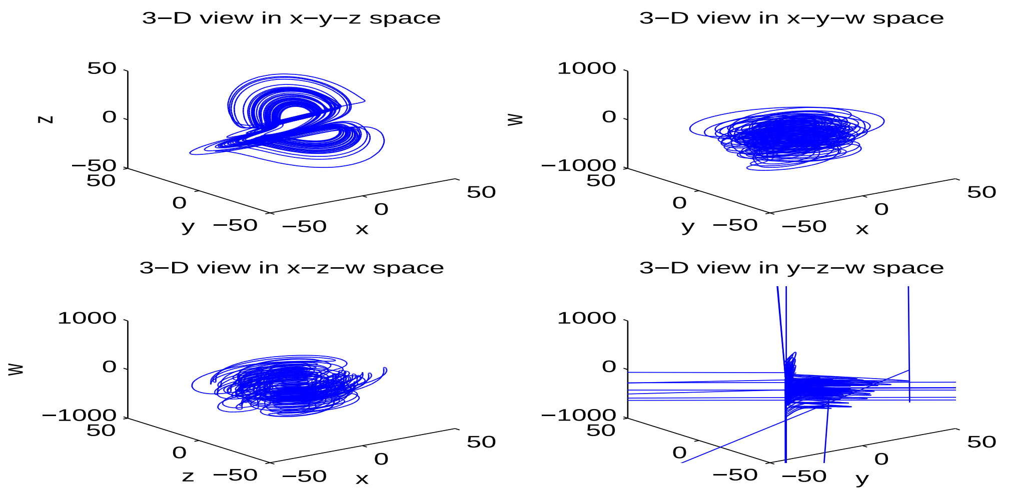


Figure 1: Phase Portrait of a new 4-D Hyper-chaotic systems.

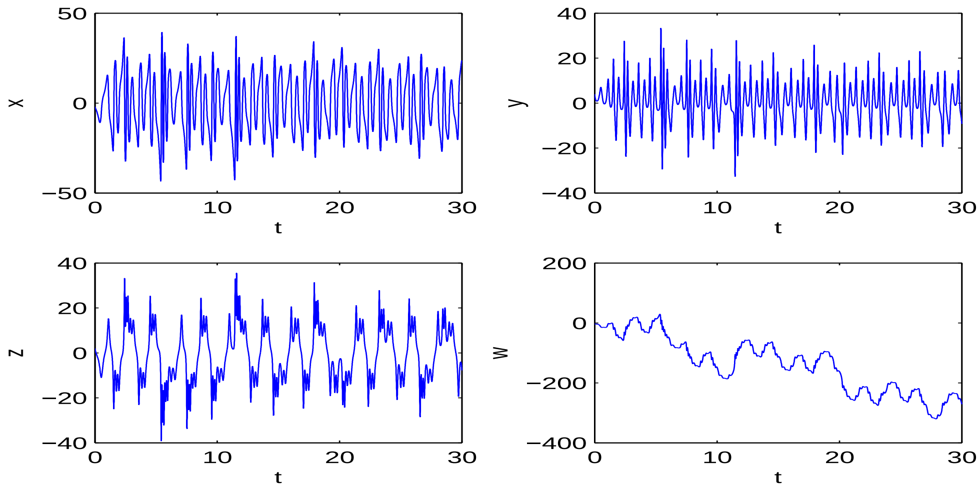


Figure 2: Time series of a new 4-D Hyper-chaotic systems.

