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Existence Results for Fractional Stochastic Differential Equation with Impulsive Effect

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Abstract: In this paper, we study the existence and uniqueness of the solution for an impulsive neutral fractional stochastic differential equation. The existence and uniqueness results are proved by using the fixed point techniques on a Hilbert space. An example is presented to verify the results of the paper.

Keywords: Fractional order differential equation; Stochastic functional differential equations; Existence results; Impulsive conditions; Fixed point theorems

1 Introduction

The fractional calculus is a classical mathematical notion and is a generalization of ordinary differentiation and integration to an arbitrary order. Presently, study of fractional calculus has become an active area of research due to its numerous applications in various fields, such as physics, fluid mechanics, viscoelasticity, chemistry and engineering sciences etc. The theory of fractional differential equations have been studied by many author's [1–11].

The deterministic models often fluctuate due to environmental noise. A natural extension of a deterministic model is stochastic model, where relevant parameters are modeled as suitable stochastic processes. Due to this fact, most of the problems in practical life situations are basically modeled by stochastic equations rather than deterministic. Therefore, it is of great significance to introduce the concept of stochastic effects in the investigation of differential equations [12]. It is also a well known fact that many dynamical system not only depends on present and past states but also involved derivatives with delays. To describe such type of systems, neutral functional differential equations are used for instance see the papers [13–15] and references therein.

However, it is known that the impulsive effects exist widely in different areas of real world problems such as mechanics, electronics, neural networks, telecommunications, finance and economics etc. [16, 17]. Due to this fact, the states of many evolutionary processes are often subject to instantaneous perturbations and experience abrupt changes at certain moments of time. The duration of these changes is very short and negligible in comparison with the duration of the process considered and can be thought of as impulses. Therefore, it is important to consider the effect of impulses in the investigation of stochastic delay differential equations. For recent contribution, we cite the papers [18–21].

C. Li et al. in [20] considered the following problem

$$\begin{cases} dy(t) = F(t, y(t), y(t-\tau(t)))dt + G(t, y(t), y(t-\tau(t)))dw(t), t \neq t_k, \\ y(t_k^+) - y(t_k) = I_k(y(t_k)), t = t_k, k \in \mathbb{N}, \end{cases}$$

and studied the stability of the stochastic differential delay system with nonlinear impulses. Subsequently, the authors established the equivalent relation between the solution of the n-dimensional stochastic differential delay system with nonlinear impulsive and without impulsive effects.

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Dabas and Gautam in their paper [22] considered the following impulsive neutral fractional integro-differential equation

$$
\begin{aligned}
&{}^cD_t^\alpha\left[x(t)+\int_0^t\frac{(t-s)^{\alpha-1}}{\Gamma(\alpha)}g(s,x_{\rho(s,x_s)})ds\right]=f(t,x_{\rho(t,x_t)},B(x)(t)),\ t\in[0,T],\\
&\Delta x(t_k)=I_k(x(t_k^-)),\Delta x'(t_k)=Q_k(x(t_k^-)),k=1,2,\ldots,m,\\
&x(t)=\phi(t)\ ,t\in[-d,0],\ ax'(0)+bx'(T)=\int_0^T q(x(s))ds,\ a+b\neq 0,\ b\neq 0,
\end{aligned}
$$

where $\alpha\in(1,2)$ and B is a Volterra integral operator. The authors established the results of existence of solution by using the Banach and Krasnoselkii's fixed point theorems.

Wang et al. in [23] studied the following fractional differential equation with boundary and impulsive conditions:

$$
\begin{aligned}
&{}^CD_t^q u(t)=f(t,u(t)),\ \ t\in[0,T],\ q\in(1,2)\\
&\Delta u(t_k)=I_k(u(t_k^-)),\Delta u'(t_k)=J_k(u(t_k^-)),k=1,2,\cdots,m,\\
&u(0)=u_0,\ u'(0)=\overline{u}_0,
\end{aligned}
$$

the authors first established the definition of the solution for the considered problem and proved the existence and uniqueness results by using Banach and Krasnoselskii's fixed point theorems.

