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Some Types of Solitary Wave Solutions in the Generalized Nonlinear Schrödinger Equation

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Abstract: Generalized Nonlinear Schrödinger equation is one of the most famous model equations in nonlinear science. Today we obtain some solitary wave solutions in the generalized nonlinear Schrödinger equation. In different region, we find that this equation admits different form of solutions, including such as peaked waves, looped and cusped waves.

Keywords: Generalized Nonlinear Schrödinger equation; Peaked waves; Looped waves; Cusped waves.

1 Introduction

In this paper, we study the generalized nonlinear Schrödinger equation

$$iu_t + (u|u|^{n-1})_{xx} + \mu u|u|^{m-1} = 0, \tag{1}$$

which is an important model and the generalized type can describe more complicate situations [1-17]. So it is meaningful to study the generalized equation. In Ref. [2], the solitary wave solutions are studied for the generalized equation (1) in the special condition, that is $\mu a < 0$.

Our method is an improvement of the work in [5]. The method in [5] can deal with the existence of solutions to a type of nonlinear PDEs. After integration, these equation has a term like $P(\varphi) = \varphi^2(b\varphi^2 + c\varphi + d)$. Solutions would be determined by analyzing the zeros of $P(\varphi)$. While in our work, the polynomial form is $P(\varphi) = \varphi^2(a\varphi^3 + b\varphi^2 + c\varphi + d)$ and the situation of zeros is more complicated. So the previous method in [20] does not work well. We solve the problem by checking the sign of some function values at selected points.

The purpose of our paper is to study the solitary wave solutions of equation (1) when $\mu a > 0$. Furthermore, we want to find the different form of the solitary waves for the generalized nonlinear Schrödinger equation.

2 Different solitary wave of equation (1)

By using the transformation $u(x,t) = \phi(x)e^{i\sigma t}$, equation (1) has the form

$$-\sigma\phi + \phi_{xx} + 2a\phi_x^2 + 2a\phi\phi_{xx} + \mu\phi^3 = 0. \tag{2}$$

Equation (2) also has the following form

$$\frac{d\phi}{dx} = y, \qquad \frac{dy}{dx} = \frac{\sigma\phi - 2ay^2 - \mu\phi^3}{1 + 2a\phi}. \tag{3}$$

After the first integral, we have

$$\phi_x^2 = F(\phi) = \frac{-\phi^2(24\mu a\phi^3 + 15\mu\phi^2 - 40a\sigma\phi - 30\sigma)}{30(1 + a\phi)^2}. \tag{4}$$

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Defining $P(\phi)$ and $G(\phi)$ by $P(\phi) = \phi^2 G(\phi)$, we have

$$G(\phi) = -(24\mu a\phi^3 + 15\mu\phi^2 - 40a\sigma\phi - 30\sigma),$$

and

$$G'(\phi) = -(72\mu a\phi^2 + 30\mu\phi - 40a\sigma).$$

Defining $\Delta = 225\mu^2 + 2880a^2\mu\sigma$ and

$$\phi_m = -\frac{15}{72a} + \frac{\sqrt{225\mu^2 + 2880a^2\mu\sigma}}{72\mu a}, \quad \phi_n = -\frac{15}{72a} - \frac{\sqrt{225\mu^2 + 2880a^2\mu\sigma}}{72\mu a},$$

where ϕ_m and ϕ_n are the roots of $G'(\phi) = 0$.

Lemma 1 *The qualitative behavior of solutions of $\phi_x^2 = F(\phi)$ near points where F has a zero or a pole is as follows.*

(1)If $F(\phi)$ has a simple zero at $\phi = m$, that is $F(\phi) = (\phi - m)G(\phi)$.
Then

$$\phi_x^2 = (\phi - m)F'(m) + O((\phi - m)^2) \quad as \quad \phi \to m. \tag{5}$$

Hence

$$\phi(x) \sim m + \lambda((x - x_0)^2) \quad as \quad x \to x_0,$$

where λ is some constant.