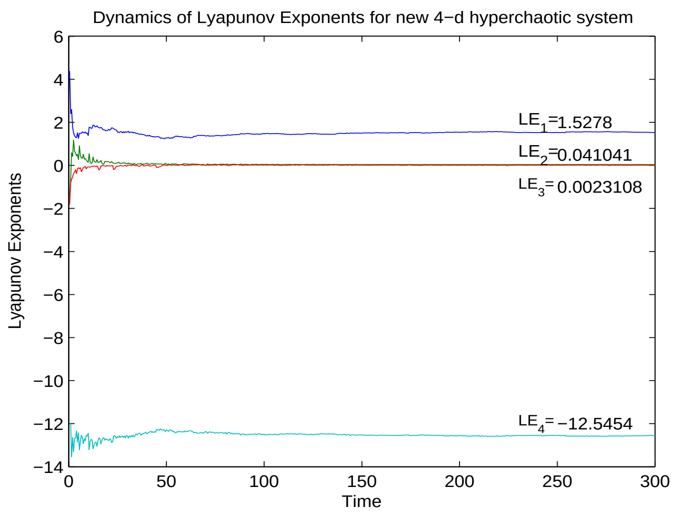


Figure 3: Lyapunov exponents graph for new 4-D Hyper-chaotic systems.

## 2.2 Dissipation

The divergence of a vector field **F** of the system (2) can be obtained as:

$$\bigtriangledown.\mathbf{F} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z} + \frac{\partial f_4}{\partial w} = a - c - d + h, \tag{2}$$

where

$$\begin{aligned}\mathbf{F} &= (f_1, f_2, f_3, f_4) \\ &= (ax - byz, -cy + xz, kx - dz + xy, hw + xy). \end{aligned} \tag{3}$$

For a system to be dissipative it is required that $\bigtriangledown.\mathbf{F} < 0$. When we substitute the parameter values as a=4.55, b=1.532, c=10.1, d=5.5 , k=3.5 and h = .04, we get $\bigtriangledown.\mathbf{F} = -11.01 < 0$. Thus, the system (2) is dissipative and converges with the rate $e^{-11.01t}$.This means , each volume containing the system orbit shrinks to zero as $t \to \infty$ with an exponential rate $\bigtriangledown.\mathbf{F} = -11.01$ and is independent of system states. Consequently, all system orbits will ultimately be confined to a specific subset of zero volume and the asymptotic motion dies onto an attractor. It proves, the existence of an attractor.

## 2.3 Poincaré mapping

The Poincaré map is one such analytical technique which helps to visualise the folding properties of chaos. This also provides the idea of the bifurcation. When a=4.55, b=1.532, c=10.1, d=5.5, k=3.5 and h=.04 and on taking the different crossing planes such as y=0, x=0. The corresponding Poincaré maps on the x-z and y-w planes are displayed in Figure 4. system(2) has a self-similar structure. Some sheets are folded and wing type structure is also visualized.

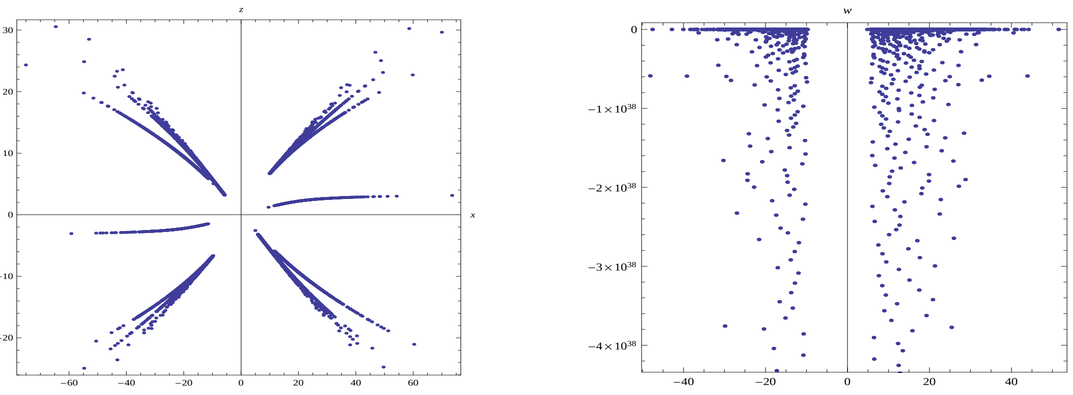


Figure 4: Poincare section for new Hyper-chaotic systems(2) with the parameters a=4.55,b=1.532,c=10.1, d=5.5,k=3.5 and h=.04.(a)Projection on x-z plane with y=0;(b) Projection on y-w plane with x=0.