Motivated by the mention work [20, 22, 23] in this article, we are concerned with the existence and uniqueness of solution for impulsive fractional functional differential equation of the form:

$$
\begin{aligned}
&{}^cD_t^\alpha\left[x(t)+\int_0^t(t-s)^\beta h(s,x_s)ds\right]=\mathbb{J}_t^{2-\alpha}\left[f(t,x_t)+g(t,x_t)\frac{dw(t)}{dt}\right],\\
&t\in J=(0,T],t\neq t_k,\ \beta\in\mathbb{Z}^+, \qquad (1)\\
&\Delta x(t_k)=I_k(x(t_k^-)),\Delta x'(t_k)=Q_k(x(t_k^-)),k=1,2,\ldots,m, \qquad (2)\\
&x(t)=\phi(t)\ ,t\in[-d,0],\ x'(0)=x_1\in\mathbb{R}, \qquad (3)
\end{aligned}
$$

where J is an operational interval and ${}^cD_t^\alpha$ denotes the Caputo's fractional derivative of order $\alpha\in(1,2)$ and $x(\cdot)$ takes the value in the real separable Hilbert space $\mathbb{H}$; $f,h:J\times PC_{\mathcal{L}}^0\to\mathbb{H},\ g:J\times PC_{\mathcal{L}}^0\to\mathcal{L}(\mathbb{K},\mathbb{H})$ and $I_k,Q_k:\mathbb{H}\to\mathbb{H}$ are appropriate functions; $\phi(t)$ is $\mathcal{F}_0$- measurable $\mathbb{H}$-valued random variable independent of w. For $0=t_0<t_1<\cdots<t_m<t_{m+1}=T$, $\Delta x(t_k)=x(t_k^+)-x(t_k^-),\ \Delta x'(t_k)=x'(t_k^+)-x'(t_k^-),\ x(t_k^+)$ and $x(t_k^-)$ denote the right and left limits of x at t_k. Similarly, $x'(t_k^+)$ and $x'(t_k^-)$ denote the right and left limits of x' at t_k respectively and $x_t\in PC_{\mathcal{L}}^0$ is defined as $x_t(\theta)=x(t+\theta)$ for $\theta\in[-d,0]$.

However, to the best of our knowledge, the existence and uniqueness of solutions for a class of impulsive neutral stochastic fractional order $\alpha\in(1,2)$ differential equations with finite delay in a Hilbert space is an untreated topic in the literature and the aim of this paper is to fill up this gap. The paper is divided into four sections, in second section we include some basic definitions and some relevant results. Third section is equipped with main results for the considered problem (1)-(3) and in the last section an example is presented to verify the results of the paper.

2 Preliminaries

Let $\mathbb{H},\mathbb{K}$ be two real separable Hilbert spaces and $\mathcal{L}(\mathbb{K},\mathbb{H})$ be the space of bounded linear operators from $\mathbb{K}$ into $\mathbb{H}$. For convenience, we will use the same notation $\|\cdot\|$ to denote the norms in $\mathbb{H},\mathbb{K}$ and $\mathcal{L}(\mathbb{K},\mathbb{H})$, and use $(\cdot,\cdot)$ to denote the inner product of $\mathbb{H}$ and $\mathbb{K}$ without any confusion. Let $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ be a complete filtered probability space satisfying that $\mathcal{F}_0$ contains all $\mathbb{P}$-null sets of $\mathcal{F}$. An $\mathbb{H}-$valued random variable is an $\mathcal{F}$- measurable function $x(t):\Omega\to\mathbb{H}$ and a collection of random variables $S=\{x(t,\omega):\Omega\to\mathbb{H}\setminus t\in J\}$ is called stochastic process. Usually we write $x(t)$ instead of $x(t,\omega)$ and $x(t):J\to\mathbb{H}$ in the space of S. $\mathbb{W}=(\mathbb{W}_t)_{t\geq 0}$ be a $\mathbb{Q}$-Wiener process defined on $(\Omega,\mathcal{F},\{\mathcal{F}_t\}_{t\geq 0},\mathbb{P})$ with the covariance operator $\mathbb{Q}$ such that $Tr\mathbb{Q}<\infty$. We assume that there exists a complete orthonormal system $\{e_k\}_{k\geq 1}$ in $\mathbb{K}$, a bounded sequence of nonnegative real numbers λ_k such that $\mathbb{Q}e_k=\lambda_k e_k,\ k=1,2,\ldots,$ and a sequence of independent Brownian motions $\{\beta_k\}_{k\geq 1}$ such that

$$(w(t),e)_{\mathbb{K}}=\sum_{k=1}^{\infty}\sqrt{\lambda_k}(e_k,e)_{\mathbb{K}}\beta_k(t),e\in\mathbb{K},t\geq 0.$$

Let $\mathcal{L}_0^2 = \mathcal{L}^2(\mathbb{Q}^{\frac{1}{2}}\mathbb{K}, \mathbb{H})$ be the space of all Hilbert Schmidt operators from $\mathbb{Q}^{\frac{1}{2}}\mathbb{K}$ to $\mathbb{H}$ with the inner product $< \varphi, \psi >_{\mathcal{L}_0^2} = Tr[\varphi \mathbb{Q} \psi *]$.

