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Threshold Dynamics for a Pertussis Model with Seasonality

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Abstract: In this paper, a periodic epidemic model is proposed in order to simulate the dynamics of pertussis transmission. We consider the waning of vaccine-induced immunity in the population. The basic reproduction number R_0 is defined. It is shown that the disease-free equilibrium is globally asymptotically stable and the disease eventually disappears if $R_0 < 1$, and there exists at least one positive periodic solution and the disease is uniformly persistent if $R_0 > 1$. Numerical simulations are provided to illustrate analytical results.

Keywords:basic reproductive number, periodic solution, seasonal fluctuation, global asymptotic stability, uniform persistence.

1 Introduction

Pertussis is a highly contagious bacterial disease of the respiratory tract, caused by Bordetella pertussis. At the beginning of the twentieth century, before vaccines were widely used, epidemics of pertussis were observed to recur at intervals of 2-3 years[1]. Since the introduction of vaccination in the 1950s, the incidence of pertussis has strongly decreased. However, even in countries with high vaccination coverage, pertussis shows epidemic peaks every 3-4 years[2]. It occurs mainly in infants and young children, and is easily transmitted from person to person, mainly through droplets. The first symptoms generally appear 7-10 days after infection, and include mild fever, runny nose, and cough, which in typical cases gradually develops into a paroxysmal cough followed by whooping (hence the common name of whooping cough).

Pertussis (whooping cough) is an important cause of infant death worldwide and continues to be a public health concern even in countries with high vaccination coverage. In the United States in the prevaccine era, reported pertussis was a universally present disease with cyclic peaks, which occurred every 2-5 years[3–5]. Estimates from WHO suggest that[6], in 2008, about 16 million cases of pertussis occurred worldwide, 95% of which were in developing countries, and that about 195 000 children died from the disease. For several decades, infant immunization programmes using pertussis vaccines of documented quality have been highly successful in preventing severe pertussis in infants all over the world.

Several reasons for the resurgence of reported pertussis in recent years have been proposed[7, 8]. These reasons include the following: genetic changes in B. pertussis, lessened potency of pertussis vaccines, waning of vaccine-induced immunity, greater awareness of pertussis, and the general availability of better laboratory tests. Of these possibilities, it is clear that what is most important is a greater awareness of pertussis.

The literature on the mathematical modeling of the transmission of pertussis is rather scant. Most of the existing studies are of clinical aspects. Herbert W. Hethcote [9] used age structured pertussis transmission models include waning of both infection-acquired and vaccine-induced immunity to simulate the pertussis epidemiology in the United States. Hanh T. H. Nguyen and Pejman Rohani[10] used household data on the incubation period to parameterize more realistic distributions of the latent and infectious periods. Shaheen Abdullah Abdulkareem [11] used the agent-based model simulate the spread of pertussis in Enschede region.

In the current work, we consider the waning of vaccine-induced immunity in our model. The aim of our study is to use mathematical modeling to gain some insights into the transmission dynamics of pertussis and to investigate the vaccination effects by analyzing the global dynamics of the model.

The paper is structured as follows. In Section 2, we present a periodic epidemic model in order to simulate the dynamics of pertussis transmission. And we define the basic reproduction number R_0. In Section 3, we obtain the global properties of the proposed model. There is a unique disease-free equilibrium and the disease always dies out if

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$R_0 < 1$; while the disease uniformly persists in the population and there is at least one positive periodic solution if $R_0 > 1$. Numerical simulation are provided to validate analytical results in section 4. In the final section, we give the brief conclusions.

2 Model formulation

In our model, we assume that vaccination effects is waning, that is to say, vaccinated compartment is infected contact with the infectious. Moreover, some studies report pertussis has seasonality which is not consistent in time and place [12–14], we also assume that the transmission is seasonal.

We split the total population (denoted by N) into five compartments, which are the susceptible to disease, S; the latent/exposed, E; the infective, I; the recovered individuals with acquired temporary immunity, R; and the vaccinated susceptible individuals V. The total population are denoted by $N = S + E + I + R + V$.

