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Wave-breaking for the Weakly Dissipative Modified Camassa-Holm Equation

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Abstract: This paper is concerned with the local well-posedness and blow-up phenomena for the weakly dissipative modified Camassa-Holm equation. Blow-up criteria are established, and a new wave-breaking mechanism for solutions with certain initial profiles is described in detail.

Keywords: the weakly dissipative modified Camassa-Holm equation; Local well-posedness; Blow-up; Wave breaking rate; Blow-up criterion

1 Introduction

The modified Camassa-Holm equation

$$m_t + \left(\left(u^2 - u_x^2\right) m\right)_x + \gamma u_x = 0, \tag{1}$$

is a model of nonlinear shallow water waves, where the variable $u(t,x)$ and $m(t,x)$ represent, respectively, the velocity of the fluid and its potential density at time $t \geq 0$ in the spatial x direction, in which γ is a constant, and

$$m = u - u_{xx}. \tag{2}$$

The equation (1) arises from an intrinsic (arc-length preserving) invariant planar curve flow in Euclidean geometry and it can be regarded as a Euclidean-invariant version of the Camassa-Holm equation in [1]. It has the form of a modified Camassa-Holm equation with cubic nonlinearity. By Fuchssteiner [2] and Olver and Rosenau[3], it can be derived as a new integrable system by applying the general method of tri-Hamiltonian duality to the bi-Hamiltonian representation of the modified Korteweg-de Vries equation. Later, it was obtained by Qiao [4] from the two-dimensional Eular equations. In [5] it was shown that equation (1) admits a Lax pair. In [1] it can be solved by the method of inverse scattering, its scaling limit equation satisfies the short-pulse equation.

The original Camassa-Holm (CH) equation

$$m_t + um_x + 2u_x m + \gamma u_x = 0, \tag{3}$$

where m is as above, (2), was derived from the Korteweg-de Vries equation by tri-Hamiltonian duality. Since the Camassa-Holm (CH) equation [6,7] was originally proposed as a model for surface waves, and has been studied extensively during the last twenty years because of its many remarkable properties: infinity of conservation laws and complete integrability [6,7,8], well-posedness and breaking waves, meaning solutions that remain bounded while their slope becomes unbounded in finite time [9,10,11,12,13]. Note that the nonlinearity in CH equation is quadratic. Since the modified CH equation has a cubic nonlinearity which the CH equation is only quadratic, one expects that the modified CH equation should also have peaked solitons and wave-breaking. In [1], the author obtained blow-up criteria for strong solutions, singularities which correspond to wave breaking and a sufficient condition for wave breaking of strong solutions in finite time.

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Two integrable CH-type equations with cubic nonlinearity have been discovered: One is the equation (1), and the second is the Novikov equation [14]

$$m_t + u^2 m_x + \frac{3}{2}\left(u^2\right)_x m = 0. \tag{4}$$

The integrability, peaked solitons, well-posedness and blow up phenomena to the Novikov equation have been studied extensively [14, 15, 16]. An alternative modified CH equation was introduced in [17].

In general, to avoid energy dissipation mechanisms in a real world is not so easy. Ott and Sudan[18] studied how the KdV equation was modified by the presence of dissipation and the effect of such dissipation on the solitary solution of such dissipation on the solitary solution of the KdV equation, and Ghidaglia [19] investigated the long time behavior of solutions to the weakly dissipative KdV equation as a finite-dimensional dynamical system. Wu, Escher and Yin have investigated the blow-up phenomena, blow-up rate of the strong solutions of the weakly dissipative CH equation [20] and DP equation [21]. Inspired by the results mentioned above, in the paper, we are going to discuss the initial-value problem associated with the generalized weakly dissipative modified Camassa-Holm equation:

$$m_t + \left(\left(u^2 - u_x^2\right) m\right)_x + \gamma u_x + \lambda m = 0, m = u - u_{xx}, t > 0, x \in R, \tag{5}$$

where $\lambda > 0$ is a constant.