The collection of all strongly measurable, square integrable, $\mathbb{H}$-valued random variables, denoted by

$$\mathcal{L}^2(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P}; \mathbb{H}) = \mathcal{L}^2(\Omega; \mathbb{H}),$$

is a Banach space equipped with norm $\|x(\cdot)\|^2_{\mathcal{L}^2} = E\|x(\cdot, w)\|^2_{\mathbb{H}}$, where E denotes expectation defined by $E(h) = \int_\Omega h(w)d\mathbb{P}$. An important subspace is given by $\mathcal{L}_0^2(\Omega; \mathbb{H}) = \{f \in \mathcal{L}^2(\Omega, \mathbb{H}) : f$ is $\mathcal{F}_0$- is measurable$\}$.

Let $PC^0_{\mathcal{L}} = C([-d, 0], \mathcal{L}^2(\Omega; \mathbb{H}))$ be a Banach space of all continuous map from $[-d, 0]$ into $\mathcal{L}^2(\Omega; \mathbb{H})$ satisfying the condition $\sup E\|\phi(t)\|^2 < \infty$ with norm

$$\|\phi\|_{PC^0_{\mathcal{L}}} = \sup_{t\in[-d,0]} \{E\|\phi(t)\|_{\mathbb{H}}\}, \phi \in PC^0_{\mathcal{L}}.$$

Consider $C^1(J, \mathcal{L}^2(\Omega; \mathbb{H}))$ be a Banach space of all continuously differentiable map from J into $\mathcal{L}^2(\Omega; \mathbb{H})$ satisfying the condition $\sup E\|x(t)\|^2 < \infty$ with norm defined

$$\|x\|^2_{C^1} = \sup_{t\in J} \sum_{j=0}^{1} \left\{E\|x^j(t)\|^2_{\mathbb{H}}\right\}, x \in C^1(J, \mathcal{L}^2(\Omega; \mathbb{H})).$$

To study the impulsive conditions, we consider

$$PC^2_{\mathcal{L}} = PC^1([-d, T], \mathcal{L}^2(\Omega; \mathbb{H})),$$

be a Banach space of all such continuous functions $x : [-d, T] \to \mathcal{L}^2(\Omega; \mathbb{H})$, which are continuously differentiable on $[0, T]$ except for a finite number of points $t_i \in (0, T),\ i = 1, 2, \ldots, \mathbb{N}$, at which $x'(t_i^+)$ and $x'(t_i^-) = x'(t_i)$ exists and endowed with the norm

$$\|x\|^2_{PC^2_{\mathcal{L}}} = \sup_{t\in J} \sum_{j=0}^{1} \left\{E\|x^j(t)\|^2_{\mathbb{H}}\right\}, x \in PC^2_{\mathcal{L}}.$$

Definition 1 *The Reimann-Liouville fractional integral operator for order $\alpha > 0$, of a function $f : \mathbb{R}^+ \to \mathbb{R}$ and $f \in L^1(\mathbb{R}^+, X)$ is defined by*

$$J_t^0 f(t) = f(t),\ J_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f(s) ds, \quad \alpha > 0,\ t > 0,$$

where $\Gamma(\cdot)$ is the Gamma function.

Definition 2 *Caputo's derivative of order $\alpha > 0$ for a function $f : [0, \infty) \to \mathbb{R}$ is defined as*

$$D_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t (t-s)^{n-\alpha-1} f^{(n)}(s) ds = J^{n-\alpha} f^{(n)}(t),$$

for $n - 1 < \alpha < n,\ n \in N$. If $0 < \alpha < 1$, then

$$D_t^\alpha f(t) = \frac{1}{\Gamma(1-\alpha)} \int_0^t (t-s)^{-\alpha} f^{(1)}(s) ds.$$

Obviously, Caputo's derivative of a constant is equal to zero.

We mention the statement of Krasnoselskii's fixed point theorem from [24] and using it to prove one of the result of this paper.