The differential equations for pertussis model are

$$\begin{cases} \frac{dS}{dt} = \Lambda - \frac{\beta_1(t)SI}{N} - \mu S - pS + \delta R, \\ \frac{dE}{dt} = \frac{\beta_1(t)SI}{N} + \frac{\beta_2(t)VI}{N} - (\mu + k)E, \\ \frac{dI}{dt} = kE - (\mu + \alpha + \gamma)I, \\ \frac{dR}{dt} = \gamma I - \mu R - \delta R, \\ \frac{dV}{dt} = pS - \frac{\beta_2(t)VI}{N} - \mu V, \\ N = S + E + I + R + V, \end{cases} \tag{1}$$

with initial condition $(S(0), E(0), I(0), R(0), V(0)) = (S_0, E_0, I_0, R_0, V_0) \in \mathbb{R}^5_+$. Here Λ is the recruitment rate of susceptible individuals into the population by birth or by immigration and μ is the natural death rate. The average exposed and infectious periods are given by $1/k$ and $1/\gamma$, respectively. p is the fraction of the susceptible vaccinated, α represent the disease-induced death rate. $\beta_1(t)$ and $\beta_2(t)$ are the transmission rate coefficient of the susceptible and the vaccinated, respectively. δ is the progression rate of the recovered individuals R. We assume that $\beta_i(t)(i = 1, 2)$ is periodic positive continuous functions in t with period ω for some $\omega > 0$, other parameters are positive constant.

It is obvious that any solution of system (1) with nonnegative initial values is unique and nonnegative.

Lemma 1 *System* (1) *has a unique and bounded solution with the initial value*

$$(S_0, E_0, I_0, R_0, V_0) \in X := \mathbb{R}^5_+.$$

Further, the compact set

$$G := \{(S, E, I, R, V) \in X \ : \ S + E + I + R + V \leq \Lambda/\mu\}$$

is positively invariant set, which attracts all positive orbits in X.

Proof. From (1), the total population $N(t)$ satisfies the following equation:

$$\frac{dN}{dt} = \Lambda - \alpha I - \mu N \leq \Lambda - \mu N.$$

It is easy to see that the linear differential equation $\frac{dN}{dt} = \Lambda - \mu N$ has a unique equilibrium $N_0 = \Lambda/\mu$, which is globally asymptotically stable. The comparison principle [15, Theorem B.1] implies that $N(t)$ is ultimately bounded, and hence, the solutions of system (1) exist globally on the interval $[0, \infty)$.

And $\frac{dN}{dt} \leq \Lambda - \mu N \leq 0$. if $N(t) \geq \frac{\Lambda}{\mu}$, which implies that G is positively invariant with respect to system (1). This proves the lemma. ■

Let $(\mathbb{R}^n, \mathbb{R}^n_+)$ be the standard ordered $n-$dimensional Euclidean space with a norm $\| \cdot \|$. For $u, v \in \mathbb{R}^n$, we denote $u \geq v$, if $u - v \in \mathbb{R}^n_+$; $u > v$, if $u - v \in \mathbb{R}^n_+ \backslash \{0\}$; and $u \gg v$, if $u - v \in Int(\mathbb{R}^n_+)$.

Let A(t) be a continuous, cooperative, irreducible, and periodic $n \times n$ matrix function with period $\omega > 0$, $\Phi_A(t)$ be the fundamental solution matrix of the linear ordinary differential equation

$$\dot{x} = A(t)x.$$

Let $r(\Phi_A(\omega))$ be the spectral radius of $\Phi_A(\omega)$. By Perron-Frobenius theorem, $r(\Phi_A(\omega))$ is the principle eigenvalue of $\Phi_A(\omega)$, in the sense that it is simple and admits an eigenvector $v^* \gg 0$. We present the following lemma from [16] for our discussion in the next section.

