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Global Chaos Synchronization of a Perturb Nonlinear Finance System by Active Control

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Abstract: This paper addresses the chaos synchronization problems between two identical finance chaotic systems and two non-identical finance and Wang chaotic system. It is assumed that the chaotic systems are perturbed by the model uncertainties and external disturbances. In this study, using the active control technique, the linear controller gains are estimated based on the Lyapunov direct method that guarantee the globally asymptotically stable synchronization in both cases. It has been shown that the state trajectories are synchronized with a good synchronization quality and rates in the presence of model uncertainties and external disturbances. All simulation results are carried out to validate the effectiveness of the proposed control strategy and possible feasibility in synchronizing two identical as well as two nonidentical chaotic systems by using Mathematica 10. The effect of total disturbances is under our discussions.

Keywords: Lyapunov stability theory; active control; finance chaotic system; Wang 3-D chaotic system

1 Introduction

Chaos synchronization is one of the most fascinating phenomenon related to complex nonlinear dynamical systems. Mathematically, a chaotic system is a nonlinear deterministic system that displays an unpredictable and complex behavior [1]. Due to these features, chaos synchronization has received much attention [2, 3] especially, after the pioneering work discussed in [4]. Chaos synchronization has been widely investigated in different scientific fields. These include financial systems [3], biological systems [5], laser physics [6], physical systems [7], secure communications [8], and other engineering systems [9] (to name but a few). In this light, a wide range of control techniques and methods have been proposed for carrying out chaos synchronization such as sliding mode control [10], nonlinear active control [11], projective synchronization [12], adaptive control [13], active control [14] and linear error state feedback control [15] (are worth citing here among others). Among the above worth mentioning techniques, active control techniques have attracted the attention of many researchers due to its numerical as well as practical implementation [16–18]. The active control techniques have been widely accepted as an efficient control techniques that synchronize chaotic systems.

Recently, there have been many work reported in the literature for the synchronization of chaotic systems using the active control techniques. Ho and Hung [18] investigated the generalized synchronization between the Lorenz chaotic system and the Rossler chaotic system under the effect of external noise. Vincent [19] investigated the synchronization of two identical and two non-identical 4-D chaotic systems using the generalized active control startegy based on the Routh-Hurwitz criterion [20] and the Lyapunov stability theory [21]. In 2009, Ablay and Aldemir [22] presented the exponential synchronization problem between two different chaotic systems using the generalized active control. The exponential stability criterion is investigated by the fact that the eigenvalues of the matrix associated with closed-loop system are in the left half of the complex plane. Hammami et al. [23] addressed the synhcronization problem between two identical 4-D Lorenz Stenflo chaotic systems and two nonidentical Lorenz Stenflo and Qi 4-D chaotic systems. Idowu and Vincent [24] achieved identical synchronization and chaos stablization in a Quasi-1D bose-Einstein Condensate. Very recently,

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Ahmad et al. [25] reported the synchronization control of two identical and two non-identical chaotic systems using the active control approach based on the Lyapunov direct method.

From the literature review, it is noticed that, most of the active control techniques are focused on synchronizing two chaotic systems (either identical or non-identical) and the synchronization speed can be modified by selecting high control gains and excessive energy is required to achieve chaos synchronization. However, by selecting high controller gains may increase the synchronization transient speed but in practice, this may create large vibrations within the system that can excite the system at the start of the controller. Resulting in the breaking of synchronization behavior completely. In addition, one cannot disregard the effect of both model uncertainties and environmental disturbances and detach the original information in the receiver. Therefore, it is important and also interesting to design suitable linear controller gains that minimizes the controller complexity, applicable to a variety of chaotic systems and robust against the total disturbances.

The globally asymptotic synchronization in the presence of total disturbance with the determination of proper linear control gains is an interesting subject and limited research work is devoted in this direction. In this paper, based on the Lyapunov direct method and using the active control technique, a simple and efficient approach will be proposed to estimate suitable linear controller gains to achieve globally asymptotic stable synchronization in the presence of total disturbances. Motivated by the above discussions, there are three main objectives of this research study that can be summarized by the authors as follows:

1. To employ the active control technique to study and investigate the identical and nonidentical synchronization problems of finance chaotic system [26].

2. Based on the Lyapunov direct method, suitable linear control gains will be estimated that would guarantee the globally asymptotic stability of the closed- loop systems in the presence of total disturbances.

3. Numerical simulations and graphs will be imparted to show the performance and efficiency of the proposed approach.

The rest of the paper is organized as follows. In section 2, we formulate the synchronization control theory and give a brief description of the system under consideration. In section 3, the problem of synchronization between two identical finance chaotic systems is solved. In section 4, synchronization between two non-identical finance and Wang chaotic systems is given. In section 5, numerical simulations are carried out to show the efficiency of the proposed approach and support the analytical results. Finally the concluding remarks are given in section 6.

2 Synchronization theory using the active control

2.1 Problem statement

Let us consider an autonomous chaotic system that is described as follows:

$$\dot{X}(t) = X(t)A + f\left(X\left(t\right)\right), \tag{1}$$

where $X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^n$ is the corresponding state vector, $f \in C^1\left(R^n, R^n\right)$ is nonlinear uniformly continuous function, and $A \in R^{n\times 1}$ is a matrix whose coefficients are the parameters of the chaotic system in (1).

