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A Variation of (G'/G)-Expansion Method: Travelling Wave Solutions to Nonlinear Equations

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Abstract: A variation of (G'/G)-expansion method is proposed to seek exact travelling wave solutions of nonlinear partial differential equations. The method is implemented on the Zakharov - Kuznetsov - BBM (ZKBBM) equation, the Boussinesq equation, the modified Camassa - Holm (mCH) equation and the (2+1) dimensional Potential Kadomstev-Petviashvili (PKP) equation to illustrate the validity and advantages of the proposed method. As a result, many new and more general exact solutions have been obtained for the equations. A generalization of this method is introduced to find exact travelling wave solutions in terms of Jacobi elliptic functions. This method can also be applied to other nonlinear evolution equations in mathematical physics.

Keywords: Nonlinear equations, (G'/G)-expansion method, travelling wave solutions, Jacobi elliptic functions, Riccati equation

1 Introduction

Nonlinear phenomena can be seen in a broad variety of scientific applications such as plasma physics [1], chemical physics [2], fluid mechanics [3], solid state physics [4], optical fibres [5] etc. Thus finding of exact travelling wave solutions of those problems has a great interest.
Several powerful methods have been established and improved such as the inverse scattering transform method [6], the exp-function method [7], the Backlund transform method [8, 9], the truncated Painlevé expansion method [10, 11], the Hirota's bilinear operators [12], the Weierstrass elliptic function method [13], the Jacobi elliptic function method [14, 15], the tanh-function method [16–18], the homogeneous balance method [19], the sine-cosine method [20], the rank analysis method [21], the ansatz method [22–24], the (G'/G)-expansion method [25].
Among these methods the (G'/G)-expansion method has got much popularity due to its elementary idea, straightforwardness and effectiveness. Bekir [26] , and Zedan [27] applied the method on some nonlinear partial differential equations and obtained some new exact travelling wave solutions. Zhang et al. [28] proposed the extension of the method to get more general travelling wave solutions. Zayed [29, 30] proposed an alternative approach of the (G'/G)-expansion method, namely changed the auxiliary equations, in which $G = G(\xi)$ satisfies the Jacobi elliptic equation $[G'(\xi)]^2 = e_2G^4(\xi) + e_1G^2(\xi) + e_0, \quad e_2, e_1, e_0$ are arbitrary constants, and the Riccati equation $G'(\xi) = A + B\,G^2(\xi), \quad A, B$ are arbitrary constants. Guo and Zhou [31] presented the extended (G'/G)-expansion method, in which the solutions are presented in the form $u = a_0 + \sum_{i=1}^{n}\{a_i(G'/G)^i + b_i(G'/G)^{i-1}\sqrt{\sigma(1 + (\frac{1}{\mu})(G'/G)^2)}\}$. Lü, Liu and Niuj [32] changed the auxiliary equation to $G'(\xi) = h_0 + h_1G(\xi) + h_2G^2(\xi) + h_3G^3(\xi), \quad h_0, h_1, h_2, h_3$ are arbitrary constants. Ma [33] proposed the modified (G'/G)-expansion method and obtained new exact travelling wave solutions of the (1+1) dimensional KDV equation. Recently we have considered (2+1) dimensional PKP equation and have obtained [34] several new exact solutions using an extension of (G'/G)-expansion method.
In the present letter, we shall propose a variation of the (G'/G)-expansion method which will play an important role to

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find exact travelling wave solutions of nonlinear partial differential equations. To show the reliability of the method, we applied it on some nonlinear partial differential equations, namely the Zakharov - Kuznetsov - BBM (ZKBBM) equation, the Boussinesq equation, the modified Camassa - Holm (mCH) equation and the (2+1) dimensional Potential Kadomstev-Petviashvili (PKP) equation and find out their exact travelling wave solutions.
The rest of the paper is arranged as follows: In section 2, we describe the main idea of the variation of the (G'/G)-expansion method. In section 3, we applied the proposed method on the above mentioned four equations and obtained new exact travelling wave solutions. In section 4, we applied a generalization of the method to find exact travelling wave solutions of the ZKBBM equation in terms of Jacobi elliptic functions. In section 5, conclusions are given.