Lemma 2 *Let $\rho = \frac{1}{\omega}\ln r(\Phi_A(\omega))$. Then there exists a positive $\omega-$periodic function $v(t)$ such that $e^{\rho t}v(t)$ is a solution of $\dot{x} = A(t)x$.*

It is easy to see that system (1) has exactly one disease-free equilibrium $P_0(S_0, E_0, I_0, R_0, V_0) = (\Lambda/(\mu+p), 0, 0, 0, \Lambda p/(\mu(\mu+p)))$. In what follows, we introduce the basic reproduction number R_0 for system (1) according to the general procedure presented in [17]. We obtain

$$F(t) = \begin{pmatrix} 0 & \beta_1(t)\frac{\mu}{\mu+p} + \beta_2(t)\frac{p}{\mu+p} \\ 0 & 0 \end{pmatrix},$$

and

$$V(t) = \begin{pmatrix} \mu+k & 0 \\ -k & \mu+\alpha+\gamma \end{pmatrix}.$$

Assume $Y(t,s), t \geq s$, is the matrix solution of the linear ω-periodic system

$$\frac{dy}{dt} = -V(t)y. \tag{2}$$

That is, for each $s \in \mathbb{R}$, the 2×2 matrix $Y(t,s)$ satisfies

$$\frac{d}{dt}Y(t,s) = -V(t)Y(t,s), \ \ \forall\, t \geq s, \quad Y(s,s) = I,$$

where I is the 2×2 identity matrix. Thus, the monodromy matrix $\Phi_{-V}(t)$ of (2) is equal to $Y(t,0), t \geq 0$.

Let C_ω be the ordered Banach space of all ω-periodic functions from $\mathbb{R}$ to $\mathbb{R}^2$, which is equipped with the maximum norm $||\cdot||$ and the positive cone $C_\omega^+ := \{\phi \in C_\omega \ : \ \phi(t) \geq 0, \forall\, t \in \mathbb{R}\}$. Then we can define a linear operator $L : C_\omega \to C_\omega$ by

$$(L\phi)(t) = \int_0^\infty Y(t, t-a)F(t-a)\phi(t-a)da, \quad \forall\, t \in \mathbb{R}, \quad \phi \in C_\omega. \tag{3}$$

Following [17], we call L the next infection operator, and define the basic reproduction number as $R_0 := r(L)$, the spectral radius of L.

It is easy to verify that system (1) satisfies assumptions $(A1)$-$(A7)$ in [17]. Thus, we have the following result, which will be used in the proof of our main result in section 3.

Lemma 3 *([17, Theorem 2.2]) The following statements are valid:*

(i) $R_0 = 1$ *if and only if* $r(\Phi_{F-V}(\omega)) = 1$.

(ii) $R_0 > 1$ *if and only if* $r(\Phi_{F-V}(\omega)) > 1$.

(iii) $R_0 < 1$ *if and only if* $r(\Phi_{F-V}(\omega)) < 1$.

Thus, the disease-free equilibrium P_0 is locally asymptotically stable if $R_0 < 1$, and unstable if $R_0 > 1$.

3 Threshold dynamics

In this section, we will use the method developed in [17] to analyze the threshold dynamics of system (1).

Theorem 4 *If the basic reproduction number $R_0 < 1$, then the unique disease-free equilibrium $P_0(\Lambda/(\mu+p), 0, 0, 0, \Lambda p/(\mu(\mu+p)))$ is globally asymptotically stable and if $R_0 > 1$, it is unstable.*

Proof. From Lemma 3, we know that if $R_0 < 1$, then P_0 is locally asymptotically stable and if $R_0 > 1$, P_0 is unstable. We now prove the global attractivity of P_0 for $R_0 < 1$.

If $R_0 < 1$, Lemma 3 implies $r(\Phi_{F-V}(\omega)) < 1$. We can choose $\eta > 0$ small enough such that $r(\Phi_{F_\eta - V}(\omega)) < 1$, where

$$F_\eta(t) = \begin{pmatrix} 0 & \beta_1(t)\frac{S_0+\eta}{N_0-\eta} + \beta_2(t)\frac{V_0+\eta}{N_0-\eta} \\ 0 & 0 \end{pmatrix}.$$

Equations

$$\frac{dS}{dt} = \Lambda - \mu S - pS, \frac{dV}{dt} = pS - \mu V$$

has a unique equilibrium $(S_0, V_0) = (\frac{\Lambda}{\mu+p}, \frac{\Lambda p}{\mu(\mu+p)})$ which is globally attractive in $\mathbb{R}^2_+$.