A certain chaotic system is called the drive or master system and the second system is called response or slave system. Most of the synchronization techniques belong to the master-slave (drive-response) system arrangement in which the slave system is forced to track the master system under some coupling force and the two systems show similar behavior for all future states. Let us consider the following master-slave system synchronization scheme that is described by:

$$\begin{cases} \text{Master System: } \dot{X}(t) = X(t)A_1 + f\left(X(t)\right) + d(t), \\ \text{Slave System: } \dot{Y}(t) = Y(t)A_2 + g\left(Y(t)\right) + D(t) + \eta(t), \end{cases} \tag{2}$$

where $X(t) = [x_1(t), x_2(t), \ldots, x_n(t)]^T \in R^n$ and $Y(t) = [y_1(t), y_2(t), \ldots, y_n(t)]^T \in R^n$ are the corresponding state vectors, $f, g \in C^1\left(R^n, R^n\right)$ are the nonlinear uniformly continuous functions, $A_1, A_2 \in R^{n\times 1}$ are the $(n \times n)$ constant coefficients matrices which contains the parameters of the master and slave systems alternatively. $D(t) \in R^{n\times 1}$ and

$d(t) \in R^{n\times 1}$ are the model uncertainties and external disturbances present in the master and slave systems respectively, where $\eta(t) \in R^{n\times 1}$ is the control input vector that force the slave system to track the master system and can eliminate the effect of total disturbances. If $f(\cdot) \neq g(\cdot)$ and/or $A_1 \neq A_2$, then $X(t)$ and $Y(t)$ are the states of two nonidentical chaotic systems.

Definition 1 *The synchronization error dynamics as the difference between the master and slave systems* (2) *that is given as under:*

$$\|e_i(t)\| = \|y_i(t) - x_i(t)\|_2\,,\ \forall e_i(t)\ \in R^n, \tag{3}$$

where $\|\cdot\|$ *represents the Euclidean form.*

The main theme of this sub-section is to focus on designing the active control technique as a way of studying and investigating the global chaos synchronization problem for two coupled chaotic systems (2) in the presence of total disturbances. From synchronization scheme (2), it follows that:

$$\begin{aligned}\dot{\mathbf{e}}(t) &= Y(t)A_2 + g\left(Y(t)\right) - X(t)A_1 - f\left(X(t)\right) + D(t) - d(t) + \eta(t)\\ &= Y(t)A_2 - X(t)A_1 + g\left(Y(t)\right) - f\left(X(t)\right) + D(t) - d(t) + \eta(t)\\ &= A_3\mathbf{e}(t) + H\left(X(t), Y(t), \mathbf{e}(t)\right) + D(t) - d(t) + \eta(t),\end{aligned} \tag{4}$$

where $H\left(X(t), Y(t), \mathbf{e}(t)\right) = g\left(Y(t)\right) - f\left(X(t)\right)$, which contains the nonlinear functions and un-common parts, $A_3 = \bar{A}_2 - \bar{A}_1$ is the $(n \times n)$ coefficient matrix of the error system (4). In these situations, the main goal is to design a feedback control strategy such that the states of the two systems (2) are synchronized in a sense that the limit of the error vector $\mathbf{e}(t)$ may tend to zero globally asymptotically.

Corollary 1 *The two coupled chaotic systems* (2) *are said to be globally asymptotically synchronized in a sense of*

$$\lim_{t\to\infty} \|e_i(t)\| = \lim_{t\to\infty} \|y_i(t) - x_i(t)\|_2\,,\ \forall\ e_i(t) \in R^n. \tag{5}$$

2.2 Controller design

The feedback controller will be constructed in two parts. The first part will eliminate the un-common terms and the terms which are not expressed in the form of $\mathbf{e}(t)$ and the second part will regulate the strength of the feedback controller to achieve globally asymptotically stable synchronization.

Theorem 2 *For the arbitrary initial conditions* $(x_1(0), x_2(0), \ldots, x_n(0)) \in R^n \neq (y_1(0), y_2(0), \ldots, y_n(0)) \in R^n$, *the master and slave system* (2) *are globally asymptotically synchronized with the following suitable feedback control function:*

$$\eta(t) = -H\left(X(t), Y(t), \mathbf{e}(t)\right) + D(t) - d(t) + \mathbf{v}(t),\ i = 1, 2, \ldots, n. \tag{6}$$

Definition 2 *The matrix* $\mathbf{v}(t) = [v_i(t)]^T$ *is defined as under*

$$\mathbf{v}(t) = [v_i(t)]^T = -K\left[e_i(t)\right]^T \text{ and } K = diag\left[k_i\right],\ i = 1, 2, \ldots, n, \tag{7}$$

where k_i *is the linear controller gain.*

Proof of Theorem 1. Using the systems of equations in (4) and (6), we have

$$\dot{\mathbf{e}}(t) = A_3\mathbf{e}(t) + \mathbf{v}(t) = A_3\mathbf{e}(t) - K\mathbf{e}(t) = (A_3 - K)\,\mathbf{e}(t) \tag{8}$$

The synchronization problem can now be addressed as the stabilization of the closed-loop system (8) at the origin. Let us construct a quadratic Lyapunov error function candidate as follows:

$$V\left(\mathbf{e}(t)\right) = \mathbf{e}^T(t)P\mathbf{e}(t) \geqslant 0, \tag{9}$$

where $P = \text{diag}\,(p_1, p_2, \ldots, p_n) \in R^n$ is a positive definite matrix (PDM) and $V\,(\mathbf{e}(t)) : R^n \to R^n$ is a positive definite construction of (6). The time derivative of (9) is given as follows:

$$\begin{aligned}\dot{V}\,(\mathbf{e}(t)) &= \dot{\mathbf{e}}^T(t)P\mathbf{e}(t) + \mathbf{e}^T(t)P\dot{\mathbf{e}}(t) \\ &= -\left((K - A_3)^T P\mathbf{e}(t) + \mathbf{e}^T(t)P\,(K - A_3)\right). \end{aligned} \tag{10}$$

At this stage, the problem is reduced to show that if the gain matrix K and a PDM P are selected such that $\dot{V}\,(\mathbf{e}(t)) \leqslant 0$, then, the selected closed-loop system (8) is globally asymptotically stable. Thus, the two coupled chaotic systems in (2) are globally asymptotically stable synchronized. ■