Moreover, periodic smooth solutions exist if $F(\phi)$has two simple zeros z_1, z_2 and $F(\phi) > 0$ for $z_1 < \phi < z_2$, $z_1 = \min_{x\in R} \phi(x), z_2 = \max_{x\in R} \phi(x)$ and $-\frac{1}{a} \notin (z_1, z_2)$.

(2) If $F(\phi)$ has a double zero at $\phi = m$, that is $F(\phi) = (\phi - m)^2 G(\phi)$.
Then

$$\phi_x^2 = (\phi - m)^2 F''(m) + O((\phi - m)^3) \quad as \quad \phi \to m. \tag{6}$$

Hence

$$\phi(x) \sim m + \eta \exp(-|x|) \quad as \quad x \to \infty \quad forsomeconstant \quad \eta.$$

Moreover, smooth solitary wave solutions exist if $F(\phi)$ has a simple zero z_1, a double zero z_2 and $F(\phi) > 0$ for $z_1 < \phi < z_2$, $z_1 = \min_{x\in R} \phi(x), z_2 = \max_{x\in R} \phi(x)$ and $-\frac{1}{a} \notin (z_1, z_2)$, where $\phi \to z_2$ exponentially as $x \to \pm\infty$.

Similar to the analysis in Ref[2], we have the following results.

(1) $\Delta < 0$.

(i) If $G(0) < 0$, we can obtain that the value of ϕ_1 is less than zero (see Fig.1.a(i)), there exists a cusped periodic wave solution pointing downwards.

(ii) If $G(0) > 0$, we can observe that the value of ϕ_1 is greater than zero (see Fig.1.a(ii)), there exists a cusped wave solution pointing downwards, a solitary wave solution, a looped wave and a cusped wave solution.

(2) $\Delta = 0$.

In this case, we deduce $\phi_m = \phi_n$, and we will consider two subcases.

Case 1. $G(\phi_n) \neq 0$.

(i) If $G(0) < 0$, we can obtain the value of ϕ_1 is less than zero (see Fig.2.a(i)), there exists a cusped periodic wave solution pointing downwards.

(ii) If $G(0) > 0$, we can observe that the value of ϕ_1 is greater than zero (see Fig.2.a(ii)), there exists a cusped wave solution pointing downwards, a solitary wave solution, a looped wave and a cusped wave solution.

Case 2. $G(\phi_n) = 0$.

(i) If $G(0) < 0$, we can obtain $a > 0$ and the value of ϕ_n is less than zero (see Fig.2.b(i)), there exists a cusped periodic wave solution pointing downwards.

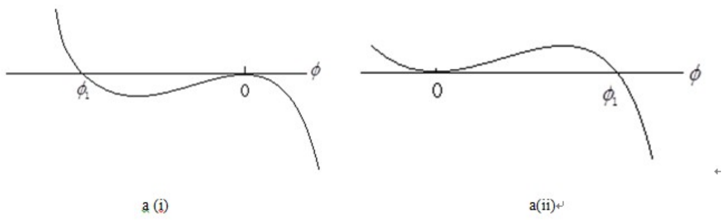

Figure 1: The graph of $P(\varphi)$ as $\Delta < 0$

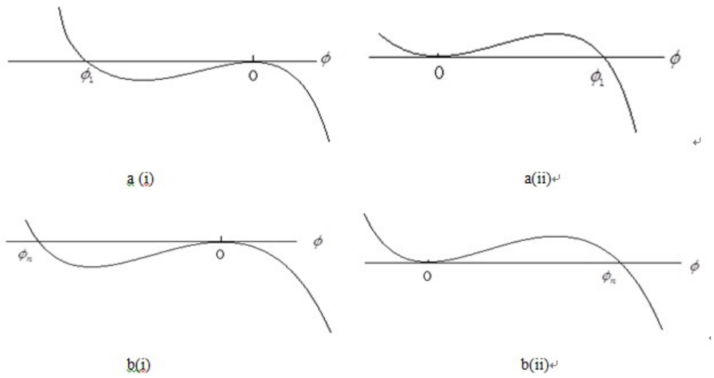

Figure 2: The graph of $P(\varphi)$ as$\Delta = 0$

(ii) If $G(0) > 0$, we get $a < 0$ and the value of ϕ_n is greater than zero (see Fig.2.b(ii)), there exists a looped wave solution, a peaked wave and a solitary wave solution.