The difference between Eq.(5) and Eq.(1) is that Eq.(5) has not the following conservation laws:

$$\frac{d}{dt}\int_R \left(u^2 + u_x^2\right) dx = 0, \quad \frac{d}{dt}\int_R \left(u^4 + 2u^2 u_x^2 - \frac{1}{3}u_x^4 + 2\gamma u^2\right) dx = 0,$$

for all $t \in [0, T)$, which play an important role in the study of Eq.(1).

In this paper, we will present the local well-posedness and blow-up phenomena for the weakly dissipative modified Camassa-Holm equation. The remainder of the paper is organized as follows. In section 2, we give the local posedness of Eq.(5). In section 3, its detailed blow-up criteria for strong solutions are established. It is shown that the solutions to the modified CH equation can only have singularities which correspond to wave breaking.

2 Local well-posedness

In this section, we will study the local posedness for the weakly dissipative modified Camassa-Holm equation on the entire line:

$$\begin{cases} m_t + \left(\left(u^2 - u_x^2\right) m\right)_x + \gamma u_x + \lambda m = 0, m = u - u_{xx}, t > 0, x \in R, \lambda > 0, \\ u(0, x) = u_0(x), x \in R, \\ u, m \to 0, |x| \to \infty. \end{cases} \tag{1}$$

Substituting the formula for m in terms of u into the partial differential equation (1) results in the following fully nonlinear partial differential equation:

$$u_t + u^2 u_x = -\left(1 - \partial_x^2\right)^{-1}\left(2u^2 u_x + 2u u_x u_{xx} + \gamma u_x + \frac{4}{3}u_x^3\right) - \lambda u + \frac{1}{3}u_x^3. \tag{2}$$

Recall that

$$u = \left(1 - \partial_x^2\right)^{-1} m = p * m, p(x) = \frac{1}{2}e^{-|x|} \tag{3}$$

and $*$ denotes the convolution product on$R$, defined by

$$(f * g)(x) = \int_R f(y) g(x - y) dy. \tag{4}$$

Using this identity, we can rewrite (1) as follows

$$\begin{cases} u_t + u^2 u_x = -p * \left(2u^2 u_x + 2u u_x u_{xx} + \gamma u_x + \frac{4}{3}u_x^3\right) - \lambda u + \frac{1}{3}u_x^3, \\ u(0, x) = u_0(x), x \in R. \end{cases} \tag{5}$$

Consider the abstract quasi-linear evolution equation:

$$\frac{du}{dt} + Au = f(u), \quad t > 0 \quad ; \quad u(0) = u_0. \tag{6}$$

Proposition 1 *(See [22].)Given the evolution equation (6), assume that the conditions (a), (b), and (c) hold. For a fixed $v_0 \in Y$, there is a maximal $T > 0$ depending only on $\|v_0\|_Y$ and a unique solution v to the abstract quasi-linear evolution equation (6) such that*

$$v = v(\cdot, v_0) \in C([0,T);Y) \cap C^1([0,T);X) \quad .$$

Moreover, the map $v_0 \to v(\cdot, v_0)$ is continuous from Y to

$$C([0,T);Y) \cap C^1([0,T);X).$$

The local well-posedness of Cauchy problem (5) with initial data $u_0 \in H^s, s > \frac{3}{2}$, by applying Kato's semigroup theory [9]. We can obtain the following local well-posedness result, as was done for the Camassa-Holm equation in [1,23].

Theorem 2 *Let $m_0 = \left(1 - \partial_x^2\right) u_0 \in H^s(R)$ withs $> \frac{1}{2}$, then there exists a time $T > 0$ such that the initial-value problem (1) has a unique strong solution $m \in C([0,T];H^s) \cap C'\left([0,T];H^{s-1}\right)$and the map $m_0 \to m$ is continuous from a neighborhood of m_0 in H^s into $C([0,T];H^s) \cap C'\left([0,T];H^{s-1}\right)$.*

3 Wave-breaking criteria

In this section, we will establish criteria for the blow up of solutions to the Cauchy problem for the modified CH equation (1). We first introduce some 1-D Moser-type estimates, [24].