2.3 Model description

Recently, Cai and Huang [26] studied and investigated a 3-D autonomous nonlinear finance system. The authors [26] briefly discussed the equilibrium points by studying the subsystem of finance chaotic system and introducing a new practical method to distinguish the chaotic, periodic and quasi-periodic orbits. The new system can produce a typical double-wing chaotic attractor. The differential equations that describe the chaotic finance system are given below:

$$\left.\begin{aligned}\dot{x}(t) &= \left(\frac{1}{b} - a\right)x(t) + z(t) + x(t)y(t), \\ \dot{y}(t) &= -by(t) - x^2(t), \\ \dot{z}(t) &= -x(t) - cz(t), \end{aligned}\right\} \tag{11}$$

where $(x(t), y(t), z(t)) \in R^3$ are the are the state variables and a, b and c are the positive parameters of the system (11). The variable $x(t)$ denotes the interest rate in the model, $y(t)$ represents the investment demand, and $z(t)$ is the price exponent. The parameters a, b and c represents the saving, the per-investment cost, and the elasticity of demands of commercials respectively. By linearizing system (11) at the equilibrium point $E_0 = (0, 0, 0)$, the Jacobean matrix is given as follows:

$$J_{(0,0,0)} = \begin{bmatrix} \frac{1}{b} - a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{bmatrix} \tag{12}$$

The characteristic equation of $J_{(0,0,0)}$ is given as below:

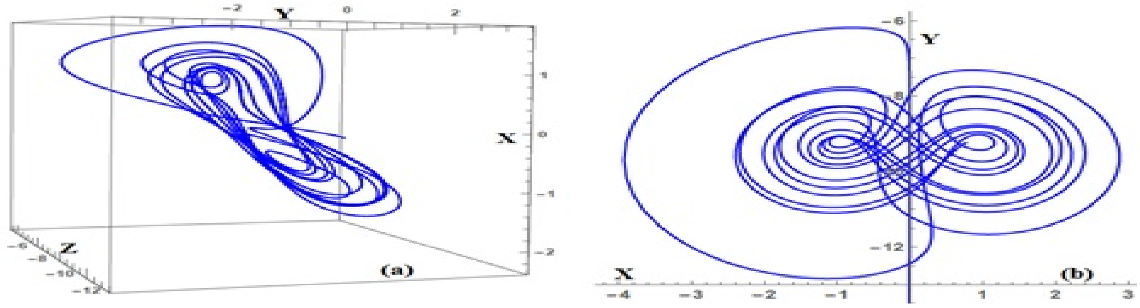
$$\lambda^2 + \left(b + c - a - \frac{1}{b}\right)\lambda^2 + (1 + c\,(b + c))\,\lambda + b - c(1 + ab) = 0. \tag{13}$$

For the parameters values $a = 0.00001$, $b = 0.1$ and $c = 1$, the three eigenvalues are $\lambda_1 = 9.90832$, $\lambda_2 = -0.908327$ and $\lambda_3 = -0.1$. In continuous nonlinear dynamical systems, the condition for the global stability is that all the eigenvalues and Lyapunov exponents must be negative. We can see that the two eigenvalues are negative and one eigenvalue is positive $(\lambda_2, \lambda_3 < 0$ and $\lambda_1 > 0)$. The Lyapunov exponents are $L_1 = 0.1735$, $L_2 = 0.0015$ and $L_3 = -0.8030$ respectively [26], which confirms that system (11) is globally unstable focus at the equilibrium point $E_0 = (0, 0, 0)$. Thus, the finance system [26] exhibits a chaotic attractor for parameters values $a = 0.00001$, $b = 0.1$ and $c = 1$, as shown in Figure 1. For the dynamical properties such as the maximum Lyapunov exponents, the spectrum analysis, the equilibrium points, the bifurcation diagram, the phase portraits etc. for system (11), one may refer to study [26].

3 Synchronization between two identical nonlinear finance systems

3.1 Problem statement

In this subsection, the main objective is to design such a feedback control function that synchronize two identical finance chaotic systems in the presence of model uncertainties and external disturbances. To achieve our objectives, the master-slave system complete synchronization scheme for chaotic systems (11) is described as under:



(a) 3D phase portrait of finance chaotic system (b) Chaotic attractor in (x, y) space

Figure 1:

$$\text{Master System: } \begin{cases} \dot{x_1}(t) = \left(\frac{1}{b} - a\right) x_1(t) + z_1(t) + x_1(t)y_1(t) + d_1(t), \\ \dot{y_1}(t) = -by_1(t) - x_1^2(t) + d_2(t), \\ \dot{z_1}(t) = -x_1(t) - cz_1(t) + d_3(t); \end{cases}$$
$$\text{Slave System: } \begin{cases} \dot{x_2}(t) = \left(\frac{1}{b} - a\right) x_2(t) + z_2(t) + x_2(t)y_2(t) + D_1(t) + \eta_1(t), \\ \dot{y_2}(t) = -by_2(t) - x_2^2(t) + D_2(t) + \eta_2(t), \\ \dot{z_2}(t) = -x_2(t) - cz_2(t) + D_3(t) + \eta_3(t); \end{cases} \tag{14}$$

where $[x_1(t), y_1(t), z_1(t)] \in R^3$ and $[x_2(t), y_2(t), z_2(t)] \in R^3$ are the state vectors, a, b and c are the parameters of the master and slave systems. Also, $d_i(t)$ and $D_i(t)$, $i = 1, 2, 3$ are the model uncertainties and external disturbances present in the master and slave systems respectively, where $\eta(t) = [\eta_1(t), \eta_2(t), \eta_3(t)] \in R^3$ is the control input injected to the slave system.

