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Generalized $\exp(-\phi)$-Expansion Method for Camassa-Holm Equation with Variable Coefficients

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Abstract: In the present study, generalized $\exp(-\phi)$-expansion method is proposed for developing various new solutions of nonlinear partial differential equations. We have successfully furnished certain new solutions of Camassa-Holm equation with variable coefficients in terms of arbitrary functions along with various parameters, which provide further freedom to simulate the desired physical situations, by using the proposed method. These exact solutions include hyperbolic function solution, trigonometric function solution and rational solution. When the parameters are taken as special values, known solitary wave solutions are derived in a very systematic manner. The most accomplishment of this study lies in the fact that this method can be used to obtain exact non-traveling wave solutions of some high-dimensional nonlinear evolution equations.

Keywords: Generalized $\exp(-\phi)$-Expansion Method; Camassa-Holm Equation with Variable Coefficients; Exact Solutions.

1 Introduction

The nonlinear partial differential equation

$$u_t - u_{txx} + 3uu_x = 2u_x u_{xx} + uu_{xxx}, \tag{1}$$

was derived by Camassa and Holm [1] as a model for the propagation of shallow water waves, with $u(x,t)$ representing the water's free surface over a bed. The Camassa-Holm equation was actually obtained much earlier as an abstract bi-Hamiltonian partial differential equation with infinitely many conservation laws by Fokas and Fuchssteiner [2]. Nevertheles, Camassa and Holm put forward its derivation as a model for shallow water waves and discovered that it is formally integrable and that its solitary waves are solitons, features that prompted an ever increasing interest in the study of this equation [3–6]. In contrast to the Korteweg-de Vries equation, which is also an integrable model for shallow water waves, the Camassa-Holm equation possesses not only solutions that are global in time but models also wave breaking. Indeed, while some initial data develop into waves that exist indefinitely in time [7], others lead to wave breaking: the solution remains bounded but its slope becomes unbounded in finite time [8]. Moreover, wave breaking is the only way in which singularities can arise in a classical solution [9].

Advancement in science and mathematical modeling dealing with complex nonlinear give rise to induction of more and more partial differential equations with constant and variable coefficients. Calculating exact and numerical solutions of nonlinear equations in mathematical physics play an important role in soliton theory. In the past several decades, many effective methods for obtaining exact solutions of NLPDEs have been presented, such as inverse scattering method [10], Hirotas bilinear method [11], auxiliary equation methods [12, 13], First integral method [14], conservation laws [15], Exp-function method [16], symmetry reduction method [17] and so on.

However, to our knowledge, most of aforementioned methods are related to the constant-coefficient models. Recently, the study of variable-coefficient NPDEs has attracted much attention as variable-coefficient version of NLPDEs [18–20] can be considered as generalization of the constant coefficients equations as there are choices for parameters and explained

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the sensitivity and stability of nonlinear phenomenon in true sense.

The present paper is motivated by the desire to propose Generalized $\exp(-\phi)$-expansion method for variable coefficients nonlinear partial differential equations to obtain more and more exact solutions in a very uniform manner. We have considered Camassa-Holm equation with variable coefficients

$$u_t + \beta(t)u_{xxt} + \sigma(t)u_x + \alpha(t)uu_{xxx} + \theta(t)uu_x + \rho(t)u_x u_{xx} = 0, \tag{2}$$

for exploring some new exact solutions by using generalized $\exp(-\phi)$-expansion method.

The paper is structured as follows: In Section 2, the description of Generalized $\exp(-\phi)$-expansion method has been given. Application of Generalized $\exp(-\phi)$-expansion method to variable coefficient version of Eq. (2) and graphical behavior of the solutions are discussed in Section 3. Some conclusions are given in Section 4.

2 Description of Generalized $\exp(-\phi)$-Expansion Method

Consider NLPDEs with dependent variable u and independent variables $X = (x, y, z, ..., t)$ in the following form:

$$F(u, u_t, u_x, u_y, u_z, ...u_{xt}, u_{yt}, u_{zt}, u_{tt}, ...) = 0, \tag{3}$$

where $u = u(x, y, z, ...t)$ is an unknown function, F is a polynomial in $u = u(x, y, z, ...t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. Here, we have given following main steps of the generalized $\exp(-\phi)$-expansion method.