2 Description of the variation of (G'/G)-expansion method

In this section, we describe the main idea of the variation of the (G'/G)-expansion method.
Consider a nonlinear partial differential equation, say, in $(n+1)$ independent variables $x_1, x_2, x_3, ..., x_n$ and t, as

$$P(u, u_t, u_{x_1}, u_{x_2}, u_{tt}, u_{x_1 t}, u_{x_2 t}, u_{x_1 x_1}, u_{x_2 x_2}, ...) = 0 \tag{2.1}$$

where $u = u(x_1, x_2, ..., x_n, t)$ is an unknown function, P is a polynomial in $u = u(x_1, x_2, ..., x_n, t)$ and its various partial derivatives, in which the highest order derivatives and nonlinear terms are involved. In the following steps, we present the main steps of the variation of the (G'/G)-expansion method.

Step 1. Consider the travelling wave transformation:

$$u = u(x_1, x_2, ..., x_n, t) = u(\xi), \quad \xi = \sum_{i=1}^{n} r_i x_i + Vt \tag{2.2}$$

where $r_i (i = 1, 2, ..., n)$ and V (speed of wave) are constants to be determined later.The travelling wave variable u in (2.2) allows us to reduce the equation (2.1) to an ordinary differential equation for $u = u(\xi)$:

$$P(u, Vu', r_1 u', r_2 u', V^2 u', r_1 V u'', r_2 V u'', r_1^2 u'', r_2^2 u'', ...) = 0, \tag{2.3}$$

Step 2. If possible integrate Eq.(2.3) term by term one or more times yields constant(s) of integration.
Step 3. Assume that the solution $u(\xi)$, of the Eq.(2.3) can be expressed as a polynomial in (G'/G) and (F'/F) as follows:

$$u(\xi) = \sum_{i=0}^{m} a_i \ (G'/G)^i + \sum_{i=1}^{m} b_i \ (G'/G)^{i-1} (F'/F) \tag{2.4}$$

where $G = G(\xi)$ and $F = F(\xi)$ expresses the solution of the coupled Riccati equation,

$$G'(\xi) = -G(\xi) \cdot F(\xi) \tag{2.5}$$

$$F'(\xi) = 1 - F^2(\xi) \tag{2.6}$$

where prime denotes derivative with respect to ξ, $a_i (i = 0, 1, ..., m), b_i (i = 1, 2, ..., m)$ are constants to be determined later.
These governing equations lead us two types of general solutions:

$$G(\xi) = \pm\,\mathrm{sech}(\xi), \quad F(\xi) = \tanh(\xi) \tag{2.7}$$

$$G(\xi) = \pm\,\mathrm{csch}(\xi), \quad F(\xi) = \coth(\xi) \tag{2.8}$$

Step 4. The positive integer m can be determined by considering the homogeneous balance between the highest order derivatives and the nonlinear terms appearing in Eq.(2.3) as follows:
If $D[u(\xi)] = m$, then $D[u^r (\frac{d^q u}{d\xi^q})^s] = mr + s(q + m)$, where D denotes the degree of the expression.
Step 5. Substituting Eq.(2.4) into Eq.(2.3) and using Eq.(2.5) and Eq.(2.6), collecting all terms with the same order of (G'/G) or (F) together, left-hand side of Eq.(2.3) is converted into another polynomial in (G'/G) or (F). Equating each coefficient of this polynomial to zero, yields a set of algebraic equations for $a_i \ (i = 0, 1, ..., m), b_i \ (i = 1, 2, ..., m), r_i \ (i =$