Theorem 1 *Let U be a closed convex and nonempty subset of a Banach space X. Let P and Q be two operators such that: (i) $Px + Qy \in U$ whenever $x, y \in U$,(ii) P is compact and continuous, (iii) Q is a contraction mapping. Then there exists $z \in U$ such that $z = Pz + Qz$.*

Definition 3 *A measurable $\mathcal{F}_t-$ adapted stochastic process $x : [-d,T] \to \mathbb{H}$ such that $x \in PC^2_{\mathcal{L}}$ is called a solution of the system (1)-(3) if $x(0) = \phi(0)$ and $x'(0) = x_1$ on $[-d,0], \Delta x|_{t=t_k} = I_k(x(t_k^-))$ and $\Delta x'|_{t=t_k} = Q_k(x(t_k^-))$, $k = 1, 2, \cdots, m$, the restriction of $x(\cdot)$ to the interval $[0,T)\backslash t_1,\ldots,t_m$, is continuous and $x(t)$ satisfies the following fractional integral equation*

$$x(t) = \begin{cases} \phi(0) + x_1 t - \int_0^t (t-s)^\beta h(s,x_s)ds + \int_0^t (t-s)f(s,x_s)ds \\ + \int_0^t (t-s)g(s,x_s)dw(s), & t \in (0,t_1], \\ \phi(0) + x_1 t + I_1(x(t_1^-)) + Q_1(x(t_1^-))(t-t_1) \\ - \int_0^t (t-s)^\beta h(s,x_s)ds + \int_0^t (t-s)f(s,x_s)ds \\ + \int_0^t (t-s)g(s,x_s)dw(s), & t \in (t_1,t_2], \\ \cdots \\ \phi(0) + x_1 t + \sum_{i=1}^k [I_i(x(t_i^-)) + Q_i(x(t_i^-))(t-t_i)] \\ - \int_0^t (t-s)^\beta h(s,x_s)ds + \int_0^t (t-s)f(s,x_s)ds \\ + \int_0^t (t-s)g(s,x_s)dw(s), & t \in (t_k,t_{k+1}], \end{cases} \tag{4}$$

for detailed steps of the result (4) one can see [16].

Further, we introduce the following assumptions to establish our results:

(H1) The nonlinear maps $f, h : J \times PC^0_{\mathcal{L}} \to \mathbb{H}$ and $g : J \times PC^0_{\mathcal{L}} \to \mathcal{L}(\mathbb{K},\mathbb{H})$ are continuous and there exits positive constants L_f, L_g, L_h such that

$$\begin{aligned} E\|f(t,\varphi) - f(t,\psi)\|^2_{\mathbb{H}} \le L_f\|\varphi - \psi\|^2_{PC^0_{\mathcal{L}}}, \\ E\|g(t,\varphi) - g(t,\psi)\|^2_{\mathbb{H}} \le L_g\|\varphi - \psi\|^2_{PC^0_{\mathcal{L}}}, \\ E\|h(t,\varphi) - h(t,\psi)\|^2_{\mathbb{H}} \le L_h\|\varphi - \psi\|^2_{PC^0_{\mathcal{L}}}, \end{aligned}$$

for all $t \in J$ and $\varphi, \psi \in PC^0_{\mathcal{L}}$.

(H2) The functions $I_k, Q_k : \mathbb{H} \to \mathbb{H}$ are continuous and there exists positive L_I, L_Q such that

$$\begin{aligned} E\|I_k(x) - I_k(y)\|^2_{\mathbb{H}} &\le L_I E\|x - y\|^2_{\mathbb{H}}, \\ E\|Q_k(x) - Q_k(y)\|^2_{\mathbb{H}} &\le L_Q E\|x - y\|^2_{\mathbb{H}}, \end{aligned}$$

for all $x, y \in \mathbb{H}$ and $k = 1, 2, \cdots, m$.

3 Existence and uniqueness results

This result is based on Banach contraction fixed point theory.

Theorem 2 *Suppose that the assumptions $(H1) - H2)$ hold and the following inequality*

$$\Theta = \left\{ 5(mL_I + mT^2L_Q) + 5T^2[\frac{T^{2\beta}}{(\beta+1)^2}L_h + \frac{T^2}{4}L_f + \frac{T}{3}L_g] \right\} < 1, \tag{5}$$

is satisfied, then the system (1)-(3) has a unique solution.