## 2.4 Bifurcation

When the parameter $'a'$ is varied, the corresponding bifurcation diagram of state y with respect to $'a'$ is obtained as shown in Figure 5. It is easy to see the chaotic behavior of new 4-D hyper-chaotic system when the parameter $'a' \in [2, 6]$ .

# 3 Hybrid projective synchronization of two identical new 4-d hyper-chaotic systems via Active control method

In order to observe the hybrid projective synchronization behavior between two identical new 4-d hyper-chaotic systems with different initial conditions, we consider master/drive system with subscript 1 in new 4-d hyper-chaotic system and

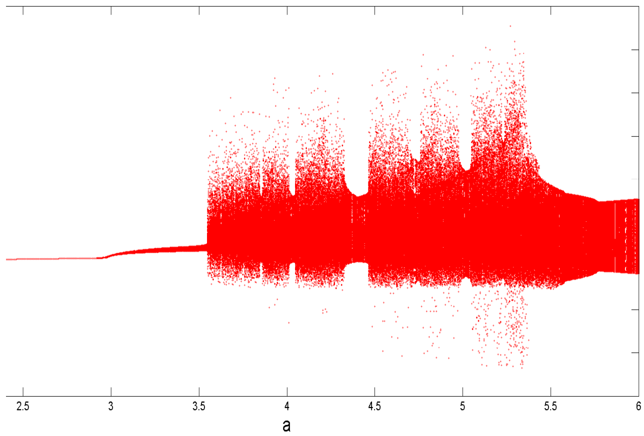


Figure 5: Bifurcation diagram of new hyper chaotic system (2) versus the parameter $'a' \in [2,6]$ when b=1.532, c=10.1, d=5.5, k=3.5 and h=.04.

slave/response system with subscript 2 in new 4-d hyper-chaotic system with different initial conditions. Then drive and response systems are given as follows:

$$\begin{aligned} \dot{x_1} &= ax_1 - by_1z_1, \\ \dot{y_1} &= -cy_1 + x_1z_1, \\ \dot{z_1} &= kx_1 - dz_1 + x_1y_1, \\ \dot{w_1} &= hw_1 + x_1y_1. \end{aligned} \tag{4}$$

and

$$\begin{aligned} \dot{x_2} &= ax_2 - by_2z_2 + U_1(t), \\ \dot{y_2} &= -cy_2 + x_2z_2 + U_2(t), \\ \dot{z_2} &= kx_2 - dz_2 + x_2y_2 + U_3(t), \\ \dot{w_2} &= hw_2 + x_2y_2 + U_4(t). \end{aligned} \tag{5}$$

where $U_1(t), U_2(t), U_3(t)$ and $U_4(t)$ are the control functions in the response system. We aim to design the control functions $U_i(t), (i = 1, 2, 3, 4)$. To observe the hybrid projective synchronization between master and slave systems given in (4)and (5)respectively . Lets us define the hybrid projective synchronization errors as :

$$e_1 = x_2 - \lambda_1 x_1, \quad e_2 = y_2 - \lambda_2 y_1, \quad e_3 = z_2 - \lambda_3 z_1 \quad and \quad e_4 = w_2 - \lambda_4 w_1. \tag{6}$$

where $\lambda_i, (i = 1, 2, 3, 4)$ are the constant parameters. Now, the error dynamics is described as :

$$\begin{aligned} \dot{e_1} &= ae_1 - by_2z_2 + \lambda_1 by_1z_1 + U_1(t), \\ \dot{e_2} &= -ce_2 + x_2z_2 - \lambda_2 x_1z_1 + U_2(t), \\ \dot{e_3} &= ke_1 + kx_1(\lambda_1 - \lambda_3) - de_3 + x_2y_2 - \lambda_3 x_1y_1 + U_3(t), \\ \dot{e_4} &= he_4 + x_2y_2 - \lambda_4 x_1y_1 + U_4(t). \end{aligned} \tag{7}$$

This error system (7) to be controlled must be a linear system with control functions. Let us redefine the control functions so that the terms in (7) which cannot be expressed as linear terms in $e_i's$ are eliminated.