Definition 3 *The error dynamics for the synchronization scheme* (14) *can be described as follows:*

$$\left.\begin{aligned} \dot{e_1}(t) &= \left(\frac{1}{b} - a\right) e_1(t) + e_3(t) + x_2(t)y_2(t) - x_1(t)y_1(t) + D_1(t) - d_1(t) + \eta_1(t), \\ \dot{e_2}(t) &= -be_2(t) + x_1^2(t) - x_2^2(t) + D_2(t) - d_2(t) + \eta_2(t), \\ \dot{e_3}(t) &= -e_1(t) - ce_3(t) + D_3(t) - d_3(t) + \eta_3(t). \end{aligned}\right\} \tag{15}$$

Theorem 3 *For the arbitrary initial conditions* $(x_1(0), y_1(0), z_1(0)) \in R^3 \neq (x_2(0), y_2(0), z_2(0)) \in R^3$ *of the master and slave systems respectively, the two coupled chaotic systems* (14) *are globally asymptotically stable synchronized with the following suitable feedback control functions:*

$$\left.\begin{aligned} \dot{\eta_1}(t) &= x_1(t)y_1(t) - x_2(t)y_2(t) + d_1(t) - D_1(t) + v_1(t), \\ \dot{\eta_2}(t) &= x_2^2(t) - x_1^2(t) + d_2(t) - D_2(t) + v_2(t), \\ \dot{\eta_3}(t) &= d_3(t) - D_3(t) + v_3(t). \end{aligned}\right\} \tag{16}$$

Proof of Theorem 2. It is assumed that the parameters of the master and slave systems (14) are available and measurable. Substituting system of equations (16) in (15) yields:

$$\left.\begin{aligned} \dot{e_1}(t) &= \left(\frac{1}{b} - a\right) e_1(t) + e_3(t) + v_1(t), \\ \dot{e_2}(t) &= -be_2(t) + v_2(t), \\ \dot{e_3}(t) &= -e_1(t) - ce_3(t) + v_3(t), \end{aligned}\right\} \tag{17}$$

where

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = -\begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}. \tag{18}$$

Rewrite the system of equations (17) as follows:

$$\begin{aligned}\begin{bmatrix} \dot e_1(t) \\ \dot e_2(t) \\ \dot e_3(t) \end{bmatrix} &= \begin{bmatrix} \frac{1}{b}-a & 0 & 1 \\ 0 & -b & 0 \\ -1 & 0 & -c \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} - \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\ &= \begin{bmatrix} \frac{1}{b}-a-k_1 & 0 & 1 \\ 0 & -b-k_2 & 0 \\ -1 & 0 & -c-k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}. \end{aligned} \tag{19}$$

Let us construct a quadratic Lyapunov exponent as given below:

$$V(\mathbf{e}(t)) = \frac{1}{2}\mathbf{e}^T(t)P\mathbf{e}(t) \geqslant 0, \tag{20}$$

where $P = \text{diag}\,[p_{ii}, i = 1, 2, 3]$ is a PDM. Now:

$$\begin{aligned}\dot V(\mathbf{e}(t)) &= \dot{\mathbf{e}}^T(t)P\mathbf{e}^T(t) + \mathbf{e}^T(t)P \\ &= \left(\begin{bmatrix} \frac{1}{b}-a-k_1 & 0 & 1 \\ 0 & -b-k_2 & 0 \\ -1 & 0 & -c-k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}\right)^T P \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\ &\quad + \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}^T P \begin{bmatrix} \frac{1}{b}-a-k_1 & 0 & 1 \\ 0 & -b-k_2 & 0 \\ -1 & 0 & -c-k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\ &= 2P\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}^T \begin{bmatrix} \frac{1}{b}-a-k_1 & 0 & 1 \\ 0 & -b-k_2 & 0 \\ -1 & 0 & -c-k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\ &= -\mathbf{e}^T(t)\begin{bmatrix} 2p_1\left(k_1+a-\frac{1}{b}\right) & 0 & 0 \\ 0 & 2p_1(k_2-b) & 0 \\ 0 & 0 & 2p_1(k_3-c) \end{bmatrix}\mathbf{e}(t) < 0. \end{aligned} \tag{21}$$

From (21), it is clear that the derivative of the Lyapunov error function is a negative definite function if the following conditions hold true:

$$k_1 > -a + \frac{1}{b},\ k_2 > b,\ k_3 > c. \tag{22}$$

Thus, the two identical chaotic systems (14) are globally asymptotically stable synchronized. ■

4 Synchronization between two non-identical finance and Wang 3-D chaotic systems

4.1 Problem statement

In order to achieve nonidentical synchronization, it is assumed that the finance chaotic system drives the Wang [27] chaotic system. Therefore, the master-slave system synchronization scheme for nonidentical synchronization is given as follows:

$$\text{Master System: } \begin{cases} \dot{x_1}(t) = \left(\frac{1}{b} - a\right) x_1(t) + z_1(t) + x_1(t)y_1(t) + d_1(t), \\ \dot{y_1}(t) = -by_1(t) - x_1^2(t) + d_2(t), \\ \dot{z_1}(t) = -x_1(t) - cz_1(t) + d_3(t); \end{cases}$$
$$\text{Slave System: } \begin{cases} \dot{x_2}(t) = \alpha\left(y_2(t) - x - 2(t)\right) - y_2(t)z_2(t) + D_1(t) + \eta_1(t), \\ \dot{y_2}(t) = -\beta y_2(t) + x_2(t)z_2(t) + D_2(t) + \eta_2(t), \\ \dot{z_2}(t) = -\gamma z_2(t) + \delta x_2(t) + x_2(t)y_2(t) + D_3(t) + \eta_3(t); \end{cases} \tag{23}$$

where $[x_1(t), y_1(t), z_1(t)] \in R^3$ and $[x_2(t), y_2(t), z_2(t)] \in R^3$ are the state vectors, a, b and c are the positive parameters of the master and slave systems alternatively. Also, $d_i(t)$ and $D_i(t)$, $i = 1, 2, 3$ are the model uncertainties and external disturbances present in the master and slave systems respectively, where $\eta(t) = [\eta_1(t), \eta_2(t), \eta_3(t)] \in R^3$ is the control input which is yet to determined.