Proof. We convert the problem (1)-(3) into fixed point problem. Let us consider an operator $N : PC^2_{\mathcal{L}} \to PC^2_{\mathcal{L}}$ defined as

$$(Nx)(t) = \begin{cases} \phi(0) + x_1 t - \int_0^t (t-s)^\beta h(s,x_s)ds \\ + \int_0^t (t-s) f(s,x_s)ds + \int_0^t (t-s) g(s,x_s)dw(s), & t \in (0,t_1], \\ \phi(0) + x_1 t + I_1(x(t_1^-)) + Q_1(x(t_1^-))(t-t_1) \\ - \int_0^t (t-s)^\beta h(s,x_s)ds + \int_0^t (t-s) f(s,x_s)ds \\ + \int_0^t (t-s) g(s,x_s)dw(s), & t \in (t_1,t_2], \\ \cdots \\ \phi(0) + x_1 t + \sum_{i=1}^k [I_i(x(t_i^-)) + Q_i(x(t_i^-))(t-t_i)] \\ - \int_0^t (t-s)^\beta h(s,x_s)ds + \int_0^t (t-s) f(s,x_s)ds \\ + \int_0^t (t-s) g(s,x_s)dw(s), & t \in (t_k,t_{k+1}]. \end{cases} \tag{6}$$

Now we show that N is a contraction map. For this we take two points x, x^* such that for $t \in (0,t_1]$, we have

$$\begin{aligned} E\|(Nx)(t) - (Nx^*)(t)\|^2_{\mathbb{H}} &\le 3E\| \int_0^t (t-s)^\beta [h(s,x_s) - h(s,x_s^*)]ds\|^2_{\mathbb{H}} \\ &\quad + 3E\| \int_0^t (t-s)[f(s,x_s) - f(s,x_s^*)]ds\|^2_{\mathbb{H}} \\ &\quad + 3E\| \int_0^t (t-s)[g(s,x_s) - g(s,x_s^*)]dw(s)\|^2_{\mathbb{H}} \\ &\le 3T^2[\frac{T^{2\beta}}{(\beta+1)^2} L_h + \frac{T^2}{4} L_f + \frac{T}{3} L_g]\|x - x^*\|^2_{PC^2_{\mathcal{L}}}. \end{aligned}$$

On similar ground, when $t \in (t_1,t_2]$, we obtain the following estimate

$$\begin{aligned} E\|(Nx)(t) - (Nx^*)(t)\|^2_{\mathbb{H}} &\le 5E\|I_1(x(t_1^-)) - I_1(x^*(t_1^-))\|^2_{\mathbb{H}} \\ &\quad + 5E\| Q_1(x(t_1^-))(t-t_1) - Q_1(x^*(t_1^-))(t-t_1)\|^2_{\mathbb{H}} \\ &\quad + 5E\| \int_0^t (t-s)^\beta [h(s,x_s) - h(s,x_s^*)]ds\|^2_{\mathbb{H}} \\ &\quad + 5E\| \int_0^t (t-s)[f(s,x_s) - f(s,x_s^*)]ds\|^2_{\mathbb{H}} \\ &\quad + 5E\| \int_0^t (t-s)[g(s,x_s) - g(s,x_s^*)]dw(s)\|^2_{\mathbb{H}} \\ &\le \Big\{ 5(L_I + T^2 L_Q) + 5T^2[\frac{T^{2\beta}}{(\beta+1)^2} L_h \\ &\quad + \frac{T^2}{4} L_f + \frac{T}{3} L_g] \Big\} \|x - x^*\|^2_{PC^2_{\mathcal{L}}}. \end{aligned}$$

Similarly for $t \in (t_k,t_{k+1}], \;\; k = 2,3,\cdots m$, we may estimate as

$$\begin{aligned} E\|(Nx)(t) - (Nx^*)(t)\|^2_{\mathbb{H}} &\le \Big\{ 5(mL_I + mT^2 L_Q) + 5T^2[\frac{T^{2\beta}}{(\beta+1)^2} L_h \\ &\quad + \frac{T^2}{4} L_f + \frac{T}{3} L_g] \Big\} \|x - x^*\|^2_{PC^2_{\mathcal{L}}} \\ &= \Theta \|x - x^*\|^2_{PC^2_{\mathcal{L}}}. \end{aligned}$$

Since $\Theta < 1$, this implies N is a contraction map and has a unique fixed point $x \in PC^2_{\mathcal{L}}$ which become the solution of the problem (1)-(3) on J. This completes the proof of the theorem. ■

Second result is based on the Krasnoselskii's fixed point theorem, for this we take the following assumptions.