Step 1: The solution of Eq. (3) are expressed as follows:

$$u = a_0(X) + \sum_{i=1}^{m} a_i(X)(\exp(-\phi(\zeta)))^i, \;\; a_m(X) \neq 0, \tag{4}$$

where $a_0(X), a_i(X), (i = 1, 2, ..., m)$ and $\zeta = \zeta(X)$ are all functions of X, to be determined later and $\phi = \phi(\zeta)$ satisfies following equation

$$\phi'(\zeta) = \exp(-\phi(\zeta)) + \mu \exp(\phi(\zeta)) + \lambda, \tag{5}$$

where $\zeta = p(t)x + q(t)$, $p(t)$ and $q(t)$ are functions to be determined.

Step 2: We firstly find the value of integer m by balancing the highest order nonlinear term(s) and the highest order partial derivative of u in Eq. (1) so that u can be determined explicitly in the form of Eq. (5).

Step 3: Substitute (4) along with Eq. (5) into Eq. (3) and collect all terms with the same order of $\exp(-\phi(\zeta))$ and $\exp(\phi(\zeta))$ together, the left hand side of Eq. (3) is converted into a polynomial in $\exp(-\phi(\zeta))$ and $\exp(\phi(\zeta))$. Then by setting each coefficient of this polynomial to zero, we derive a set of over-determined partial differential equations for $a_0(X), a_i(X)$ and ζ.

Step 4: Solve the system of over-determined partial differential equations obtained in Step 3 for $a_0(X), a_i(X)$ and ζ.

Step 5: The general solutions of equation (5) can be written in the form

(i) When $\lambda^2 - 4\mu > 0$ and $\mu \neq 0$

$$\begin{aligned} \phi(\zeta) &= \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\zeta+k)\right)-\lambda}{2\mu}\right), \\ \phi(\zeta) &= \ln\left(\frac{-\sqrt{(\lambda^2-4\mu)}\coth\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(\zeta+k)\right)-\lambda}{2\mu}\right) \end{aligned} \tag{6}$$

(ii) When $\lambda^2 - 4\mu < 0$ and $\mu \neq 0$

$$\begin{aligned} \phi(\zeta) &= \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\zeta+k)\right)-\lambda}{2\mu}\right), \\ \phi(\zeta) &= \ln\left(\frac{\sqrt{(4\mu-\lambda^2)}\cot\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(\zeta+k)\right)-\lambda}{2\mu}\right) \end{aligned} \tag{7}$$

(iii) When $\lambda^2 - 4\mu = 0$ and $\mu \neq 0$, $\lambda \neq 0$

$$\phi(\zeta) = -\ln\left(\frac{-2\left(\lambda(\zeta+k)+2\right)}{\lambda^2(\zeta+k)}\right) \tag{8}$$

(iv) When $\lambda^2 - 4\mu = 0$ and $\mu = 0$, $\lambda = 0$

$$\phi(\zeta) = \ln\left(\zeta + k\right) \tag{9}$$

Step 6: Substituting the values of $a_0(X), a_i(X)$ and ζ and the general solution of Eq (5) into Eq. (4) we have many exact solutions of the nonlinear partial differential equation (3).

3 Application of Generalized $\exp(-\phi)$-Expansion Method to Camassa-Holm Equation with Variable Coefficients

In this section, we have utilized Generalized $\exp(-\phi)$-expansion method to derive certain new solutions of with variable coefficients (2). As described in section 2, firstly we have determined the positive integer m by considering the homogeneous balance between the highest order derivatives and nonlinear terms of u in Equation (2) and we obtained $m = 2$. Thus, the solution of Eq (2) according to Eq (5) is as follows:

$$u = a_0(t) + a_1(t)\exp(-\phi(\zeta)) + a_2(t)(\exp(-\phi(\zeta)))^2, \tag{10}$$