$$\begin{aligned} U_1(t) &= by_2z_2 - \lambda_1 by_1z_1 + V_1(t), \\ U_2(t) &= -x_2z_2 + \lambda_2 x_1z_1 + V_2(t), \\ U_3(t) &= -kx_1(\lambda_1 - \lambda_3) - x_2y_2 - \lambda_3 x_1y_1 + V_3(t), \\ U_4(t) &= -x_2y_2 + \lambda_4 x_1y_1 + V_4(t). \end{aligned} \tag{8}$$

The new error system is expressed as:

$$\begin{aligned}\dot{e_1} &= ae_1 + V_1(t),\\ \dot{e_2} &= -ce_2 + V_2(t),\\ \dot{e_3} &= ke_1 - de_3 + V_3(t),\\ \dot{e_4} &= he_4 + V_4(t).\end{aligned} \tag{9}$$

The error system (9) to be controlled is a linear system with control inputs $V_1(t), V_2(t), V_3(t)$ and $V_4(t)$ as function of the error states $e_1, e_2, e_3$ and $e_4$. As long as these feedbacks stabilize the system $e_1, e_2, e_3$ and $e_4$ converge to zero as time $t$ tends to infinity. This implies that two identical new 4-D hyper-chaotic systems are synchronized with feedback control. There are many possible choices for the controls $V_1(t), V_2(t), V_3(t)$ and $V_4(t)$. We choose

$$\begin{bmatrix}V_1(t)\\V_2(t)\\V_3(t)\\V_4(t)\end{bmatrix} = A\begin{bmatrix}e_1\\e_2\\e_3\\e_4\end{bmatrix} \tag{10}$$

Here $A$ is a square matrix of order four to be determined. In order to make the closed loop system to be stable, the proper choice of the elements of the matrix $A$ is such that the feedback system must have all eigenvalues with negative real parts. We choose $A$ as:

$$A = \begin{bmatrix}-a-1 & 0 & 0 & 0\\ 0 & c-1 & 0 & 0\\ -k & 0 & d-1 & 0\\ 0 & 0 & 0 & -h-1\end{bmatrix} \tag{11}$$

Using (10)and (11),we get the values for $V_1(t), V_2(t), V_3(t)$ and $V_4(t)$,then (7) can be rewritten as:

$$\begin{bmatrix}\dot{e_1(t)}\\\dot{e_2(t)}\\\dot{e_3(t)}\\\dot{e_4(t)}\end{bmatrix} = B\begin{bmatrix}e_1(t)\\e_2(t)\\e_3(t)\\e_4(t)\end{bmatrix}$$

where

$$B = \begin{bmatrix}-1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & -1\end{bmatrix} \tag{12}$$

Hence the error system becomes

$$\dot{e_1} = -e_1, \;\; \dot{e_2} = -e_2, \;\; \dot{e_3} = -e_3 \;\; and \;\; \dot{e_4} = -e_4 \tag{13}$$

Now, we consider the Lyapunov function as

$$V(e) = 1/2e^Te = 1/2(e_1^2 + e_2^2 + e_3^2 + e_4^2) \tag{14}$$

which is a positive definite function on $R^4$. Differentiating (14) along the trajectories of (13), we get

$$\dot{V}(e) = -e_1^2 - e_2^2 - e_3^2 - e_4^2 \tag{15}$$

This shows that $\dot{V}(e)$ is a negative definite function on $R^4$. Thus, by Lyapunov stability theory [44], the error dynamics (7) is globally exponentially stable. Hence, it is proved that $lim_{t\to\infty}e_i(t) = 0, (i = 1, 2, 3, 4)$.
Thus, hybrid projective synchronization is achieved between the master and slave systems (4) and (5).