$1, 2, ..., n)$ and V.
Step 6. Determining the constants $a_i\ (i = 0, 1, ..., m), b_i\ (i = 1, 2, ..., m), r_i\ (i = 1, 2, ..., n)$ and V by solving the algebraic equations in step 5. As the general solutions of Eq.(2.5) and Eq.(2.6) are already known to us, then substituting $a_i\ (i = 0, 1, ..., m), b_i\ (i = 1, 2, ..., m), r_i\ (i = 1, 2, ..., n), V$ and the general solutions of Eq.(2.5) and Eq.(2.6), we obtain the travelling wave solutions of Eq.(2.1).

Finally we will consider a generalized version of the governing equations for G and F, which are as follows:

$$G'^2(\xi) = (1 - G^2(\xi)) \cdot (k'^2 + k^2 G^2(\xi)) \tag{2.9}$$

$$F'^2(\xi) = (1 - F^2(\xi)) \cdot (1 - k^2 F^2(\xi)) \tag{2.10}$$

where k is known as the elliptic modulus and $k' = \sqrt{1 - k^2}$ is known as the complementary elliptic modulus of the Jacobi elliptic functions [35].
The governing equations lead us to the following Jacobi elliptic expressions for G and F:

$$G(\xi) = cn(\xi, k) \tag{2.11}$$

$$F(\xi) = sn(\xi, k) \tag{2.12}$$

In section 5, we shall apply this generalization to a particular nonlinear equation.

3 Applications of the method

3.1 Example 1: the ZKBBM equation

Consider the Zakharov - Kuznetsov - BBM (ZKBBM) equation:

$$u_t + u_x - 2auu_x - bu_{xxt} = 0 \tag{3.1.1}$$

where a, b are nonzero constants.
We seek the travelling wave transformation in the form below,

$$u(x, t) = u(\xi), \xi = x + Vt \tag{3.1.2}$$

where V is a constant to be determined later. Converting Eq.(3.1.1) into an ODE for $u(\xi)$ by using Eq.(3.1.2), we have

$$(1 + V)u' - 2auu' - bVu''' = 0 \tag{3.1.3}$$

Integrating Eq.(3.1.3) with respect to ξ once, we obtain

$$(1 + V)u - au^2 - bVu'' + C = 0 \tag{3.1.4}$$

where C is the integration constant to be determined later.
Balancing the order of u'' and u^2 in Eq.(3.1.4), we get

$$2 + m = 2m, \Rightarrow m = 2 \tag{3.1.5}$$

Hence the solution of Eq.(3.1.4) as described in step 3, can be expressed as,

$$\begin{aligned} u(\xi) =& a_0 + a_1 (G'/G) + a_2 (G'/G)^2 + b_1 (F'/F) + b_2 (G'/G)(F'/F) \\ =& a_0 - a_1 F + a_2 F^2 + b_1 (F^{-1} - F) - b_2 (1 - F^2) \end{aligned} \tag{3.1.6}$$

where a_0, a_1, a_2, b_1, b_2 all are constants to be determined later.
Substituting Eq.(3.1.6) into Eq.(3.1.4) and collecting all the terms with the same powers of F together and equating each

coefficient to zero, yields a set of simultaneous algebraic equations for $a_0, a_1, a_2, b_1, b_2, C$ and V as follows:

$$\begin{aligned} F^0 :& a_0 - b_2 + C + 2aa_0b_2 + 2aa_1b - 1 - 2bVa_2 - 2bVb_2 + Va_0 - Vb_2 \\ & - aa_0^2 + 2ab_1^2 - ab_2^2 = 0 \end{aligned} \quad (3.1.7)$$

$$\begin{aligned} F^1 :& - a_1 + 2aa_0a_1 - 2aa_2b_1 - 2aa_1b_2 - 4ab_1b_2 + 2aa_0b_1 - 2bVa_1 \\ & - 2bVb_1 - Vb_1 - Va_1 - b_1 = 0 \end{aligned} \quad (3.1.8)$$