(H3) $f, h : J \times PC^0_{\mathcal{L}} \to \mathbb{H}$ and $g : J \times PC^0_{\mathcal{L}} \to \mathcal{L}(\mathbb{K}, \mathbb{H})$ are continuous and there exists continuous functions $\mu_1, \mu_2, \mu_3 : J \to (0, \infty)$ such that

$$E\|f(t,\phi)\|^2_{\mathbb{H}} \leq \mu_1(t)\|\phi\|^2_{PC^0_{\mathcal{L}}},$$
$$E\|g(t,\phi)\|^2_{\mathbb{H}} \leq \mu_2(t)\|\phi\|^2_{PC^0_{\mathcal{L}}},$$
$$E\|h(t,\phi)\|^2_{\mathbb{H}} \leq \mu_3(t)\|\phi\|^2_{PC^0_{\mathcal{L}}},$$

where $t \in J$ and $\phi \in PC^0_{\mathcal{L}}$.

(H4) The functions $I_k, Q_k : \mathbb{H} \to \mathbb{H}$ are continuous and there exists positive constants Δ, ∇ such that

$$\Delta = \max_{1\leq k\leq m,\ x\in\mathbb{H}} \{E\|I_k(x)\|^2_{\mathbb{H}}\},\ \nabla = \max_{1\leq k\leq m,\ x\in\mathbb{H}} \{E\|Q_k(x)\|^2_{\mathbb{H}}\}.$$

Theorem 3 *Let the assumptions $(H1), (H3) - (H4)$ hold with following inequalities*

$$6\left\{[\phi(0) + x_1 T] + \triangle m + \nabla m T^2 + T^2\left[\frac{T^{2\beta}}{(\beta+1)^2}\mu_3^* + \frac{T^2}{4}\mu_1^* + \frac{T}{3}\mu_2^*\right]\right\} \leq q,$$
$$3T^2\left[\frac{T^{2\beta}}{(\beta+1)^2}L_h + \frac{T^2}{4}L_f + \frac{T}{3}L_g\right] < 1, \tag{7}$$

where q is a positive real constant, $\mu_1^* = \sup_{s\in[0,t]}\mu_1(s)$, $\mu_2^* = \sup_{s\in[0,t]}\mu_2(s)$, and $\mu_3^* = \sup_{s\in[0,t]}\mu_3(s)$. Then the system (1)-(3) has at least one solution on J.

Proof. Let us consider the operators $\psi_1, \psi_2 : PC^2_{\mathcal{L}} \to PC^2_{\mathcal{L}}$ defined as

$$(\psi_1 x)(t) = \begin{cases} \phi(0) + x_1 t, & t \in (0, t_1], \\ \phi(0) + x_1 t + I_1(x(t_1^-)) + Q_1(x(t_1^-))(t - t_1), & t \in (t_1, t_2], \\ \cdots \\ \phi(0) + x_1 t + \sum_{i=1}^k [I_i(x(t_i^-)) + Q_i(x(t_i^-))(t - t_i)], & t \in (t_k, t_{k+1}], \end{cases}$$

and

$$(\psi_2 x)(t) = \begin{cases} \int_0^t (t-s)^\beta h(s, x_s)ds + \int_0^t (t-s) f(s, x_s)ds \\ + \int_0^t (t-s) g(s, x_s)dw(s), & t \in (0, t_1], \\ \cdots \\ \int_0^t (t-s)^\beta h(s, x_s)ds + \int_0^t (t-s) f(s, x_s)ds \\ + \int_0^t (t-s) g(s, x_s)dw(s), & t \in (t_k, t_{k+1}]. \end{cases}$$

In order to use Theorem 1 we will verify that ψ_1 is compact and continuous while ψ_2 is a contraction operator. For the sake of convenience, we divide the proof into several steps.

Step 1. We show that $\psi_1 x + \psi_2 x^* \in PC^2_{\mathcal{L}}$ for $x, x^* \in PC^2_{\mathcal{L}}$. For $t \in (0, t_1]$, we have

$$\begin{aligned} E\|(\psi_1 x)(t) + (\psi_2 x^*)(t)\|^2_{\mathbb{H}} \leq\ & \{4[\phi(0) + x_1 t] \\ & +4T^2[\frac{T^{2\beta}}{(\beta+1)^2}\mu_3^* + \frac{T^2}{4}\mu_1^* + \frac{T}{3}\mu_2^*]\}\|x\|^2_{PC^2_{\mathcal{L}}}. \end{aligned}$$