From the first and last equations of system (1), we have

$$\frac{dS}{dt} \le \Lambda - \mu S - pS, \frac{dV}{dt} = pS - \mu V$$

Furthermore, we obtain $S(t) \le S_0 + \eta$, $V(t) \le V_0 + \eta$ and $N(t) \ge \frac{\Lambda}{\mu} - \eta$, for sufficiently large t. From the second and third equations of system (1), for sufficiently large t, we have

$$\begin{cases} \frac{dE}{dt} \le \frac{\beta_1(t)(S_0+\eta)I}{N_0-\eta} + \frac{\beta_2(t)(V_0+\eta)I}{N_0-\eta} - (\mu+k)E, \\ \frac{dI}{dt} = kE - (\mu+\alpha+\gamma)I. \end{cases} \tag{4}$$

Consider the following auxiliary system

$$\frac{dh(t)}{dt} = (F_\eta(t) - V(t))h(t). \tag{5}$$

By Lemma 2 and the standard comparison principle, there exists a positive, ω-periodic function $\bar{h}(t)$ such that $J(t) \le e^{\theta t}\bar{h}(t)$ where $\theta = \frac{1}{\omega}\ln r(\Phi_{F_\eta - V}(\omega)) < 0$ and $J(t) = (E(t), I(t))^T$. Therefore, we get $E(t) \to 0$ and $I(t) \to 0$ as $t \to +\infty$. By the theory of asymptotic autonomous systems [18, Theorem 1.2], it then follows that $S(t) \to S_0$, $R(t) \to 0$ and $V(t) \to V_0$, as $t \to +\infty$. Therefore, the disease-free equilibrium $P_0(\Lambda/(\mu+p), 0, 0, 0, \Lambda p/(\mu(\mu+p)))$ is globally asymptotically stable. ■

Theorem 5 *If the basic reproduction number $R_0 > 1$, then the disease is uniformly persistent, i.e., there exists a $\delta > 0$ such that any solution $(S(t), E(t), I(t), R(t), V(t))$ of system (1) with initial value $(S^0, E^0, I^0, R^0, V^0) \in \{(S, E, I, R, V) \in X : E > 0, I > 0, \}$ satisfies*

$$\liminf_{t\to+\infty} E(t) \ge \delta, \text{and } \liminf_{t\to+\infty} I(t) \ge \delta,$$

and system (1) admits at least one positive periodic solution.

Proof. Define

$$X_0 := \{(S, E, I, R, V) \in X : E > 0, I > 0\}, \qquad \partial X_0 := X \backslash X_0.$$

Let P: $X \to X$ be the Poincaré map associated with system (1), i.e.,

$$P(x^0) = u(\omega, x^0), \qquad \forall\, x^0 \in X,$$

where $u(t, x^0)$ is the unique solution of system (1) with $u(0, x^0) = x^0$. It is easy to see that

$$P^m(S^0, E^0, I^0, R^0, V^0) = u(m\omega, (S^0, E^0, I^0, R^0, V^0)), \quad \forall m \ge 0.$$

We now prove that P is uniformly persistent with respect to $(X_0, \partial X_0)$. It is easy to show that X and X_0 are positively invariant, ∂X_0 is a relatively closed set in X, and P is point dissipative from Lemma 1.