(3) $\Delta > 0$.

Case 1. $G(\phi_m) > 0$ and $G(\phi_n) > 0$.

(i) If $G(0) < 0$, we can deduce that $a > 0$ for $\phi_m < 0$ and the value of ϕ_1 is less than zero (see Fig.3.a(i)), there exists a cusped periodic wave solution pointing downwards.

(ii) If $G(0) > 0$, we can deduce that the value of ϕ_1 is greater than zero (see Fig.3.a(ii)), there exists a cusped wave solution, a solitary wave and a looped wave solution.

Case 2. $G(\phi_m) > 0$ and $G(\phi_n) = 0$.

(i) If $G(0) < 0$, we can get $a > 0$ for ϕ_m and ϕ_1 (see Fig.3.b(i)), there exists a cusped wave solution pointing downwards, a solitary wave solutiona peaked periodic wave solution pointing down wardsa looped wave and a cusped wave solution.

(ii) If $\phi_n < 0$ and $G(0) > 0$, we can get ϕ_1 (see Fig.3.b(ii)), there exists a cusped wave solution pointing downwards, a double kink-like wave solution, a peaked wave solution pointing downwards, a butterfly-like wave solution, a solitary wave solution, a looped wave and a cusped wave solution. ϕ_2, ϕ_3, ϕ_m.

(iii) If $\phi_n < 0$, we can get $0 < \phi < \phi_1$ and $a < 0$ (see Fig.3.b(iii)), there exists a butterfly-like wave solution, a solitary wave solution, a peaked wave solution, a peaked periodic wave solution pointing downwards, a double kink-like wave solution, a looped wave and a cusped wave solution.

Case 3. $G(\phi_m) > 0$ and $G(\phi_n) < 0$.

(i) If $G(0) < 0$ and $\phi_m < 0$,it is easy to find that all the three simple zeros are less than zero and $a > 0$ (see Fig.3.c(i)), there exists a cusped periodic wave solution pointing downwards, a periodic wave solution, a looped periodic wave and a cusped periodic wave solution.

(ii) If $\phi_n < 0$ and $G(0) > 0$, it is easy to find that only one of the three simple zeros is on the right of the point ϕ and others left (see Fig.3.c(ii)), there exists a cusped periodic wave solution pointing downwards, a solitary wave solution pointing downwards, a cusped wave solution pointing downwards, a rotated looped wave solution, a solitary wave solution, a