$$F^{-1} : b_1 - 2aa_0b_1 + 2ab_1b_2 + 2bVb_1 + Vb_1 = 0 \quad (3.1.9)$$

$$\begin{aligned} F^2 :& a_2 - 2aa_0a_2 + 2aa_2b_2 - 2aa_0b_2 - 2aa_1b_1 + 8bVa_2 + 8bVb_2 \\ & + 2ab_2^2 + Vb_2 + Va_2 - ab_1^2 - aa_1^2 + b_2 = 0 \end{aligned} \quad (3.1.10)$$

$$F^{-2} : - ab_1^2 = 0 \quad (3.1.11)$$

$$F^3 : 2aa_2b_1 + 2aa_1a_2 + 2ab_1b_2 + 2aa_1b_2 + 2bVa_1 + 2bVb_1 = 0 \quad (3.1.12)$$

$$F^{-3} : - 2bVb_1 = 0 \quad (3.1.13)$$

$$F^4 : - 2aa_2b_2 - 6bVa_2 - 6bVb_2 - ab_2^2 - aa_2^2 = 0 \quad (3.1.14)$$

To solve this set of algebraic equations for $a_0, a_1, a_2, b_1, b_2, C$ and V by use of Maple, we get,

$$a_0 = \frac{1}{2}\left(\frac{1 + 2ab_2 + 8bV + V}{a}\right), a_1 = 0, a_2 = -\frac{ab_2 + 6bV}{a}, b_1 = 0, C = \frac{1}{4}\left(\frac{-2V + 16b^2V^2 - V^2 - 1}{a}\right) \quad (3.1.15)$$

where b_2 and V are arbitrary.
Substituting the general solutions of Eq.(2.5) and Eq.(2.6) into Eq.(3.1.6), we get two types of travelling wave solutions of the ZKBBM equation (3.1.1) with the help of the above solutions (3.1.15).

Type 1:
Considering the solution (2.7) and substituting (3.1.15) into Eq.(3.1.6), we get the solution as follows:

$$u_1(x,t) = \frac{V+1}{2a} + \frac{4bV}{a} + \left(\frac{V+1}{2a} + 2b_2 + \frac{4bV}{a}\right)\tanh^2(x + Vt) \quad (3.1.16)$$

where b_2 and V are arbitrary.

Type 2:
Considering the solution (2.8) and substituting (3.1.15) into Eq.(3.1.6), we get the solution as follows:

$$u_2(x,t) = \frac{V+1}{2a} + \frac{4bV}{a} + \left(\frac{V+1}{2a} + 2b_2 + \frac{4bV}{a}\right)\coth^2(x + Vt) \quad (3.1.17)$$

where b_2 and V are arbitrary.

3.2 Example 2: the Boussinesq equation

Consider the Boussinesq equation:

$$u_{tt} - u_{xx} - (u^2)_{xx} + u_{xxxx} = 0 \quad (3.2.1)$$

The travelling wave variable is of the form below,

$$u(x,t) = u(\xi), \xi = x - Vt \quad (3.2.2)$$

where V is a constant to be determined later. Converting Eq.(3.2.1) into an ODE for $u(\xi)$ by using Eq.(3.2.2), we have

$$(V^2 - 1)u'' - (u^2)'' + u'''' = 0 \quad (3.2.3)$$

Integrating Eq.(3.2.3) twice with respect to ξ and taking the constant of integration as zero, we obtain

$$(V^2-1)u-u^2+u''=0 \tag{3.2.4}$$

Balancing the order of u'' and u^2 in Eq.(3.2.4), we get

$$2+m=2m \Rightarrow m=2 \tag{3.2.5}$$

Hence the solution of Eq.(3.2.4) as described in step 3, can be expressed as,

$$\begin{aligned} u(\xi) =&a_0+a_1\left(G'/G\right)+a_2\left(G'/G\right)^2+b_1\left(F'/F\right)+b_2\left(G'/G\right)\left(F'/F\right) \\ =&a_0-a_1F+a_2F^2+b_1(F^{-1}-F)-b_2(1-F^2) \end{aligned} \tag{3.2.6}$$

where a_0, a_1, a_2, b_1, b_2 all are constants to be determined later.