Set

$$M_\partial := \{(S^0, E^0, I^0, R^0, V^0) \in \partial X_0 \ : \ P^m(S^0, E^0, I^0, R^0, V^0) \in \partial X_0, \forall\ m \geq 0\}.$$

We now show that

$$M_\partial = \{(S, 0, 0, R, V) \in X \ : \ S \geq 0, R \geq 0, V \geq 0\}. \tag{6}$$

In fact, it is obvious that

$$\{(S, 0, 0, R, V) \in X \ : \ S \geq 0, R \geq 0, V \geq 0\} \subseteq M_\partial. \tag{7}$$

For any $(S^0, E^0, I^0, R^0, V^0) \in \partial X_0 \backslash \{(S, 0, 0, R, V) : S \geq 0, R \geq 0, V \geq 0\}$, if $E^0 = 0, I^0 > 0$, it is clear that $S(t) > 0, I(t) > 0$ for all $t > 0$, from the second equation of (1), we have

$$E(t) = [E^0 + \int_0^t \frac{\beta_1(\sigma)S(\sigma)I(\sigma) + \beta_2(\sigma)V(\sigma)I(\sigma)}{N(\sigma)} e^{(\mu+k)\sigma} d\sigma] \times e^{-(\mu+k)t} > 0.$$

for any $t > 0$. For the case $I^0 = 0, E^0 > 0$, then $E(t) > 0$ for any $t > 0$, $\frac{dI}{dt}(0) = kE^0 > 0$.
Therefore, $(S(t), E(t), I(t), R(t), V(t)) \notin X_0$ for $t > 0$ sufficiently small. That is to say, for any $(S^0, E^0, I^0, R^0, V^0) \notin \{(S, 0, 0, R, V) \in X \ : \ S \geq 0, R \geq 0, V \geq 0\}$, $(S^0, E^0, I^0, R^0, V^0) \notin M_\partial$. This implies that

$$M_\partial \subseteq \{(S, 0, 0, R, V) \in X \ : \ S \geq 0, R \geq 0, V \geq 0\}.$$

Clearly, there is exactly one fixed point $P_0 = (\Lambda/(\mu + p), 0, 0, 0, \Lambda p/(\mu(\mu + p)))$ of P in M_∂.

Next, we prove that P is uniformly persistent with respect to $(X_0, \partial X)$. In the case where $R_0 > 1$, we have the following claims:

Claim:*There exists a $\sigma^* > 0$, such that for any $(S^0, E^0, I^0, R^0, V^0) \in X_0$ with $||(S^0, E^0, I^0, R^0, V^0) - P_0|| \leq \sigma^*$, we have*

$$\limsup_{m\to\infty} d(P^m(S^0, E^0, I^0, R^0, V^0), P_0) \geq \sigma^*. \tag{8}$$

Suppose, by contradiction, that

$$\limsup_{m\to\infty} d(P^m(S^0, E^0, I^0, R^0, V^0), P_0) < \sigma^*$$

for some $(S^0, E^0, I^0, R^0, V^0) \in X_0$. Without loss of generality, we assume that
$d(P^m(S^0, E^0, I^0, R^0, V^0), P_0) < \sigma^*$, for all $m \geq 0$. It follows that

$$||u(t, P^m(S^0, E^0, I^0, R^0, V^0)) - u(t, P_0)|| < \sigma, \forall\ m \geq 0, \forall\ t \in [0, \omega].$$

For any $t \geq 0$, let $t = m\omega + t'$, where $t' \in [0, \omega)$, and m is the largest integer less than or equal to $\dfrac{t}{\omega}$. Therefore, we have

$$\begin{aligned} &||u(t, (S^0, E^0, I^0, R^0, V^0)) - u(t, P_0)|| \\ &= ||u(t', P^m(S^0, E^0, I^0, R^0, V^0)) - u(t', P_0)|| < \sigma, \forall\ t \geq 0. \end{aligned}$$

Note that $(S(t), E(t), I(t), R(t), V(t)) = u(t, (S^0, E^0, I^0, R^0, V^0))$. It then follows that $\frac{\Lambda}{\mu+p} - \sigma \leq S(t) \leq \frac{\Lambda}{\mu+p} + \sigma, 0 \leq E(t) \leq \sigma, 0 \leq I(t) \leq \sigma, 0 \leq R(t) \leq \sigma, \frac{\Lambda p}{\mu(\mu+p)} - \sigma \leq V(t) \leq \frac{\Lambda p}{\mu(\mu+p)} + \sigma$, Then $\frac{S}{N} \geq \frac{\frac{\Lambda}{\mu+p} - \sigma}{\frac{\Lambda}{\mu} + 5\sigma}$, $\frac{V}{N} \geq \frac{\frac{\Lambda p}{\mu(\mu+p)} - \sigma}{\frac{\Lambda}{\mu} + 5\sigma}$. Thus, from system (1) we obtain