Substituting (10) into (2) and using (5), collecting all terms with the same order of $\exp(-\phi(\zeta))$ together, the left-hand side of Eq. (2) is converted into polynomial in $(\exp(-\phi(\zeta)))^j$, $(j = 0, 1, ...)$. We derived following system of overdetermined differential equations for $a_0(t), a_1(t), a_2(t), p(t)$ and $q(t)$ by setting each coefficient of this polynomial to zero. Solving this set of equations, the following results are furnished:

$$\begin{aligned} &p\left(t\right)=k_3,\ \ a_0\left(t\right)=k_2,\\ &a_1\left(t\right)=k_1\lambda,\ \ a_2\left(t\right)=k_1,\\ &\alpha\left(t\right)=-\tfrac{1}{2}\,\rho\left(t\right),\\ &\sigma\left(t\right)=\frac{\left(-8\left(\left(\frac{1}{8}\,\lambda^2+\mu\right)k_1-\frac{3}{2}\,k_2\right){k_3}^2\beta(t)-k_1\right)q'(t)-2\,\rho(t)\left(\left(\mu+\frac{1}{2}\,\lambda^2\right)k_1-3\,k_2\right)(k_1\mu-k_2){k_3}^3}{k_3k_1},\\ &\theta\left(t\right)=-\frac{\left(3\left(\frac{d}{dt}q(t)\right)\beta(t)+\rho(t)\left(\left(\frac{1}{8}\,\lambda^2+\mu\right)k_1-\frac{3}{2}\,k_2\right)k_3\right)4k_3}{k_1}, \end{aligned} \tag{11}$$

where k_i, $i = 1, 2, 3$ are arbitrary constants.
By substituting the general solutions of (5) into (10) and using (11), we arrived at following three types of solutions of (2):

When $\lambda^2 - 4\mu > 0$, we found hyperbolic function solution in the form

$$\begin{aligned} u_1(x,t) &= k_2 + k_1\lambda\left(\frac{2\mu}{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(k_3x+q(t)+k_4)\right)-\lambda}\right)\\ &\quad+k_1\left(\left(\frac{2\mu}{-\sqrt{(\lambda^2-4\mu)}\tanh\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(k_3x+q(t)+k_4)\right)-\lambda}\right)\right)^2,\\ u_2(x,t) &= k_2 + k_1\lambda\left(\frac{2\mu}{-\sqrt{(\lambda^2-4\mu)}\coth\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(k_3x+q(t)+k_4)\right)-\lambda}\right)\\ &\quad+k_1\left(\left(\frac{2\mu}{-\sqrt{(\lambda^2-4\mu)}\coth\left(\frac{\sqrt{(\lambda^2-4\mu)}}{2}(k_3x+q(t)+k_4)\right)-\lambda}\right)\right)^2 \end{aligned} \tag{12}$$