Definition 4 *The error dynamics for the synchronization scheme* (23) *can be described as follows:*

$$\left.\begin{aligned} \dot{e_1}(t) &= -\alpha e_1(t) + e_3(t) - \left(\alpha - a + \frac{1}{b}\right) x_1(t) + \alpha y_2(t) - z_2(t) - x_1(t)y_1(t) - y_2(t)z_2(t) + D_1(t) - d_1(t) + \eta_1(t), \\ \dot{e_2}(t) &= -\beta e_2(t) + (b - \beta)\, y_1(t) + x_1^2(t) + x_2(t)z_2(t) + D_2(t) - d_2(t) + \eta_2(t), \\ \dot{e_3}(t) &= -\gamma e_3(t) + \delta e_1(t) + (\delta + 1)\, x_1(t) + (c - \gamma)\, z_1(t) + x_2(t)y_2(t) + D_3(t) - d_3(t) + \eta_3(t). \end{aligned}\right\} \tag{24}$$

Theorem 4 *For the arbitrary initial conditions* $(x_1(0), y_1(0), z_1(0)) \in R^3 \neq (x_2(0), y_2(0), z_2(0)) \in R^3$ *of the master and slave systems, the two coupled chaotic systems* (14) *are globally asymptotically stable synchronized with the following suitable feedback control functions:*

$$\left.\begin{aligned} \dot{\eta_1}(t) &= \left(\alpha - a + \frac{1}{b}\right) x_1(t) - \alpha y_2(t) + z_2(t) + x_1(t)y_1(t) + y_2(t)z_2(t) - D_1(t) + d_1(t) + v_1(t), \\ \dot{\eta_2}(t) &= (\beta - b)\, y_1(t) - x_1^2(t) - x_2(t)z_2(t) - D_2(t) + d_2(t) + v_2(t), \\ \dot{\eta_3}(t) &= -(\delta + 1)\, x_1(t) + (\gamma - c)\, z_1(t) - x_2(t)y_2(t) - D_3(t) + d_3(t) + v_3(t). \end{aligned}\right\} \tag{25}$$

Proof of Theorem 3. Let us assume that the parameters of the master and slave systems (23) are available and measurable. Substituting system of equations (25) in (24) yields:

$$\left.\begin{aligned} \dot{e_1}(t) &= -\alpha e_1(t) + e_3(t) + v_1(t), \\ \dot{e_2}(t) &= -\beta e_2(t) + v_2(t), \\ \dot{e_3}(t) &= \delta e_1(t) - \gamma e_3(t) + v_3(t), \end{aligned}\right\} \tag{26}$$

where

$$\begin{bmatrix} v_1(t) \\ v_2(t) \\ v_3(t) \end{bmatrix} = - \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}. \tag{27}$$

Rewrite the system of equations (26) as follows:

$$\begin{aligned} \begin{bmatrix} \dot{e}_1(t) \\ \dot{e}_2(t) \\ \dot{e}_3(t) \end{bmatrix} &= \begin{bmatrix} \alpha & 0 & -1 \\ 0 & -\beta & 0 \\ \delta & 0 & -\gamma \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} - \begin{bmatrix} k_1 & 0 & 0 \\ 0 & k_2 & 0 \\ 0 & 0 & k_3 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\ &= \begin{bmatrix} \alpha - k_1 & 0 & -1 \\ 0 & -\beta - k_2 & 0 \\ \delta & 0 & -\gamma - k_3 \end{bmatrix} \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}. \end{aligned} \tag{28}$$

Let us construct a quadratic Lyapunov exponent as given belows:

$$V(\mathbf{e}(t)) = \frac{1}{2}\mathbf{e}^T(t)P\mathbf{e}(t) \geqslant 0, \tag{29}$$

where $P = \text{diag}\,[p_{ii}, i = 1, 2, 3]$ is a PDM. Now:

$$\begin{aligned}
\dot{V}(\mathbf{e}(t)) &= \dot{\mathbf{e}}^T(t)P\mathbf{e}^T(t) + \mathbf{e}^T(t)P \\
&= \left(\begin{bmatrix} k_1-\alpha & 0 & 0 \\ 0 & \beta+k_2 & 0 \\ 0 & 0 & \gamma+k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}\right)^T P \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\
&\quad + \begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}^T P \begin{bmatrix} k_1-\alpha & 0 & 0 \\ 0 & \beta+k_2 & 0 \\ -1 & 0 & \gamma+k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\
&= 2P\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix}^T \begin{bmatrix} k_1-\alpha & 0 & 0 \\ 0 & \beta+k_2 & 0 \\ 0 & 0 & \gamma+k_3 \end{bmatrix}\begin{bmatrix} e_1(t) \\ e_2(t) \\ e_3(t) \end{bmatrix} \\
&= -\mathbf{e}^T(t)\begin{bmatrix} 2p_1(k_1-\alpha) & 0 & 0 \\ 0 & 2p_2(\beta+k_2) & 0 \\ 0 & 0 & 2p_3(\gamma+k_3) \end{bmatrix}\mathbf{e}(t) < 0.
\end{aligned} \tag{30}$$

From system of equations (30), it is clear that the derivative of the Lyapunov error function will be a negative definite function if the following conditions hold true:

$$k_1 - \alpha > 0,\ k_2 + \beta > 0,\ k_2 + \gamma > 0. \tag{31}$$

This complete the proofs. ■

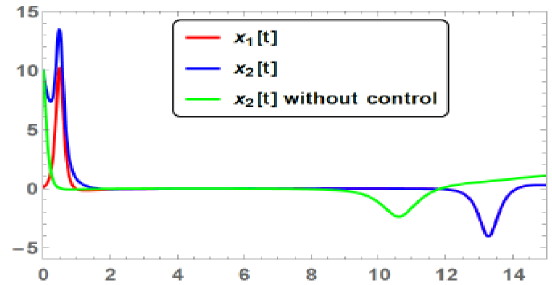
5 Numerical simulations and discussions

5.1 Example 1

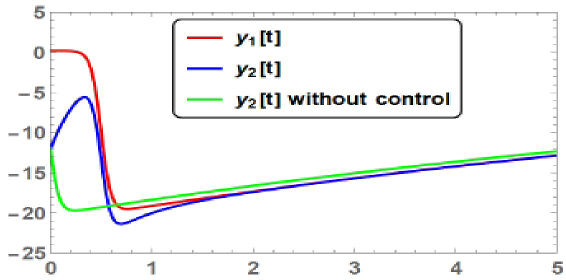
Numerical simulations are carried out using the Mathematica 10. The advantages and effectiveness of the proposed schemes can be demonstrated by discussing the simulation results in synchronizing two identical finance chaotic systems [26] with the presence of model uncertainties and external disturbances. The parameters for the finance chaotic system [26] are selected as $a = 0.0001$, $b = 0.1$, $c = 1$, with initial conditions being taken as $(0.1, 0.23, 0.31)$ and $(10, -12, 0)$ respectively. In numerical simulations, the linear controller gains which satisfy the conditions (22) are selected as $k_1 = 12$, $k_2 = 2.5$, and $k_3 = 2$, and the PDM is chosen as $P = \text{diag}\,[0.5, 0.5, 0.5]$. The following model of uncertainties and external disturbances are applied to the master and slave systems respectively:

$$\begin{aligned}
&d_1(t) = -0.02\sin(\pi x_1(t)) + 0.001\cos(10t), && D_1(t) = 0.04\cos\left(\frac{\pi}{6}x_2(t)\right) - 0.001\cos(45t), \\
&d_2(t) = -0.04\sin\left(\frac{5\pi}{18}y_1(t)\right) + 0.001\cos(30t), \quad \text{and} && D_2(t) = 0.01\cos\left(\frac{3\pi}{2}y_2(t)\right) - 0.01\sin(30t), \\
&d_3(t) = 0.02\sin\left(\frac{5\pi}{20}z_1(t)\right) - 0.001\sin(20t); && D_3(t) = -0.03\cos\left(\frac{\pi}{2}z_2(t)\right) + 0.01\cos(60t).
\end{aligned} \tag{32}$$

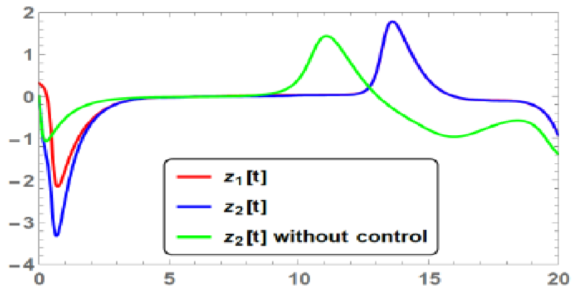
For identical finance chaotic systems, the time histories of the state vectors of synchronized and unsynchronized trajectories are depicted in Figures 2(a), 2(b), and 2(c). Figure 2(d) illustrates the synchronized error signals when the controllers are switched on at $t \approx 1.5s$. It has been shown that the error signals for the coupled identical finance chaotic systems converged to the zero state smoothly with faster converging rates. This approves that the designed control strategy is robust against the total disturbances.



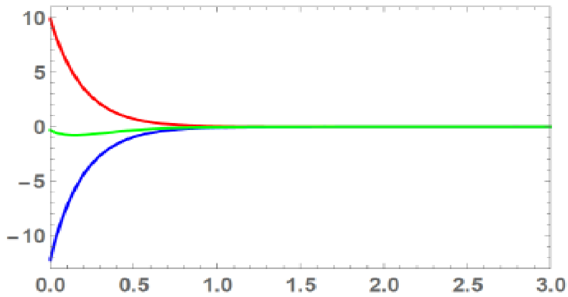
(a) Time series of the state vectors $x_1(t)$ and $x_2(t)$ $(t(s))$



(b) Time series of the state vectors $y_1(t)$ and $y_2(t)$ $(t(s))$



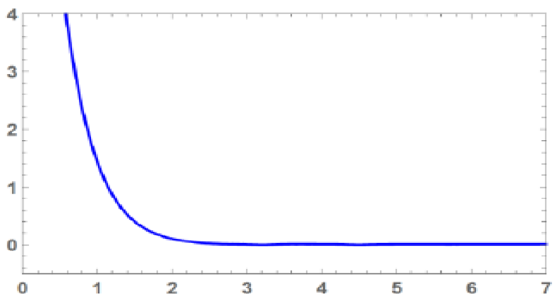
(c) Time series of the state vector $z_1(t)$ and $z_2(t)$ $(t(s))$



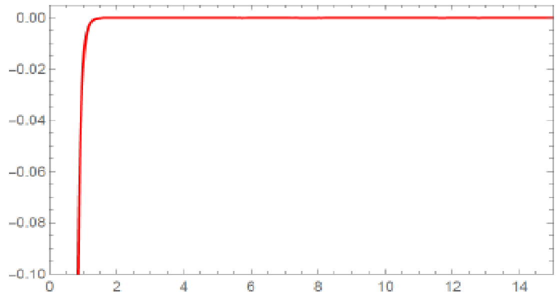
(d) Time series of the synchronized error signals $(t(s))$

Figure 2:

The dynamics of the error convergence is defined as $e(t) = \sqrt{e_1^2(t) + e_2^2(t) + e_3^2(t)}$, whose time evolution is depicted in Figure 3(a), while to observe the stability of the closed-loop system, the derivative of the Lyapunov function (21) at the equilibrium point is given in Figure 3(b). One can see that the convergence of the synchronization error and the stability of the closed-loop system are achieved with good accuracy despite the total disturbances, which is stable with our expectations.