$$\begin{cases} \dfrac{dE}{dt} \geq \beta_1(t) \frac{\frac{\Lambda}{\mu+p} - \sigma}{\frac{\Lambda}{\mu} + 5\sigma} I + \beta_2(t) \frac{\frac{\Lambda p}{\mu(\mu+p)} - \sigma}{\frac{\Lambda}{\mu} + 5\sigma} I - (\mu + k)E, \\ \dfrac{dI}{dt} \geq kE - (\mu + \alpha + \gamma)I. \end{cases} \tag{9}$$

By Lemma 3, we know that $r(\Phi_{F-V}(\omega)) > 1$, then we can choose $\sigma > 0$ small enough such that $r(\Phi_{F_\sigma - V}(\omega)) > 1$. Again by Lemma 2 and the standard comparison principle, we know that there exists a positive, ω-periodic function $v_2(t)$

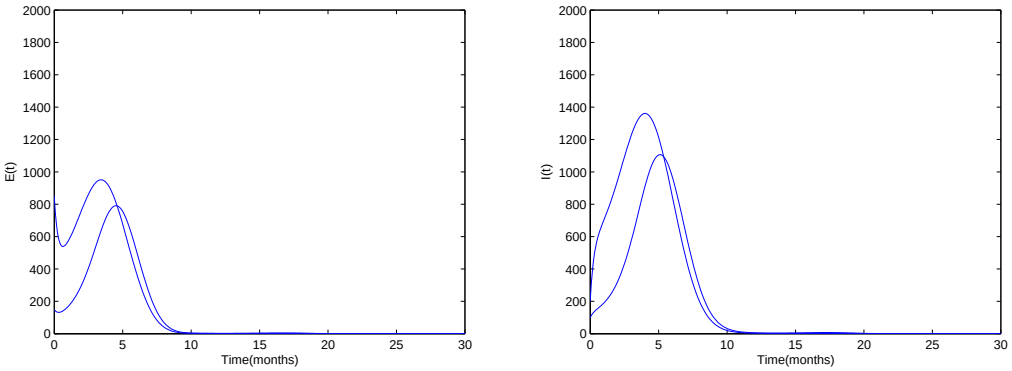

Figure 1: The global asymptotic stability of the disease-free equilibrium P_0 when $R_0 = 0.2916, b_0 = 1.3$.

such that $J(t) \leq v_2(t)e^{p_2 t}$ where $J(t) = (E(t), I(t))^T$ and $p_2 = \frac{1}{\omega} \ln r(\Phi_{F_\sigma - V}(\omega)) > 0$, which implies that $E(t) \to \infty$ and $I(t) \to \infty$ as $n \to \infty$. This leads to a contradiction.

The above claim implies that $P_0(\Lambda/(\mu+p), 0, 0, 0, \Lambda p/(\mu(\mu+p)))$ is isolated invariant set in X and $W^s(P_0) \cap X_0 = \phi$. Note that every orbit in M_∂ approaches to P_0, and P_0 is acyclic in M_∂. By [19, Theorem 1.3.1], we obtain that P is uniformly persistent with respect to $(X_0, \partial X_0)$. It follows from [19, Theorem 3.1.1], the solutions of system (1) are uniformly persistent with respect to $(X_0, \partial X_0)$, that is, there exists a $\delta > 0$ such that any solution $(S(t), E(t), I(t), R(t), V(t))$ of system (1) with initial value $(S^0, E^0, I^0, R^0, V(t)) \in X_0$ satisfies $\liminf_{t \to +\infty} E(t) \geq \delta$ and $\liminf_{t \to +\infty} I(t) \geq \delta$.