Proposition 3 *For $s \geq 0$, the following estimates hold:*

$$\begin{aligned} \|fg\|_{H^s(R)} &\leq C\left(\|f\|_{H^s(R)}\|g\|_{L^\infty(R)} + \|f\|_{L^\infty(R)}\|g\|_{H^s(R)}\right) \quad , \\ \|f\partial_x g\|_{H^s(R)} &\leq C\left(\|f\|_{H^{s+1}(R)}\|g\|_{L^\infty(R)} + \|f\|_{L^\infty(R)}\|\partial_x g\|_{H^s(R)}\right), \end{aligned} \tag{1}$$

where the $C's$ are constants independent of f and g.

The following estimates for solutions to the one-dimensional transport equation have been needed in [24, 25].

Lemma 4 *(See [2]) Consider the one-dimensional linear transport equation*

$$\partial_t f + v\partial_x f = g \quad , \quad f|_{t=0} = f_0 \quad . \tag{2}$$

Let $0 \leq \sigma < 1$, and suppose that

$$\begin{aligned} &f_0 \in H^\sigma(R) \quad , \quad g \in L^1([0,T];H^\sigma(R)); \\ &v_x \in L^1([0,T];L^\infty(R)), \quad f \in L^1([0,T];H^\sigma(R)) \cap C([0,T];S'(R)), \end{aligned}$$

then $f \in C([0,T];H^\sigma(R))$. More precisely, there is a constant C depending only on σ such that, for every $0 < t \leq T$,

$$\|f(t)\|_{H^\sigma} \leq \|f_0\|_{H^\sigma} + C\int_0^t \|g(\tau)\|_{H^\sigma} d\tau + C\int_0^t \|f(\tau)\|_{H^\sigma} V'(\tau)\, d\tau \tag{3}$$

and hence,

$$\|f(t)\|_{H^\sigma} \leq e^{CV(t)}\left(\|f_0\|_{H^\sigma} + C\int_0^t \|g(\tau)\|_{H^\sigma} d\tau\right), V(t) = \int_0^t \|\partial_x v(\tau)\|_{L^\infty} d\tau.$$

Let us rewrite the modified CH equation (1) as a transport equation in terms of m with the flow generated by $u^2 - u_x^2$

$$m_t + \left(u^2 - u_x^2\right) m_x = -2u_x m^2 - \gamma u_x - \lambda m \tag{4}$$

the transport equation theory makes it certian that, if the slope

$$\left(u^2 - u_x^2\right)_x = 2u_x m \tag{5}$$

is bounded, the solution will remain regular, and can't blow up in finite time.

In[1], the following blow-up criterion was obtained (with a slight modification).

Theorem 5 *Let $m_0 = \left(1 - \partial_x^2\right) u_0 \in H^s(R)$ be as in Theorem 2 with $s > \frac{1}{2}$. Let m be the corresponding solution to (1). Assume $T_{m_0}^* > 0$ is the maximum time of existence. Then*

$$T_{m_0}^* < \infty \Rightarrow \int_0^{T_{m_0}^*} \|m(\tau)\|_{L^\infty}^2 d\tau = \infty. \tag{6}$$

Remark 6 *For a strong solution $m = u - u_{xx}$ in Theorem 2, that is*

$$\begin{aligned} \frac{d}{dt}\int_R \left(u^2 + u_x^2\right) dx &= -2\lambda \int_R \left(u^2 + u_x^2\right) dx \quad , \\ \|u\|_{H^1}^2 \int_R \left(u^2 + u_x^2\right) dx &= \|u_0\|_{H^1}^2 e^{-2\lambda t} \le \|u_0\|_{H^1}^2 . \end{aligned} \tag{7}$$

The following blow-up criterion shows that the wave-breaking depends only on the infimum of mu_x :

Theorem 7 *let $m_0 \in H^s(R)$ be as in Theorem 2 with $s > \frac{1}{2}$. Then the corresponding solution m to (1) blows up infinite time $T_{m_0}^* > 0$ if and only if*

$$\lim_{t \to T_{m_0}^*} \inf_{x \in R} \{(mu_x)(t,x)\} = -\infty. \tag{8}$$

Proof. Since the existence time $T_{m_0}^*$ is independent of s, we only need to consider the case $s = 3$, which relies on a simple density argument.