Substituting Eq.(3.2.6) into Eq.(3.2.4) and collecting all the terms with the same powers of F together and equating each coefficient to zero, yields a set of simultaneous algebraic equations for a_0, a_1, a_2, b_1, b_2 and V as follows:

$$\begin{aligned} F^0 :& -a_0+2a_2+3b_2+2a_1b_1+V^2a_0-V^2b_2-a_0^2+2a_0b_2 \\ &+2b_1^2-b_2^2=0 \end{aligned} \tag{3.2.7}$$

$$\begin{aligned} F^1 :&3a_1-2a_2b_1-V^2a_1-4b_1b_2+2a_0a_1-2a_1b_2+2a_0b_1 \\ &-V^2b_1+3b_1=0 \end{aligned} \tag{3.2.8}$$

$$F^{-1} : -3b_1+V^2b_1+2b_1b_2-2a_0b_1=0 \tag{3.2.9}$$

$$\begin{aligned} F^2 :& -9a_2-2a_0a_2+V^2b_2-2a_1b_1+V^2a_2+2a_2b_2-2a_0b_2 \\ &-9b_2-a_1^2-b_1^2+2b_2^2=0 \end{aligned} \tag{3.2.10}$$

$$F^{-2} : -b_1^2=0 \tag{3.2.11}$$

$$F^3 :2a_1b_2+2a_2b_1+2b_1b_2+2a_1a_2-2a_1-2b_1=0 \tag{3.2.12}$$

$$F^{-3} :2b_1=0 \tag{3.2.13}$$

$$F^4 : -2a_2b_2-a_2^2-b_2^2+6a_2+6b_2=0 \tag{3.2.14}$$

Solving the set of algebraic equations for a_0, a_1, a_2, b_1, b_2 and V by Maple, we get

Case 1:

$$a_0=-2+b_2, a_1=0, a_2=-b_2+6, b_1=0, V=\pm\sqrt{5} \tag{3.2.15}$$

where b_2 is arbitrary.

Case 2:

$$a_0=-6+b_2, a_1=0, a_2=-b_2+6, b_1=0, V=\pm\sqrt{3}\,\mathrm{i} \tag{3.2.16}$$

where b_2 is arbitrary.

Substituting the general solutions of Eq.(2.5) and Eq.(2.6) into Eq.(3.2.6), we get two types of travelling wave solutions of the Boussinesq equation (3.2.1) corresponding to above mentioned two cases. Furthermore, based on the two types of general solution of the governing equations Eq.(2.5) and Eq.(2.6), there will be two different classes of solutions for each of two types.

Type 1:

Substituting Eq.(3.2.15) into Eq.(3.2.6) we have the solution of Eq.(3.2.1) as follows:

Class I: When $G(\xi)=\pm\,\mathrm{sech}(\xi)$, $F(\xi)=\tanh(\xi)$

$$u_{11}(x,t)=-2+6\tanh^2(x\mp\sqrt{5}t) \tag{3.2.17}$$

Class II: When $G(\xi) = \pm\,\text{csch}(\xi),\;\; F(\xi) = \coth(\xi)$

$$u_{12}(x,t) = -2 + 6\,\coth^2(x \mp \sqrt{5}t) \tag{3.2.18}$$

Type 2:
Substituting Eq.(3.2.16) into Eq.(3.2.6) we have the solution of Eq.(3.2.1) as follows:
Class I: When $G(\xi) = \pm\,\text{sech}(\xi),\;\; F(\xi) = \tanh(\xi)$