Furthermore, [19, theorem 1.3.6] implies that P has a fixed point

$(S^*(0), E^*(0), I^*(0), R^*(0), V^*(0)) \in X_0$. Then $S^*(0) \geq 0$, $E^*(0) > 0, I^*(0) > 0$, and $R^*(0) \geq 0, V^*(0) \geq 0$. We further prove that $S^*(0) > 0$, $R^*(0) > 0$ and $V^*(0) > 0$. Suppose not, if $S^*(0) = 0$, then we obtain

$$S^*(t) = \exp\left(\int_0^t (\mu + \beta_1(\xi)I(\xi)/N(\xi) + p)\, d\xi\right) \times \Lambda \int_0^t \exp\left(\int_0^\xi (\mu + \beta_1(\zeta)I(\zeta)/N(\zeta) + p)\, d\zeta\right) d\xi > 0, \forall t > 0.$$

The periodicity of $S^*(t)$ implies $S^*(n\omega) = S^*(0) = 0$, it is a contradiction. Similarly, we can prove that $R^*(0) > 0$ and $V^*(0) > 0$. Therefore, $(S^*(t), E^*(t), I^*(t), R^*(t), V^*(t))$ is a positive ω-periodic solution of system (1). ■

4 Numerical simulations

From our theoretical results we see that R_0 is a threshold parameter to determine whether or not pertussis persists in the population. Our numerical simulations in this section will demonstrate the asymptotical behavior of (1) in different cases.

Most of the parameters values are taken from[10]. The model parameters with time unit as 1 month are taken as: $\Lambda = 600, \mu = 0.0014$(the average life is 60 years), $p = 0.6, \gamma = 1.4286$(the average recover time is 21 days), $\alpha = 1.2 \times 10^{-7}$, $k = 2.1429$(the average incubating time is 14 days), $\delta = 0.0028$(the average immune time is 30 years), and $\beta_1(t) = a_0(1.1 + \sin\frac{\pi t}{6}), \beta_2(t) = b_0(1.1 + \sin\frac{\pi t}{6})$.

We take $a_0 = 2.3$, b_0 is used in the simulation to demonstrate the asymptotical behavior of the solutions. For the small $b_0(b_0 = 1.3)$ the basic reproductive number is 0.2916. The simulation shows that the disease dies out (see Figure 1) . The simulation results are the same as what we got in Theorem 3.1. For the large $b_0(b_0 = 1.8)$ the basic reproductive number is 8.4140. The disease keeps persistent in the population and the simulation suggests that in the case where $R_0 > 1$, every solution with nontrivial initial data is asymptotic to a periodic solution (see Figure (2)). From the numerical point of view, there exists a unique global attractive positive periodic solution. It is worth studying the uniqueness and stability of positive periodic solution of model (1) in the case where $R_0 > 1$. We leave these challenging problems for further investigation. Figures 1 and 2 also show that vaccination effects is an important reason for the prevalence of pertussis.

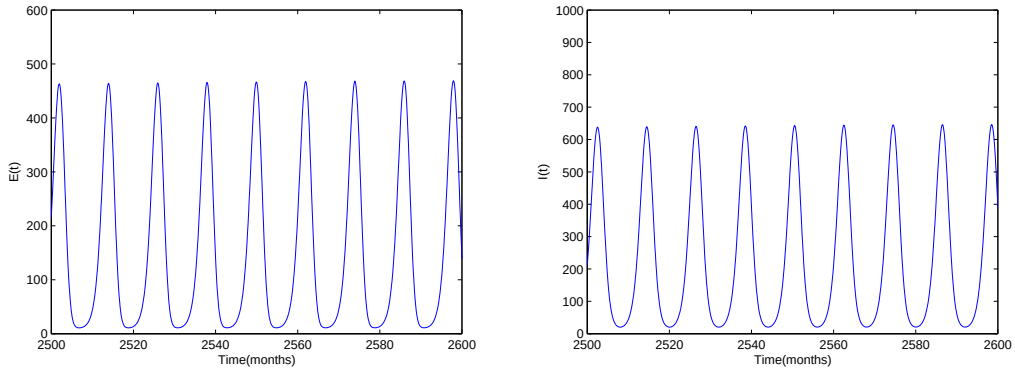

Figure 2: The existence of a periodic solution when $R_0 = 8.4140, b_0 = 1.8$.