# 4 Simulation results

Numerical results are presented to demonstrate the effectiveness of the proposed technique. We select the parameters of new 4-d hyper chaotic system as $a = 4.55, b = 1.532, c = 10.1, d = 5.5, k = 3.5$ and $h = .04$, for these values the system shows hyper-chaos. The initial values of the master and slave systems are $x_1(0) = -2, y_1(0) = 4, z_1(0) = 2, w_1(0) = -3, x_2(0) = 2, y_2(0) = -3, z_2(0) = -1, w_2(0) = 2$. On choosing parameters $\lambda_1 = 1, \lambda_2 = 2, \lambda_3 = -2, \lambda_4 = -1$ for hybrid projective synchronization, the initial states of the error system (13) become $e_1(0) = 4, e_2(0) = 5, e_3(0) = -5, e_4(0) = -1$. The time waveform diagram of master and slave systems states variables are illustrated in (Figure 6). The dynamics of synchronization error functions for the drive and response systems with time "t" is shown (Figure 7 ) by trajectories $e_1(t), e_2(t), e_3(t)$ and $e_4(t)$.

**Remark 1** *Also,it is easy to see that complete synchronization , anti synchronization and projective synchronization are the special cases of hybrid projective synchronization with parameters* $\lambda_i = 1, (i = 1, 2, 3, 4)$ , $\lambda_i = -1, (i = 1, 2, 3, 4)$ *and* $\lambda_i = \alpha, (i = 1, 2, 3, 4 and \alpha = constant\ parameters)$ *respectively.*

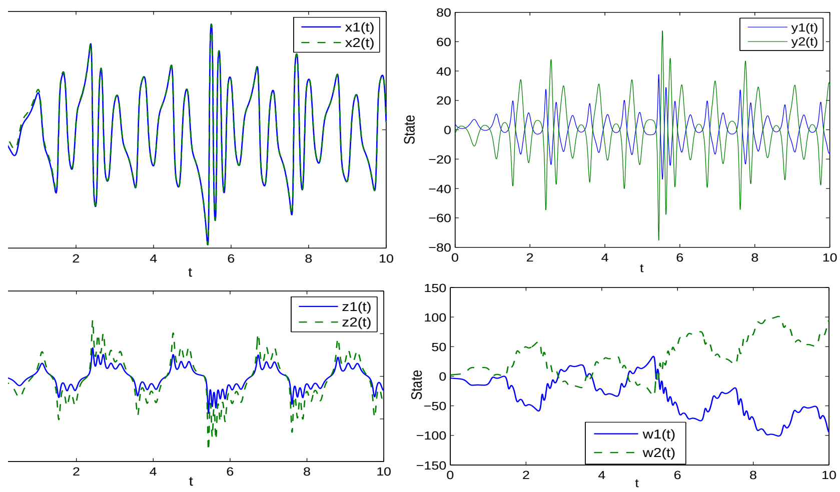


Figure 6: The time waveform diagram of the two identical new 4-d hyper-chaotic system with different initial conditions by using active control method in hybrid projective synchronization.

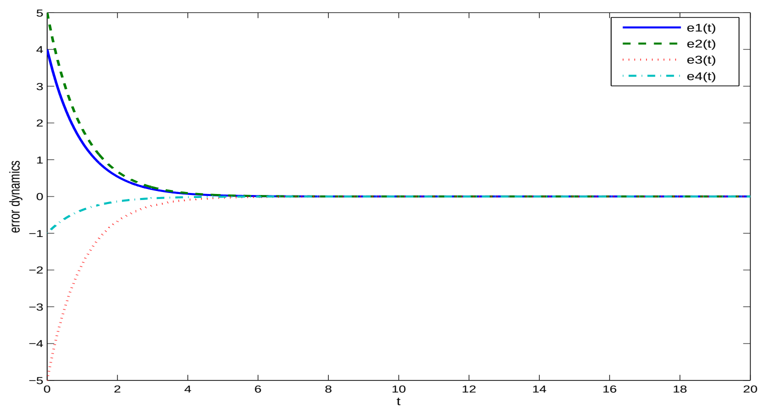


Figure 7: Dynamics of synchronization errors states $e_1(t)$, $e_2(t)$, $e_3(t)$ & $e_4(t)$ of four state variable with respect to time.

## 5 Conclusion

In this manuscript, we have presented hybrid projective synchronization between two identical new 4-d hyper-chaotic systems evolving from different initial conditions using the Active Control Technique which is based on Lyapunov Stability Theory. The effectiveness and feasibility of results are validated in numerical simulations which are performed by using Matlab software. Remarkably, our analytic and computational results are in an excellent agreement.