$$u_{21}(x,t) = -6 + 6\,\tanh^2(x \mp \sqrt{3}\,\text{i}t) \tag{3.2.19}$$

Class II: When $G(\xi) = \pm\,\text{csch}(\xi),\;\; F(\xi) = \coth(\xi)$

$$u_{22}(x,t) = -6 + 6\,\coth^2(x \mp \sqrt{3}\,\text{i}t) \tag{3.2.20}$$

3.3 Example 3: the modified Camassa-Holm(mCH)equation

Consider the the modified Camassa-Holm(mCH)equation:

$$u_t - u_{xxt} + 3u^2u_x = 2u_xu_{xx} + uu_{xxx} \tag{3.3.1}$$

Consider the travelling wave variable in the form below,

$$u(x,t) = u(\xi), \xi = x - Vt \tag{3.3.2}$$

where V is a constant to be determined later. Converting Eq.(3.3.1) into an ODE for $u(\xi)$ by using Eq.(3.3.2), we have

$$-Vu' + Vu''' + 3u^2u' = 2u'u'' + uu''' \tag{3.3.3}$$

Integrating Eq.(3.3.3) with respect to ξ once, we obtain

$$V(u'' - u) + u^3 - uu'' - \frac{1}{2}(u')^2 + C = 0 \tag{3.3.4}$$

where C is the integration constant.
Balancing the order of uu'' and u^3 in Eq.(3.3.4), we get

$$m + (2 + m) = 3m \Rightarrow m = 2 \tag{3.3.5}$$

Hence the solution of Eq.(3.3.4) as described in step 3, can be expressed as,

$$\begin{aligned} u(\xi) =& a_0 + a_1\,(G'/G) + a_2\,(G'/G)^2 + b_1\,(F'/F) + b_2\,(G'/G)\,(F'/F) \\ =& a_0 - a_1F + a_2F^2 + b_1(F^{-1} - F) - b_2(1 - F^2) \end{aligned} \tag{3.3.6}$$

where a_0, a_1, a_2, b_1, b_2 all are constants to be determined later.
Substituting Eq.(3.3.6) into Eq.(3.3.4) and collecting all the terms with the same powers of F together and equating each coefficient to zero, yields a set of simultaneous algebraic equations for $a_0, a_1, a_2, b_1, b_2, C$ and V as follows:

$$\begin{aligned}
F^0 :& C + 6a_1b_1b_2 + 2Va_2 + 3Vb_2 - Va_0 - 2a_0b_2 - 3b_1^2 + 2b_2^2 - 3a_0^2b_2 \\
&- 6a_0b_1^2 + 3a_0b_2^2 - 2a_2a_0 + 2a_2b_2 - \frac{1}{2}a_1^2 - 3a_1b_1 - 6a_0a_1b_1 \\
&+ 3a_2b_1^2 + 9b_1^2b_2 + a_0^3 - b_2^3 = 0 \qquad (3.3.7) \\
F^1 :& 6a_0a_1b_2 + 6a_2a_0b_1 + 12a_0b_1b_2 - 6a_2b_1b_2 + 6a_1b_2 + 10a_2b_1 \\
&+ 10a_2b_1 + 12b_1b_2 + 3b_1^3 + 3Vb_1 - 2a_0b_1 - 2a0a_1 + 3Va_1 \\
&- 3a_0^2b_1 - 3a_0^2a_1 + 6a_1b_1^2 - 3a_1b_2^2 + 3a_1^2b_1 - 9b_1b_2^2 + 4a_2a_1 = 0 \qquad (3.3.8) \\
F^{-1} :& - 6a_0b_1b_2 - 4b_1b_2 - 3b_1^3 + 2a_0b_1 - 3Vb_1 + 3a_0^2b_1 - 3a_1b_1^2 \\
&+ 3b_1b_2^2 - 2a_2b_1 = 0 \qquad (3.3.9) \\
F^2 :& - 6a_2a_0b_2 + 6a_0a_1b_1 - 6a_2a_1b_1 - 12a_1b_1b_2 - 16a_2b_2 \\
&+ 3a_1^2 + 4b_1^2 - 12b_2^2 + 3b_2^3 - 4a_2^2 - 9Va_2 + 7a_1b_1 + 8a_0b_2 \\
&+ 8a_2a_0 + 3a_0a_1^2 - 9Vb_2 + 3a_0b_1^2 - 6a_0b_2^2 + 3a_0^2b_2 \\
&+ 3a_0^2a_2 - 3a_1^2b_2 - 6a_2b_1^2 + 3a_2b_2^2 - 9b_1^2b_2 = 0 \qquad (3.3.10) \\
F^{-2} :& 4b_1^2 + 3a_0b_1^2 - 3b_1^2b_2 + a_1b_1 = 0 \qquad (3.3.11) \\
F^3 :& - 6a_0b_1b_2 - 6a_2a_0b_1 - 6a_0a_1a_2 - 6a_0a_1b_2 + 6a_2a_1b_2 \\
&+ 12a_2b_1b_2 - 20b_1b_2 - 18a_2b_1 - 14a_2a_1 - 2Va_1 - a_1^3 \\
&- b_1^3 - 16a_1b_2 - 2Vb_1 - 3a_1b_1^2 + 6a_1b_2^2 - 3a_1^2b_1 + 3a_2^2b_1 \\
&+ 9b_1b_2^2 + 2a_0a_1 + 2a_0b_1 = 0 \qquad (3.3.12) \\
F^{-3} :& b_1^3 + 2Vb_1 - 2a_0b_1 + 2b_1b_2 = 0 \qquad (3.3.13)
\end{aligned}$$

$$\begin{aligned}
F^4 :& 6Vb_2 + 6a_2a_0b_2 + 6a_1b_1b_2 + 6a_2a_1b_1 + 12a_2^2 + 18b_2^2 \\
&- 3b_2^3 - \frac{5}{2}a_1^2 - \frac{5}{2}b_1^2 + 6Va_2 + 30a_2b_2 + 3a_0a_2^2 \\
&+ 3a_0b_2^2 + 3a_1^2a_2 + 3a_1^2b_2 - 3a_2^2b_2 + 3a_2b_1^2 \\
&- 6a_2b_2^2 + 3b_1^2b_2 - 5a_1b_1 - 6a_2a_0 - 6a_0b_2 = 0 \qquad (3.3.14) \\
F^{-4} :& - \frac{5}{2}b_1^2 = 0 \qquad (3.3.15) \\
F^5 :& - 6a_2a_1b_2 - 6a_2b_1b_2 - 3a_1a_2^2 - 3a_1b_2^2 - 3a_2^2b_1 - 3b_1b_2^2 + 10a_2a_1 \\
&+ 10a_1b_2 + 10a_2b_1 + 10b_1b_2 = 0 \qquad (3.3.16)
\end{aligned}$$

Solving the set of algebraic equations for a_0, a_1, a_2, b_1, b_2 and V by Maple, we get,

$$a_1 = 0, a_2 = -\frac{30b_2 - 2a_0 - 3b_2^2 - 3a_0b_2 + 6a_0^2}{28 + 9a_0 - 9b_2}, b_1 = 0,$$

$$V = \frac{-24b_2 + 24a_0 + 20b_2^2 - 40a_0b_2 + 20a_0^2 - 9a_0^2b_2 + 9a_0b_2^2 - 3b_2^3 + 3a_0^3}{28 + 9a_0 - 9b_2},$$

$$C = -\frac{2(- 240b_2^3a_0 + 360b_2^2a_0^2 - 240a_0^3b_2 - 90a_0^2b_2^3 + 90a_0^3b_2^2 - 45a_0^4b_2 + 45a_0b_2^4 + 60b_2^4 + 60a_0^4 - 9b_2^5 + 9a_0^5 + 688a_0b_2 - 150a_0^2b_2 + 150a_0b_2^2 - 344b_2^2 + 50a_0^3 - 50b_2^3 - 344a_0^2)}{(28 + 9a_0 - 9b_2)^2} \qquad (3.3.17)$$

where a_0 and b_2 are arbitrary.