When $\lambda^2-4\mu<0$, we have trigonometric function solution

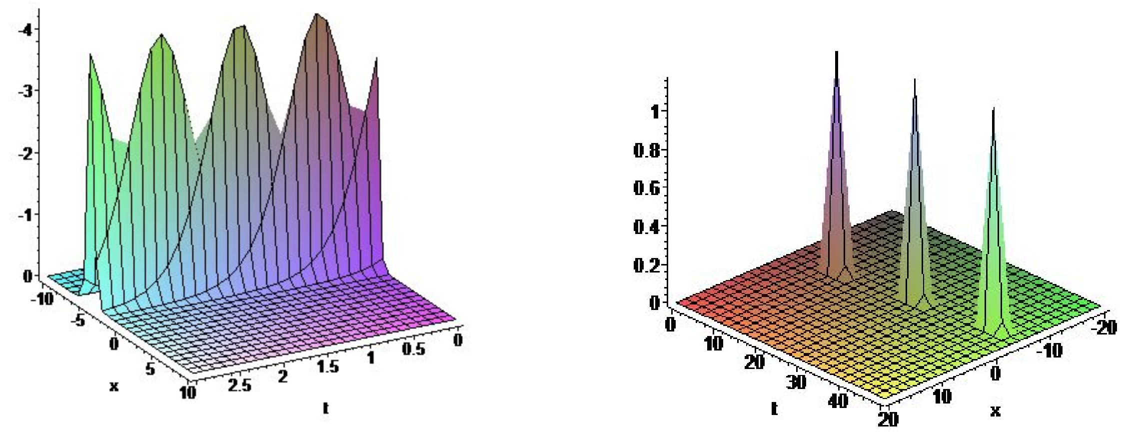
$$\begin{aligned}u_3(x,t)&=k_2+k_1\lambda\left(\frac{2\mu}{\sqrt{-(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(k_3x+q(t)+k_4)\right)-\lambda}\right)\\&+k_1\left(\left(\frac{2\mu}{\sqrt{-(4\mu-\lambda^2)}\tan\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(k_3x+q(t)+k_4)\right)-\lambda}\right)\right)^2,\\u_4(x,t)&=k_2+k_1\lambda\left(\frac{2\mu}{\sqrt{-(4\mu-\lambda^2)}\cot\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(k_3x+q(t)+k_4)\right)-\lambda}\right)\\&+k_1\left(\left(\frac{2\mu}{\sqrt{-(4\mu-\lambda^2)}\cot\left(\frac{\sqrt{(4\mu-\lambda^2)}}{2}(k_3x+q(t)+k_4)\right)-\lambda}\right)\right)^2\end{aligned}\tag{13}$$

When $\lambda^2-4\mu=0,\ \mu\neq0,\ \lambda\neq0$, we obtained rational solution as follows:

$$u_5(x,t)=k_2+k_1\lambda\left(-\frac{\lambda^2(k_3x+q(t)+k_4)}{2(\lambda((k_3x+q(t)))+2)}\right)+k_1\left(\frac{\lambda^2(k_3x+q(t)+k_4)}{2(\lambda((k_3x+q(t)))+2)}\right)^2.\tag{14}$$

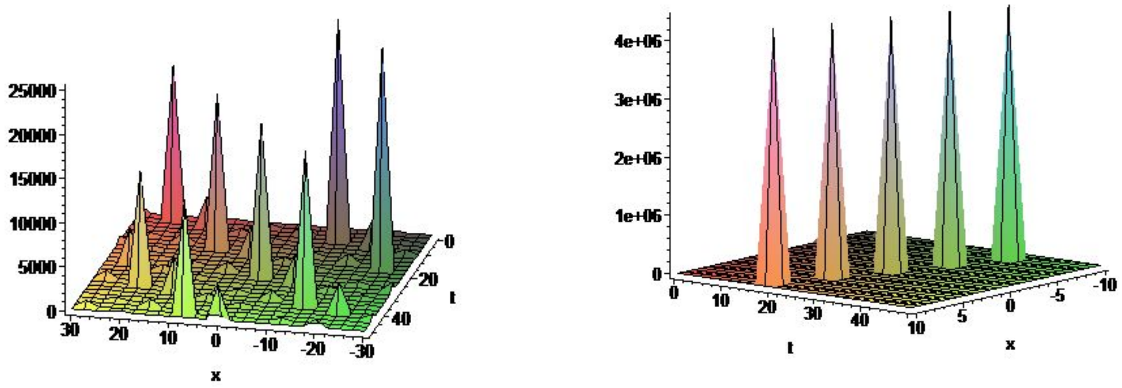
When $\lambda^2-4\mu=0,\ \mu\neq0,\ \lambda\neq0$, we got following solution:

$$u_6(x,t)=k_2+k_1\lambda\left(\frac{1}{(k_3x+q(t)+k_4)}\right)+k_1\left(\frac{1}{(k_3x+q(t)+k_4)}\right)^2,\tag{15}$$