## References

[1] C. Sparrow. The Lorenz Equations: Bifurcations Chaos and Strange Attractors. *Springer, New York,*1982.
[2] Rössler O. E.. An equation for continuous chaos. *Phys. Lett. A,* 57, 397-398 (1976)
[3] Chua, L., Komuro, M., Matsumoto, T.. The double scroll family. *IEEE Trans. Circuits Syst.*, 33: (1986), 1073–1118.
[4] Chen, G. R., Ueta, T.. Yet another chaotic attractor. *Int. J. Bifurc. Chaos*, 9:(1999), 1465–1466.
[5] Lü, J. H., Chen, G.. A new chaotic attractor coined. *Int. J. Bifurc. Chaos*, 12:(2002), 659–661 .
[6] Qi, G., Chen, G., Du, S., Chen, Z., Yuan, Z.. Analysis of a new chaotic system. *Phys. A*, 352:(2005), 295–308.
[7] Yang, Q. G., Chen, G. R., Huang, K. F.: Chaotic attractors of the conjugate Lorenz-type system. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering* , 17:(2007), 3929–3949.
[8] Liu, Y., Pang, W.. Dynamics of the general Lorenz family.*Nonlinear Dyn.*, 67:(2012), 1595–1611.
[9] Zarei, Amin. Complex dynamics in a 5-D hyper-chaotic attractor with four-wing, one equilibrium and multiple chaotic attractors. *Nonlinear Dynamics*, 81(1-2): (2015), 585–605.
[10] O. Rossler . An equation for hyperchaos. *Phys. Lett. A*, 71:(1979),155-157.
[11] G. R. Chen and X. Dong. From Chaos to Order: Methodologies, Perspectives and Applications.*World Scientific, Singapore,* 1998.
[12] Y. Li, W. K. S. Tang, and G. Chen. Generating hyperchaos via state feedback control. *International Journal of Bifurcation and Chaos in Applied Sciences and Engineering*, 15( 10): 2005, 3367-3375.
[13] A. Chen, J. Lu, J. Lu, and S. Yu. Generating hyperchaotic Lu attractor via state feedback control. *Physica A,* 364:( 2006), 103-110.
[14] S. Nikolov and S. Clodong. Occurrence of regular, chaotic and hyperchaotic behavior in a family of modified Rossler hyperchaotic systems. *Chaos, Solitons & Fractals,* 22(2):2004 ,407-431 .
[15] X. Wang and M. Wang. A hyperchaos generated from Lorenz systeM. *Physica A,* 387(14):2008 ,3751-3758.
[16] R. Barboza. Dynamics of a hyperchaotic Lorenz system. *International Journal of Bifurcation and Chaos,* 17(12):(2007), 4285-4294.
[17] L.M. Pecora, T.L. Carroll. Synchronization in chaotic systems. *Phys. Rev. Lett.,* 64: (1990), 821-24.
[18] H.N. Agiza. Chaos synchronization of L dynamical system. *Nonlinear Anal.,* 58 :(2004), 11-20.
[19] C.D. Li, X.F. Liao, R. Zhang. Impulsive synchronization of nonlinear coupled chaotic systems.*Phys. Lett. A,* 328: (2004), 47-50.
[20] Xiaohua Yuan, Chuandong Li, Tingwen Huang. Projective Lag Synchronization of Delayed Chaotic Systems with Parameter Mismatch via Intermittent Control.*International Journal of Nonlinear Science,* 23(1):(2017),3-10.
[21] L. Junge, U. Parlitz. Phase synchronization of coupled Ginzburg-Landau equations. *Phys. Rev. E.*, 62 :(2000), 320-24.
[22] J. Hu, S.H. Chen, L. Chen. Adaptive control for anti-synchronization of Chuas chaotic system.*Phys. Lett. A*, 339: (2005), 455-60.
[23] R. Femat, G. Solis-Perales. On the chaos synchronization phenomena. *Phys. Lett. A,* 262: (1999), 50-60.
[24] L. Kacarev, U. Parlitz. Generalized synchronization, predictability, and equivalence of unidirectionally coupled dynamical systems. *Phys. Rev. Lett.,* 76 :(1996), 1816-819.
[25] E.M. Shahverdiev, S. Sivaprakasam, K.A. Shore. Lag synchronization in time-delayed systems. *Phys. Lett. A,* 292: (2002), 320-24.
[26] Z.Y. Yan. Q-S (lag or anticipated) synchronization back stepping scheme in a class of continuous-time hyperchaotic systems: a symbolic numeric computation approach. *Chaos*, 15 :(2005), 023902.
[27] R. Mainieri, J. Rehacek. Projective synchronization in the three-dimensional chaotic systems. *Phys. Rev. Lett.,* 82: (1999),3042-45.