Similarly, for $t \in (t_k, t_{k+1}], k = 1, 2, \ldots, m$, we get

$$\begin{aligned} E\|(\psi_1 x)(t) + (\psi_2 x^*)(t)\|^2_{\mathbb{H}} \leq\ & \{6[\phi(0) + x_1 t] + 6\triangle m + 6\nabla m T^2 + \\ & +6T^2[\frac{T^{2\beta}}{(\beta+1)^2}\mu_3^* + \frac{T^2}{4}\mu_1^* + \frac{T}{3}\mu_2^*]\}\|x\|^2_{PC^2_{\mathcal{L}}} \\ \leq\ & q. \end{aligned}$$

This implies that $\|(\psi_1 x)(t)+(\psi_2 x^*)(t)\|_{PC^2_{\mathcal{L}}} \leq q$, means $(\psi_1 x)(t)+(\psi_2 x^*)(t) \in PC^2_{\mathcal{L}}$.

Step 2. Here we show that the map ψ_1 is continuous on $PC^2_{\mathcal{L}}$. Let $\{x^n\}_{n=1}^{\infty}$ be a sequence in $PC^2_{\mathcal{L}}$ with $\lim x^n \to x \in PC^2_{\mathcal{L}}$. Then for $t \in (t_k, t_{k+1}], k = 0, 1, \ldots, m$, we have

$$\begin{aligned} E\|(\psi_1 x^n)(t)-(\psi_1 x)(t)\|^2_{\mathbb{H}} &\leq 2mE\|I_i(x^n(t_i^-)) - I_i(x(t_i^-))(t)\|^2_{\mathbb{H}} \\ &\quad +2mT^2E\|Q_i(x^n(t_i^-)) - Q_i(x(t_i^-))(t)\|^2_{\mathbb{H}}, \end{aligned}$$

since the functions $I_i, Q_i, i = 1, 2, ..., m$, are continuous hence $\lim_{n\to\infty} E\|\psi_1 x^n - \psi_1 x\|^2_{\mathbb{H}} = 0$, which implies that the mapping ψ_1 is continuous on $PC^2_{\mathcal{L}}$.

Step 3. Now we show that ψ_1 maps bounded sets into bounded sets in $PC^2_{\mathcal{L}}$.
Let us prove that for $q > 0$ there exists $\widehat{r} > 0$ such that for each $x \in PC^2_{\mathcal{L}}$, we have $E\|(\psi_1 x)(t)\|^2_{\mathbb{H}} \leq \widehat{r}$ for $t \in (t_k, t_{k+1}],\ k = 0, 1, \ldots, m$. Now, we have

$$\begin{aligned} E\|(\psi_1 x)(t)\|^2_{\mathbb{H}} &\leq 3E\|\phi(0)+x_1 t\|^2_{\mathbb{H}} \\ &\quad +3E\sum_{i=1}^{k}\|I_i(x(t_i^-))\|^2_{\mathbb{H}} + 3E\sum_{i=1}^{k}\|(t-t_i)Q_i(x(t_i^-))\|^2_{\mathbb{H}} \\ &\leq 3(\phi(0)+x_1 T) + 3m\Delta + 3mT^2\nabla = \widehat{r}, \end{aligned}$$

which proves the desired result.

Step 4. The map ψ_1 is equicontinuous.

Let $u, v \in (t_k, t_{k+1}], t_k \leq u < v \leq t_{k+1},\ k = 0, 1, 2, \ldots, m, x \in PC^2_{\mathcal{L}}$ we obtain

$$\begin{aligned} &E\|(\psi_1 x)(v) - (\psi_1 x)(u)\|^2_{\mathbb{H}} \\ &\leq 2E\|(v-u)x_1\|^2_{\mathbb{H}} + 2m\|(v-t_i)-(u-t_i)\|^2 E\|Q_i(x(t_i^-))\|^2_{\mathbb{H}}. \end{aligned}$$

As $v \to u$, then $\lim_{u\to v} E\|(\psi_1 x)(v) - (\psi_1 x)(u)\|^2_{\mathbb{H}} = 0$, which implies that ψ_1 is equicontinuous. Finally, combining Step 1 to Step 4 together with Ascoli's theorem, we conclude that the operator ψ_1 is compact.