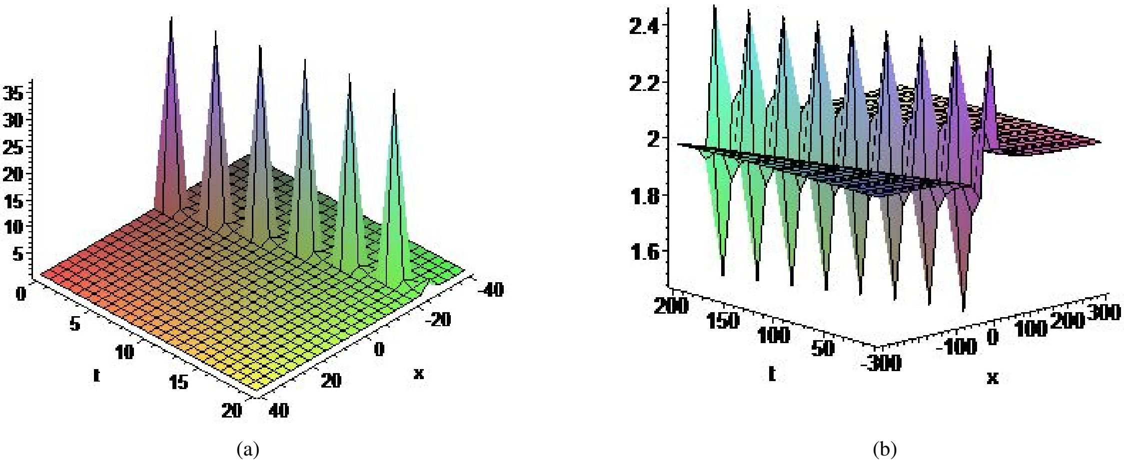
(a) (b)

Figure 1: The graphs of several kinds of travelling wave solutions of Eq. (2). (a) The peaked compacton (corresponding to $u_1(x,t)$): $\mu=2,\ \lambda=5,\ k_1=1,\ k_2=2,\ k_3=1,\ k_4=0.5,\ q(t)=t$. (b) The compacton (corresponding to $u_2(x,t)$): $\mu=2,\ \lambda=5,\ k_1=1,\ k_2=2,\ k_3=10,\ k_4=5,\ q(t)=t$.



(a) (b)

Figure 2: The graphs of several kinds of travelling wave solutions of Eq. (2). (a) The solitary wave (corresponding to $u_3(x,t)$): $\mu = 4,\ \lambda = 2,\ k_1 = 1,\ k_2 = 2,\ k_3 = 1,\ k_4 = 0.5,\ q(t) = t$. (b) The peaked solitary wave (corresponding to $u_4(x,t)$): $\mu = 4,\ \lambda = 2,\ k_1 = 1,\ k_2 = 2,\ k_3 = 1,\ k_4 = 0.5,\ q(t) = t$.



(a) (b)

Figure 3: The graphs of several kinds of travelling wave solutions of Eq. (2). (a) The kink-like wave (corresponding to $u_5(x,t)$): $\mu = 4,\ \lambda = 2,\ k_1 = 1,\ k_2 = 2,\ k_3 = 1,\ k_4 = 5,\ q(t) = t$ (b) The kink-like wave (corresponding to $u_6(x,t)$): $\mu = 4,\ \lambda = 2,\ k_1 = 1,\ k_2 = 2,\ k_3 = 1,\ k_4 = 5,\ q(t) = t$.

4 Concluding Remarks

Generalized $\exp(-\phi)$-expansion method has been proposed and then used to construct exact and explicit analytic solutions with arbitrary parameters to Camassa-Holm equation with variable coefficients. These solutions are expressed in terms of the hyperbolic functions, trigonometric functions and rational functions. In almost all the cases, the solutions obtained are such that one can choose the arbitrary function $q(t)$, along with various other arbitrary parameters, in a suitable manner, to attain physical situations with some desired features. Thus in some parameter conditions, many new exact parametric representations of the travelling waves such as peaked compacton, compacton, peaked solitary wave, solitary wave and kink-like wave etc. in explicit form and implicit form are obtained. It has been observed that the applied method is quite powerful and is practically well suited for exploring solutions of other NLPDEs, those arise in mathematical physics.

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