(a) Time series of convergence of errors $(t(s))$



(b) Time series of $\dot{V}(\mathbf{e}(t))$ $(t(s))$

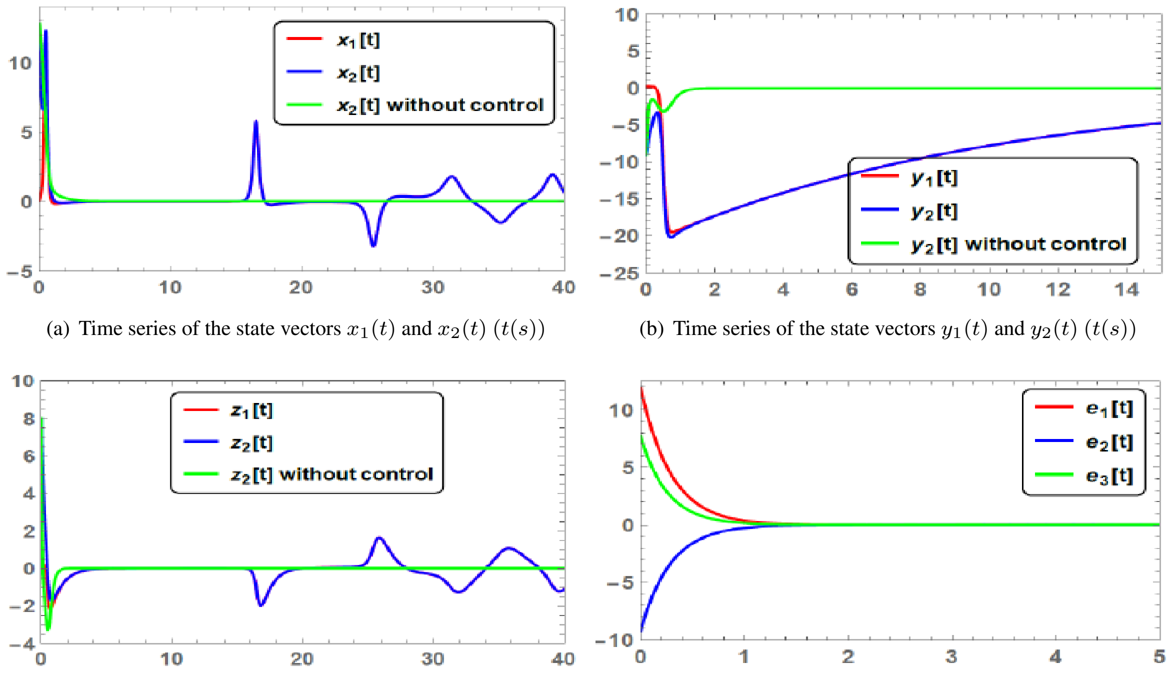
Figure 3:

5.2 Example 2

In this subsection, numerical simulation results are presented to verify the effectiveness and feasibility of the proposed approach in synchronizing two nonidentical finance and Wang 3-D chaotic systems with model uncertainties and external disturbances. The parameters of the finance chaotic system [26] are set as $a = 0.0001$, $b = 0.1$, $c = 1$, with the assumption of initial conditions being taken as $(0.1, 0.23, 0.31)$ and $(10, -12, 0)$. The parameters for the Wang chaotic system [27] are set as $\alpha = 1$, $\beta = 5.5$, $\gamma = 5$, and $\delta = 0.06$, with initial conditions being taken as $(12, -9, 8)$ and $(-2, 15, 20)$. The linear controller gains that satisfy conditions (31) are selected as $k_1 = 3$, $k_2 = -2.2$, and $k_3 = -1$, where the PDM is chosen as $P = \text{diag}\left[\frac{1}{2}, \frac{1}{2}, \frac{1}{0.012}\right]$. The following model of uncertainties and external disturbances are applied to the master and slave systems respectively:

$$\begin{aligned} d_1(t) &= -0.02\sin(\pi x_1(t)) + 0.001\cos(10t), & D_1(t) &= 0.04\cos\left(\frac{\pi}{6}x_2(t)\right) - 0.001\cos(45t), \\ d_2(t) &= -0.04\sin\left(\frac{5\pi}{18}y_1(t)\right) + 0.001\cos(30t), \quad \text{and} & D_2(t) &= 0.01\cos\left(\frac{3\pi}{2}y_2(t)\right) - 0.01\sin(30t), \\ d_3(t) &= 0.02\sin\left(\frac{5\pi}{20}z_1(t)\right) - 0.001\sin(20t); & D_3(t) &= -0.03\cos\left(\frac{\pi}{2}z_2(t)\right) + 0.01\cos(60t). \end{aligned} \tag{33}$$

For nonidentical finance and Wang chaotic systems, the time histories of the synchronized and unsynchronized state trajectories are illustrated in Figures 4(a), 4(b) and 4(c). Figure 4(d), illustrates the synchronized error signals. The error signals converged to the zero state when the controllers are switched on at $t < 2s$. This confirms the robustness of the proposed synchronization approach.



(a) Time series of the state vectors $x_1(t)$ and $x_2(t)$ $(t(s))$

(b) Time series of the state vectors $y_1(t)$ and $y_2(t)$ $(t(s))$

(c) Time series of the state vector $z_1(t)$ and $z_2(t)$ $(t(s))$

(d) Time series of the synchronized error signals $(t(s))$

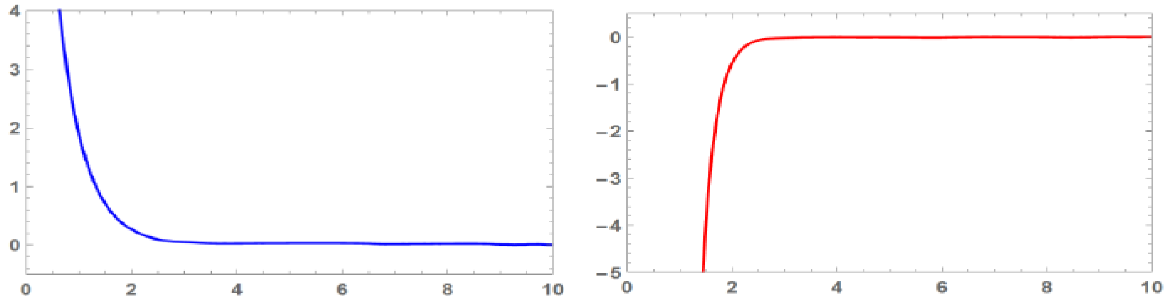
Figure 4:

The time history for the dynamics of error convergence is depicted in Figure 5(a), while Figure 5(b) illustrates the derivative of the Lyapunov function (30). The proposed control strategy is robust against the parameters mismatches, model uncertainties and external disturbances.