5 Conclusions

In this paper, we have formulated a compartmental pertussis model with seasonality. The dynamics of the pertussis disease transmission are analyzed, and the basic reproductive number R_0 is determined. It is proved that $R_0 = 1$ is the threshold to distinguish the disease extinction or persistence. It shows that the disease-free equilibrium is globally asymptotically stable if $R_0 < 1$, while the disease persists if $R_0 > 1$ and the system has one positive periodic solution.

Numerical simulations have been done. The simulation result illustrate the analytical results. We obtain our model is more realistic than the model with constant coefficients. Furthermore, numerical simulations indicate that there may be a unique positive periodic solution which is globally asymptotically stable, which will be another further work.

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References

[1] Young M. An investigation into the periodicity of epidemics of whooping cough from 1870-1910 by means of the periodogram. *Proceedings of the Royal Society of Medicine*,(1920):207-236.

[2] Broutin H, et al. Large-scale comparative analysis of pertussis population dynamics: periodicity, synchrony and impact of vaccination. *American Journal of Epidemiology*, 161(2005):1159-1167.

[3] Mattoo S, Cherry JD. Molecular pathogenesis, epidemiology, and clinical manifestations of respiratory infections due to Bordetella pertussis and other Bordetella subspecies. *Clin Microbiol Rev*, 18(2005):326-382.

[4] Cherry JD. The epidemiology of pertussis and pertussis immunization in the United Kingdom and the United States: a comparative study. *Curr Probl Pediatr*, 14(1984):1-78.

[5] Cherry JD, Brunell PA, Golden GS, Karson DT. Report of the task force on pertussis and pertussis immunization–1988. *Pediatrics*, 81(1988):933-984.

[6] http://www.who.int/immunization/topics/pertussis/en/index.html

[7] Cherry JD. The epidemiology of pertussis: a comparison of the epidemiology of the disease pertussis with the epidemiology of Bordetella pertussis infection. *Pediatrics*,115(2005):1422-1427.

[8] Cherry JD. Epidemiology of pertussis. *Pediatr Infect Dis J*, 25(2006):361-362.

[9] Herbert W. Hethcote, Simulations of pertussis epidemiology in the United States: e.ects of adult booster vaccinations,*Mathematical Biosciences*, 158(1999):47-73.

[10] Hanh T. H. Nguyen and Pejman Rohani, Noise, nonlinearity and seasonality: the epidemics of whooping cough revisited,*J. R. Soc. Interface*, 5(2008):403-413.

[11] Shaheen Abdullah Abdulkareem, Simulating the spread of pertussis in Enschede Region using Agent-Based Modeling,*Thesis*, (3)2010.

[12] Fine PE, Clarkson JA. Seasonal influences on pertussis. International Journal of Epidemiology ; 15(1986):237- 247.
[13] Gomes MC, et al. Diphtheria, pertussis, and measles in Portugal before and after mass vaccination: a time series analysis. *European Journal of Epidemiology*, 15(1999):791-798.
[14] Tanaka M, et al. Trends in pertussis among infants in the United States,*Journal of the American Medical Association*, 290(2003):2968-2975.
[15] H. L. Smith, Monotone Dynamical Systems: An Introduction to the Theory of Competitive and Cooperative Systems, *Mathematical Surveys and Monographs*,(41)1995.
[16] F. Zhang and X.-Q. Zhao, A periodic epidemic model in a patchy environment, *J. Math. Anal. Appl*, (25(2007):496-516.
[17] W. Wang and X.-Q. Zhao, Threshold dynamics for compartmental epidemic models in periodic environments, *J. Dyn. Diff. Equat*, 20(2008):699-717.
[18] H. R. Thieme, Convergence results and a Poincaré-Bendison trichotomy for asymptotical autonomous differential equations, *J. Math. Biol*, (30)(1992):755-763.
[19] X.-Q. Zhao,Dynamical Systems in Population Biology, *Springer-Verlag, New York* ,2003.