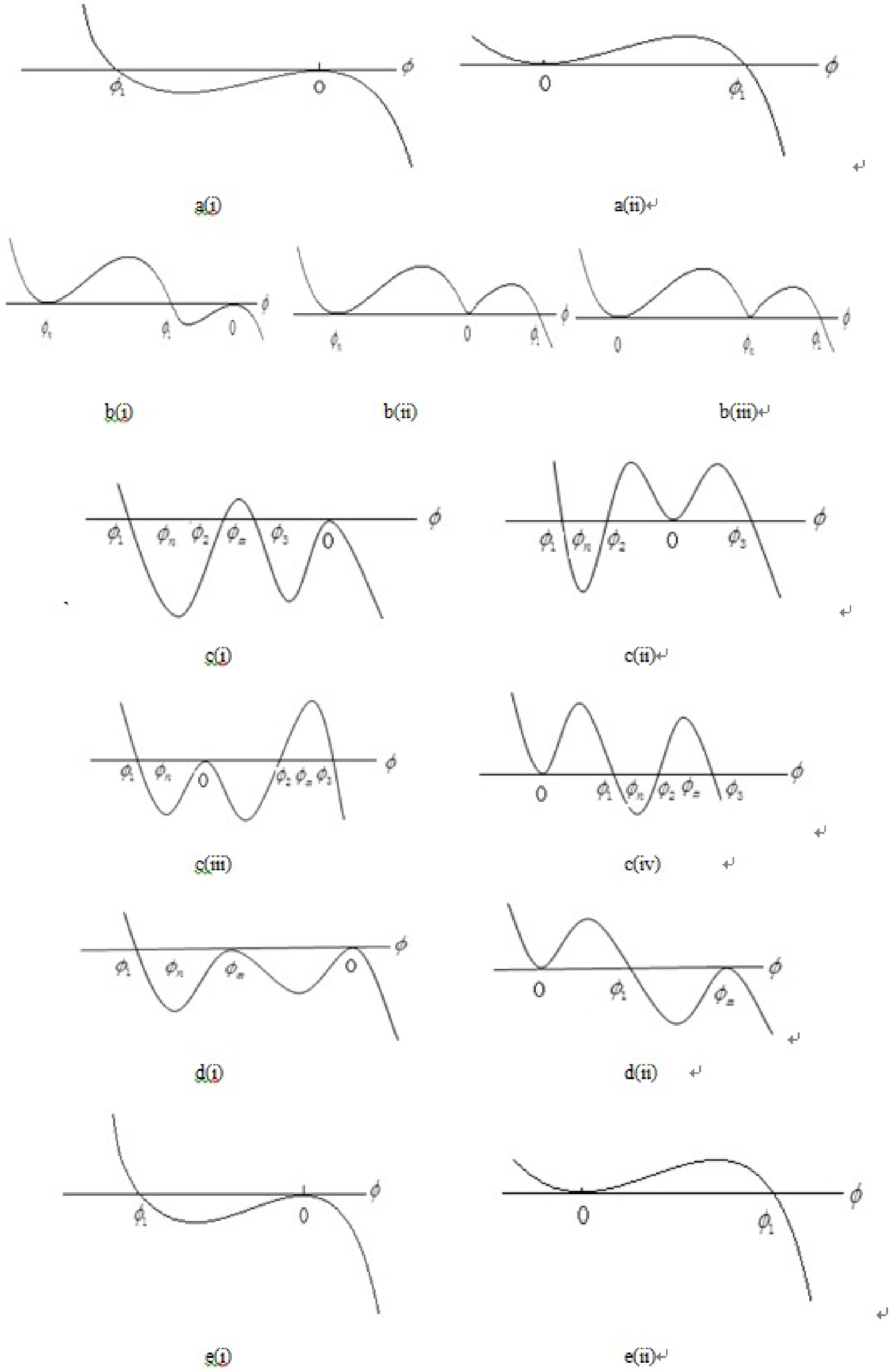

Figure 3: The graph of $P(\varphi)$ as $\Delta > 0$.

looped wave and a cusped wave solution.

(iii) If $\phi_m > 0$ and $G(0) < 0$, it is easy to find that two simple zero on the right of the point $\phi = 0$ and others left (see Fig.3.c(iii)), there exists a cusped periodic wave solution pointing downwards, a periodic wave solution, a cusped periodic wave and a looped periodic wave solution.

(iv) If $\phi_n > 0$ and $G(0) > 0$, it is easy to find that all the three simple zeros are on the right of the point zero and $a < 0$ (see Fig.3.c(iv)), there exists a looped wave solution, a cusped wave solution, a solitary wave solution, a periodic wave solution, a cusped periodic wave solution pointing downwards, a looped periodic wave and a cusped periodic wave solution.

Case 4. $G(\phi_m) = 0$ and $G(\phi_n) < 0$

(i) If $G(0) < 0$, we can get $a > 0$ and $\phi_1 < \phi_m < 0$ (see Fig.3.d(i)), there exists a cusped periodic wave solution pointing downwards.

(ii) If $G(0) > 0$, we can get $\phi_1 < 0 < \phi_n$ (see Fig.3.d(ii)), there exists a looped wave solution, a cusped wave and a solitary wave solution.

Case 5. $G(\phi_m) < 0$ and $G(\phi_n) < 0$.

(i) If $G(0) < 0$, it is easy to find that $\phi_1 < 0$ (Fig.3.e(i)), there exists a cusped periodic wave solution pointing downwards.

(ii) If $G(0) > 0$, it is easy to find that $a < 0$ for $\phi_n > 0$, and ϕ_1 (see Fig.3.e(ii)), there exists a looped wave solution, a cusped wave and a solitary wave solution.