6 Conclusion

This paper investigated the global chaos synchronization problem between two identical finance chaotic systems and two nonidentical finance and Wang chaotic systems. Based on the Lyapunov stability theory and using the active control technique, suitable linear controller gains were designed to achieve the global stability of the closed-loop systems in the presence of the model uncertainties and external disturbances.

There are two main advantages that the authors have achieved in this research work. Firstly, as the effect of the chaotic system depends sensitively on a tiny perturbation of the trajectories, the effect of total disturbances have been taken into account. Secondly, based on the Lyapunov direct method, suitable linear control gains are estimated that ensured globally



(a) Time series of convergence of errors $(t(s))$

(b) Time series of $\dot{V}(\mathbf{e}(t))$ $(t(s))$

Figure 5:

asymptotic stability of the closed-loop systems in both cases. It has been shown that the error signals converged to the origin very smoothly with minimum rates of decay and sufficient transient speed. These features give an advantage to our proposed synchronization schemes.

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References

[1] J. Wisdom. Spin-orbit secondary resonance dynamics of Encedalus. *The Astronomical Journey*, 128(2004):484–491.

[2] R. Z. Luo and F. Zhang. Finite-time modified projective synchronization of unknown Rossler and Coullet systems. *Communication in Control Science and Engineering*, 1(4)(2013):51–56.

[3] M. Shahzad and I. Ahmad. Experimental study of synchronization & Antisynchronization for spin orbit problem of Enceladus. *International Journal of Control Science and Engineering*, 3(2)(2013):41–47.

[4] L. Pecora and T. Carrol. Synchronization in Chaotic Systems. *Physical Review Letters*, 64(1990):821–823.

[5] S. Al-Hadhrami, A. Saaban, A. Ibrahim, M. Shazad and I. Ahmad. Linear active control algorithm to synchronize a nonlinear HIV/AIDS dynamical system. *Asian Journal of Applied Sciences and Engineering*, 3(2)(2014):96–115.

[6] A. N. Pisarchik, F. T. Arecchi, R. Meucci and A. D. Garbo. Synchronization of Shilnikov chaos in a CO_2 laser with feedback. *Laser Physics*, 11(11)(2014):1235–1239.

[7] X. Wang, T. Wang and W. Xu. Hopf bifurcation and limit cycle analysis of the Rikitake system. *International Journal of Nonlinear Science*, 14(2)(2012):211–215.

[8] A. N. Njah. Synchronization via active control of parametrically and externally excited ϕ6 Van der Pol and Duffing oscillators and application to secure communications. *Journal of Vibration and Control*, 17(4)(2011):493–504.

[9] H. Chen. Chaos synchronization of a symmetric gyro with linear-plus cubic damping. *Journal of Sound and Vibrations*, 255(2002):719–740.

[10] M. Aghababa and H. Aghababa. Adaptive finite-time stabilization of uncertain nonautonomous chaotic electromechanical gyrostate systems with unknown parameters. *Mechanical Research Communications*, 2(38)(2011):500–505.

[11] L. Yao and G. Chai. Chaos synchronization of a new hyperchaotic finance system via a novel chatter free sliding mode control strategy. *International Journal of Nonlinear Science*, 17(2)(2014):176–181.

[12] G. Cai and Z. Tan. Chaos synchronization of a new chaotic system via nonlinear control. *Journal of Uncertain Systems*, 1(3)(2007):235–240.

[13] G. Cai and S. Zheng. Projective synchronization of a new chaotic system. *Journal of Uncertain Systems*, 2(1)(2008):67–74.

[14] S. Miao, S. Hua and W. Hao. Adaptive synchronization of the fractional-order $L\ddot{U}$ hyperchaotic system with uncertain parameters and its circuit simulation. *Journal of Uncertain Systems*, 6(1)(2012):11–19.

[15] M. M. Al-Sawalha and M. S. M. Noorani. Anti-synchronization between two different hyperchaotic systems. *Journal of Uncertain Systems*, 3(3)(2009):192–200.

[16] G. P. Jiang and W. K. S. Tang. A global synchronization criterion for coupled chaotic systems via unidirectional linear error feedback approach. *International Journal of Bifurcation and Chaos*, 12(10)(2002):2239–2253.

[17] A. N. Njah and U. E. Vincent. Synchronization and anti-synchronization of chaos in an extended Bonhoffer-van der Pol oscillator using active control. *Journal of Sound and Vibrations*, 319(2009):41–49.

[18] M. C. Ho and Y. C. Hung. Synchronization of two different systems by using the generalized active control. *Physics Letters A*, 301(2002):424–428.

[19] U. E. Vincent. Synchronization of identical and nonidentical 4-D chaotic systems using active control. *Chaos, Solitons and Fractals*, 37(2008):1065–1075.

[20] R. C. Dorf and R. H. Bishop. Modern Control Systems. Princeton Hall, NJ USA. 2001.

[21] H. K. Khalil. Non Linear Systems. Prentice Hall, NJ USA. 2002.

[22] G. Ablay and T. Aldemir. Synchronization of different chaotic systems using generalized active control. 6th Int. Conference on Electrical and electronics Engineering (ELECO), Bursa, Turkey. 5-8 Nov, 2009.

[23] S. Hammami, K. Saad and M. Benrejeb. On the synchronization of identical and non-identical 4-D chaotic systems using arrow form matrix. *Chaos, Solitons and Fractals*, 42(2009):101–112.

[24] B. A. Idowu and U. E. Vincent. Synchronization and stabilization of chaotic dynamics in a quasi-1D Bose-Einstein condensate. *Journal of Chaos*, 2013(2013): 8 pages, doi:10.1155/2013/723581.

[25] I. Ahmad, A. Saaban, A. Ibrahim and M. Shahzad. Global chaos synchronization of new chaotic system using linear active control. *Complexity*, 2014: doi:10.1002/cplx.21573.

[26] G. Cai and J. Huang. A new finance chaotic attractor. *International Journal of Nonlinear Science*, 3(3)(2007):213–220.

[27] L. Wang. 3-scroll and 4-scroll chaotic attractors generated from a new 3-D quadratic autonomous system. *Nonlinear Dynamics*, 56(2009):453–462.