[28] D.L. Xu, C.Y. Chee. Controlling the ultimate state of projective synchronization in chaotic systems of arbitrary dimension. *Phys. Rev. E.*, 66: (2002), 046218.

[29] G.L. Wen, D.L. Xu. Observed-based control for full state projective synchronization of a general class of chaotic maps in any dimension. *Phys. Lett. A,* 333: (2004), 420-25.

[30] M.F. Hu, Z.Y. Xu. Adaptive feedback controller for projective synchronization. *Nonlinear Anal.: RWA,* 9 :(2008),1253-260.

[31] M.F. Hu , Z.Y. Xu, R. Zhang, A.H. Hu. Parameters identification and adaptive full state hybrid projective synchronization of chaotic (hyper-chaotic) systems. *Phys. Lett. A,* 361 :(2007), 231-37 .

[32] A. El-Gohary, I.A. Alwasel. The chaos and optimal control of cancer model with complete unknown parameters. *Chaos Soliton Fract.*, 42:(2009),2865-2874.

[33] Israr Ahmad, Azizan Bin Saaban, Adyda Ibrahim, Mohammad Shahzad. Global Chaos Synchronization of a Perturb Nonlinear Finance System by Active Control.*International Journal of Nonlinear Science*,22 (3):(2016), 167-178.

[34] Shikha Ayub Khan. Mixed Tracking and Projective Synchronization of 6D Hyperchaotic System Using Active Control. *International Journal of Nonlinear Science*, 22(1):(2016), 44-53.

[35] L. Yao and G. Chai. Chaos synchronization of a new hyperchaotic finance system via a novel chatter free sliding mode control strategy.*International Journal of Nonlinear Science,* 17(2):(2014):176-81.

[36] S. Effati, J. Saberi-Nadjafi, Saberi Nik. Optimal and adaptive control for a kind of 3D chaotic and 4D hyper-chaotic system. *Appl. Math. Modell*, 38:(2014), 759-774.

[37] C.H. Chen, L.J. Sheu, H.K. Chen, J.H. Chen, H.C. Wang, Y.C. Chao, Y.K. Lin. A new hyper-chaotic system and its synchronization. *Nonlinear Anal. RWA.*, 10:(2009), 2088-2096.

[38] H. Saberi Nik, M. Golchaman. Neural Computing and Applications. (2013), doi: 10.1007/s00521-013-1539-z.

[39] W.G. Yu. *Phys. Lett. A*, 374:(2010), 1488-1492.

[40] M. Roopaei, B.R. Sahraei, T.C. Lin. *Commun. Nonlinear Sci. Numer. Simulat.*, 15:(2010), 4158-4170.

[41] Y.G. Yu, S.C. Zhang. Adaptive back stepping synchronization of uncertain chaotic system. *Chaos Solitons Fractals,* 21 :(2004), 643-49.

[42] G. Chen, X. Dong. Form Chaos to Order: Perspectives. *Methodologies and Applications, World Scientific, Singapore*, (1998).

[43] Khan, Ayub, and Arti Tyagi. Analysis and hyper-chaos control of a new 4-D hyper-chaotic system by using optimal and adaptive control design. *International Journal of Dynamics and Control*, (2016), 1–9.

[44] W. Hahn. The Stability of Motion. *Springer-Verlag, Berlin,* (1967).