Multiplying equation (4) by m and integrating over R with respect to$x$, and then Integration by parts, produces

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\int_R m^2 dx &= -\int_R \left(u^2 - u_x^2\right) m m_x dx - 2\int_R u_x m^3 dx - \gamma \int_R u_x m dx - \lambda \int_R m^2 dx \\ &= -\int_R \left(mu_x + \lambda\right) m^2 dx \\ &= -\int_R u_x m^3 dx - \lambda \int_R m^2 dx. \end{aligned}$$

By differentiating (4) once with respect to x, we have

$$\begin{aligned} m_{xt} + \left(u^2 - u_x^2\right) m_{xx} &= -3u_x \left(m^2\right)_x - 2u_{xx} m^2 - \gamma u_{xx} - \lambda m_x \\ m_{xt} &= -2u_{xx} m^2 - 6u_x m m_x - \left(u^2 - u_x^2\right) m_{xx} - \gamma u_{xx} - \lambda m_x \\ &= -2um^2 + 2m^3 - 6u_x m m_x - \left(u^2 - u_x^2\right) m_{xx} - \gamma u_{xx} - \lambda m_x. \end{aligned}$$

Multiplying by m_x and integrating over R with respect to x, leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\int_R m_x^2 dx \\ &= -\int_R \left(u^2 - u_x^2\right) m_x m_{xx} dx - 2\int_R u_x m^2 m_x dx - 6\int_R u_x m m_x^2 dx + 2\int_R m^3 m_x dx \\ &- \gamma \int_R u_{xx} m_x dx - \lambda \int_R m_x^2 dx \\ &= \frac{1}{2}\int_R \left(u^2 - u_x^2\right)_x m_x^2 dx + \frac{2}{3}\int_R u_x m^3 dx - 6\int_R u_x m m_x^2 dx - \frac{\gamma}{2}\int_R \left(u_x^2 - u_{xx}^2\right)_x dx - \lambda \int_R m_x^2 dx \\ &= -5\int_R u_x m m_x^2 dx + \frac{2}{3}\int_R u_x m^3 dx - \lambda \int_R m_x^2 dx, \end{aligned}$$

therefore,

$$\frac{d}{dt}\int_R \left(m^2+m_x^2\right)dx=-10\int_R u_x m m_x^2 dx-\frac{2}{3}\int_R u_x m^3 dx-\lambda\int_R \left(m^2+m_x^2\right)dx.$$

If mu_x is bounded from below on $\left[0,T_{m_0}^*\right)\times R$, i.e., there exists a positive constant $C_1>0$ such that $mu_x\geq -C_1$ on $\left[0,T_{m_0}^*\right)\times R$, then the above estimate implies

$$\begin{aligned}\frac{d}{dt}\int_R \left(m^2+m_x^2\right)dx &\leq -10\int_R \left(m^2+m_x^2\right)dx-2\lambda\int_R \left(m^2+m_x^2\right)dx\\ &=\left(10C_1-2\lambda\right)\int_R \left(m^2+m_x^2\right)dx.\end{aligned}$$

Applying Gronwall's inequality then yields, for $t\in\left[0,T_{m_0}^*\right)$,

$$\|m\left(t\right)\|_{H^1}^2\leq\int_R \left(m^2+m_x^2\right)dx\leq e^{(10C_1-2\lambda)t}\left\|m_0\right\|_{H^1}^2$$

which ensures that the solution $m\left(t,x\right)$ does not blow up in finite time.
On the other hand, if

$$\liminf_{t\to T_{m_0}^*}\left[\inf_{x\in R}\left(mu_x\right)\left(t,x\right)\right]=-\infty,$$

by theorem 5 for the existence of local strong solutions and the Sobolev embedding theorem, we infer that the solution will blow-up in finite time. ■