3 Conclusions

By an improved method, we obtain some solitary wave solutions in the generalized nonlinear Schrödinger equation. Those solutions include looped wave solutions, cusped wave solutions, peaked wave solutions, double kinked waves and butterfly-like waves. Based on the study, it might be concluded that the improved method is useful and efficient. It can be widely applied to other nonlinear wave equations.

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References

[1] K. Nakkeeran, P.K.A. Wai, Generalized projection operator method to derive the pulse parameters equations for the nonlinear Schrödinger equation, *Optics Communications*, 244 (2005): 377-382.

[2] J. Yin et al., Butterfly-like waves in the nonlinear Schrödinger equation with a combined dispersion term, *Nonlinear Analysis: RWA*, 14 (2013): 1276-1285.

[3] F. Ndzana, A. Mohamadou, T. C. Kofane, Modulational instability in the cubic-quintic nonlinear Schrödinger equation through the variational approach, *Optics Communications,* 275 (2007): 421-428.

[4] S. M. Hoeini, T. R. Marchant, Evolution of solitary waves for a perturbed nonlinear Schrödinger equation, *Applied Mathematics and Computation*, 216 (2010): 3642-3651.

[5] J. Lenells, Travelling wave solutions of the Camassa-Holm equation, *J. Differential Equations*, 217 (2005): 393-430.

[6] A. de Bouard, A. Debussche, The nonlinear Schröinger equation with white noise dispersion, *J. Functional Analysis*, 259 (2010): 1300-1321.

[7] Z. Yan, Envelope compactons and solitary patterns, *Phys. Lett. A*, 335 (2006): 212-215.

[8] J. Yin, L. Tian, X. Fan, Classification of optical wave solutions to the nonlinearly dispersive Schröinger equation, *Commun. Nonlinear Sci. Numer. Simulat.*, 17 (2012) 1224-1232

[9] P. D. Green, A. Biswas, Bright and dark optical solitons with time-dependent coefficients in a non-Kerr law media, *Commun. Nonlinear Sci. Numer. Simulat.*, 15 (2010): 3865-3873.

[10] E. Topkara, D. Milovic, A. K. Sarma, E. Zerrad, A. Biswas, Optical solitons with non-Kerr law nonlinearity and inter-modal dispersion with time-dependent coefficients, *Commun. Nonlinear Sci. Numer. Simulat.*, 15 (2010): 2320-2330.

[11] J.M.S. Benjamin, A.L. Dawn and A. Biswas, Dynamics of topological optical solitons with time-dependent dispersion, nonlinearity and attenuation, *Commun. Nonlinear Sci. Numer. Simulat.*, 14 (2009): 3305-3308.

[12] R. Kohl, A. Biswas, D. Milovic and E. Zerrad, Optical soliton perturbation in a non-Kerr law media, *Optics Laser Technology*, 40 (2008): 647-662.

[13] C.M. Khalique, A. Biswas, A Lie symmetry approach to nonlinear Schr?inger's equation with non-Kerr law nonlinearity,*Commun. Nonlinear Sci. Numer. Simulat.*, 14 (2009): 4033-4040.

[14] P. Green, D. Milovic, A.K. Sarma, D. Lott and A. Biswas, Dynamics of super-sech solitons in optical fibers, *J. Nonlinear Optical Physics and Materials*, 19 (2010): 339-370.

[15] A. Biswas, D. Milovic and E. Zerrad, Optical soliton perturbation with log law nonlinearity by He's semi-inverse variational principle, *Optics and Photonics Letters*, 3 (2010): 1-5.

[16] E. Topkara, D. Milovic, A.K. Sarma, E. Zerrad and A. Biswas, Optical soliton perturbation with full nonlinearity in non-Kerr law media, *J. Optical and Fiber Communications Research*, 7(1-4) 2010: 43-59.

[17] A. Biswas, J. E. Watson, C. Cleary and D. Milovic, Optical Solitons with Higher Order Dispersion in a Log Law Media, *J. Infrared, Millimeter and Terahertz Waves*, 31 (2010): 1057-1062.