Step 5. Now, we show that the map ψ_2 is a contraction mapping. Let $x, x^* \in PC^2_{\mathcal{L}}$ and $t \in (t_k, t_{k+1}],\ k = 0, 1, \ldots, m$, we have

$$\begin{aligned} E\|(\psi_2 x)(t) - (\psi_2 x^*)(t)\|^2_{\mathbb{H}} &\leq 3E\|\int_0^t (t-s)^{\beta}[h(s,x_s) - h(s,x_s^*)]ds\|^2_{\mathbb{H}} \\ &\quad +3E\|\int_0^t (t-s)[f(s,x_s) - f(s,x_s^*)]ds\|^2_{\mathbb{H}} \\ &\quad +3E\|\int_0^t (t-s)[g(s,x_s) - g(s,x_s^*)]dw(s)\|^2_{\mathbb{H}}, \\ &\leq 3T^2[\frac{T^{2\beta}}{(\beta+1)^2}L_h + \frac{T^2}{4}L_f + \frac{T}{3}L_g]\|x - x^*\|^2_{PC^2_{\mathcal{L}}}. \end{aligned}$$

By the condition given in the equation (7), we obtain that ψ_2 is a contraction mapping. Therefore, by Krasnoselskii's fixed point theorem we can conclude that the problem (1)-(3) has at least one solution on J. This completes the proof of the theorem. ■

4 Example

We consider the following partial differential equation with fractional derivative of the form

$$\frac{\partial^\alpha}{\partial t^\alpha}\left[u(t,x)+\int_0^t (t-s)\frac{\|u(s-d,x)\|}{25+\|u(s-d,x)\|}ds\right]=\frac{1}{\Gamma(2-\alpha)}\int_0^t (t-s)^{1-\alpha}$$
$$\times\left[\frac{\|u(s-d,x)\|}{36+\|u(s-d,x)\|}ds+\frac{\|u(s-d,x)\|}{49+\|u(s-d,x)\|}dw(s)\right],$$
$$t\in[0,1],\ x\in(0,\pi),\ t\neq\frac{1}{2}. \tag{8}$$
$$u(t,x)=\phi(t,x),\ u'(0,x)=0,\ t\in[-d,0],\ x\in[0,\pi], \tag{9}$$
$$u(t,0)=u(t,\pi)=0,\ u'(t,0)=u'(t,\pi)=0,\ t\geq 0, \tag{10}$$
$$\Delta u|_{t=\frac{1}{2}^-}=\sin(\frac{1}{20}\|u(\frac{1}{2}^-,x)\|);\ \Delta' u|_{t=\frac{1}{2}^-}=\cos(\frac{1}{20}\|u(\frac{1}{2}^-,x)\|), \tag{11}$$

where $\frac{\partial^\alpha}{\partial t^\alpha}$ is Caputo's fractional derivative of order $\alpha\in(1,2),\ 0<t_1<1$ are prefixed numbers and $\phi\in PC_{\mathcal{L}^2}$. Let $X=L^2[0,\pi]$ and setting $u(t)(x)=u(t,x)$, such that

$$\begin{aligned}
E\|f(t,x_t)-f(t,y_t)\|^2_{\mathbb{H}} &\leq \frac{1}{36}E\|x-y\|^2_{\mathbb{H}},\\
E\|g(t,x_t)-g(t,y_t)\|^2_{\mathbb{H}} &\leq \frac{1}{49}E\|x-y\|^2_{\mathbb{H}},\\
E\|h(t,x_t)-h(t,y_t)\|^2_{\mathbb{H}} &\leq \frac{1}{25}E\|x-y\|^2_{\mathbb{H}},\\
E\|I_k(x(t^-))-I_k(y(t^-))\|^2_{\mathbb{H}} &\leq \frac{1}{20}E\|x-y\|^2_{\mathbb{H}},\\
E\|Q_k(x(t^-))-Q_k(y(t^-))\|^2_{\mathbb{H}} &\leq \frac{1}{20}E\|x-y\|^2_{\mathbb{H}},
\end{aligned}$$

then with these settings the problem (8)-(11) can be rewritten in the abstract form of the equations (1)-(3). Further more, we have $L_f=\frac{1}{36},\ L_g=\frac{1}{49},\ L_h=\frac{1}{25},\ L_I=\frac{1}{20},\ L_Q=\frac{1}{20},\beta=1,\ m=1,\ T=1$, put these values in the condition given in the Theorem 2 as

$$\Theta=\left\{5(mL_I+mT^2L_Q)+5T^2[\frac{T^{2\beta}}{(\beta+1)^2}L_h+\frac{T^2}{4}L_f+\frac{T}{3}L_g]\right\},$$

we get $\Theta=0.61<1$, which implies that the problem (8)-(11) has a unique solution in $[0,1]$.

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