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References

[1] G. L. Gui et al.. Wave-breaking and peakons for a modified Camassa-Holm eqution. *Communication in Mathematical Physics.* 319(2013)(3): 731-759.
[2] B. Fuchssteiner. Some tricks from the symmetry-toolbox for nonlinear equations:generalizations of the Camassa-Holm equation. *Physica D.* 95(1996): 229-243.
[3] P. J. Olver and P.Roesenau. Tri-Hamiltonian duality between solitons and solitary-wave solutions having compact support.. *Phys.Rev. E.* 53(1996): 1900-1906.
[4] Z. Qiao. A new integrable equation with cuspons and W/M-shape-peaks solitons *J.Math. Phys.* 47(2006): 112701.
[5] J. C. Brunelli. The short pulse hierarchy. *J. MATH. Phys.* 46(2005): 123507.
[6] R. Camassa and D. D. Holm. An integrable shallow water equation with peaked solitons. *Phys.Rev.Lett.* 71(1993): 1661–1664.
[7] B. Fuchssteiner and A. S. Fokas. Symplectic structures, their Backlund transformations and hereditary symmetries. *Physica D.* 4(1981): 47-66.
[8] M. Fisher and J. Schiff. The Camassa-Holm equation: conserved quantities and the initial value problem. *Phys.Lett.A.* 259(1999): 371-376.
[9] A. Constantin. Existence of permanent and breaking waves for a shallow water equation: a geometric approach. *Ann.Inst.Fourier(Grenoble).* 50(2000): 321-362.
[10] A. Constantin and J. Escher. Wave breaking for nonlinear nonlocal shallow water equations. *Acta Math.* 181(1998): 229-243.
[11] A. Constantin and J. Escher. On the blow-up rate and the blow-up set of breaking waves for a shallow warter equation. *Math. Z.* 233(2000): 75-91.
[12] A. Constantin and J. Escher. Global existences and blow-up for a shallow water equation. *Ann.Scuola Norm.Sup.Pisa* 26(1998): 303-328.

[13] Y. A. Li and P. J. Olver. Well-posedness and blow-up solutions for an integrable nonlinearly dispersive model wave equation. *J. Diff. Eq.* 162(2000): 27-63.
[14] V. Novikov. Generalizations of the Camassa-Holm equation. *J. Phys. A.* 42(2009): 342002.
[15] A. N. Hone et al. Explicit multipeakon solutions of Novikov's cubically nonlinear integrable Cmassa-Holm type equation. *Dyn. Partial Diff. Eq.* 6(2009): 253-289.
[16] A. N. Hone and J. P. Wang. Integrable peakon equations with cubic nonlinearity. *J. Phys. A* 41(2008): 372002.
[17] P. Gorka and E. G. Reyes. The modified Camassa-Holm equation. *Int, Math.Res.Notes.* 2011(2011): 2617-2649.
[18] E. Ott and R. N. Sudan. Damping of so;itary waves. *Phys.Fluids* 13(1970): 1432-1434.
[19] J. M. Ghidaglia. Weakly damped forced Korteweg-de Vries equations behavior as a finite dimensional dynamical system in the long time. *J.Differential Equations.* 74(1988): 369-390.
[20] S. Wu and Z. Yin. Blow-up, blow-up rate and decay of the weakly dissipative Camassa-Holm equation.. *J. Math. Phys.* 47(2006): 0135041-01350412.
[21] S. Wu et al.. Global existence and blow-up phenomena for a weakly dissipative Degaspris-Procesi equation. *Discrete Contin.Dyn. Syst.Ser.B.* 12(2009): 633-645.
[22] A. Constantin and D. Lannes. The hydrodynamical relevance of the Camassa-Holm and Degasperis-Procesi equations. *Arch.Ration.Mech.Anal.* 192(2009): 165-186.
[23] L. D. Ni and Y. Zhou. Well-posedness and persistence properties for the Novikov equation. *Journal of Differential Equations.* 250(2011): 3002-3021.
[24] G. Gui and Y. Liu. On the global existence and wave-breaking criteria for thetwo-component Camassa-Holm system. *Journal of Functional Analysis.* 258(2010): 4251-4278.