Substituting the general solutions of Eq.(2.5) and Eq.(2.6) into Eq.(3.3.6), we get two types of travelling wave solutions of the mCH equation (3.3.1) with the help of the above solutions (3.3.17).

Type 1:
Considering the solution (2.7) and substituting (3.3.17) into Eq.(3.3.6), we get the solution as follows:

$$u_1(x,t) = a_0 - b_2 - \frac{2(a_0 - b_2)(3a_0 - 1 - 3b_2)}{28 + 9a_0 - 9b_2} \tanh^2(x - Vt) \tag{3.3.18}$$

where a_0 and b_2 are arbitrary and the value of V is given in (3.3.17)

Type 2:
Considering the solution (2.8) and substituting (3.3.17) into Eq.(3.3.6), we get the solution as follows:

$$u_2(x,t) = a_0 - b_2 - \frac{2(a_0 - b_2)(3a_0 - 1 - 3b_2)}{28 + 9a_0 - 9b_2} \coth^2(x - Vt) \tag{3.3.19}$$

where a_0 and b_2 are arbitrary and the value of V is given in (3.3.17).

3.4 Example 4: the (2+1) dimensional PKP equation

Consider the (2+1) dimensional PKP equation:

$$\frac{1}{4}u_{xxxx} + \frac{3}{2}u_x u_{xx} + \frac{3}{4}u_{yy} + u_{xt} = 0 \tag{3.4.1}$$

The travelling wave variable is of the form below,

$$u(x,t) = u(\xi), \xi = ax + by + ct \tag{3.4.2}$$

where a, b, c are constant to be determined later. Converting Eq.(3.4.1) into an ODE for $u(\xi)$ by using Eq.(3.4.2), we have

$$\frac{1}{4}a^4 u'''' + \frac{3}{2}a^3 u' u'' + \frac{3}{4}b^2 u'' + acu'' = 0 \tag{3.4.3}$$

Integrating Eq.(3.4.3) once with respect to ξ , we obtain

$$\frac{1}{4}a^4 u''' + \frac{3}{4}a^3 (u')^2 + (\frac{3}{4}b^2 + ac)u' = K \tag{3.4.4}$$

where K is the constant of integration.
Balancing the order of u''' and $(u')^2$ in Eq.(3.4.4), we get

$$m + 3 = 2 + 2m, \qquad \Rightarrow m = 1 \tag{3.4.5}$$

Hence the solution of Eq.(3.4.4) as described in step 3, can be expressed as,

$$\begin{aligned} u(\xi) =& a_0 + a_1\,(G'/G) + b_1\,(F'/F) \\ =& a_0 - a_1 F + b_1 (F^{-1} - F) \end{aligned} \tag{3.4.6}$$

where a_0, a_1, b_1 all are constants to be determined later.
Substituting Eq.(3.4.6) into Eq.(3.4.4) and collecting all the terms with the same powers of F together and equating each

coefficient to zero, yields a set of simultaneous algebraic equations for a_0, a_1, b_1, a, b and c as follows:

$$F^0 : -K - aca_1 - \frac{3}{2}a^3a_1b_1 + \frac{1}{2}a^4a_1 + \frac{3}{4}a^3a_1^2 - \frac{3}{2}a^3b_1^2 - \frac{3}{4}b^2a_1 = 0 \tag{3.4.7}$$

$$F^2 : -\frac{3}{2}a^3a_1b_1 + aca_1 + acb_1 - 2a^4b_1 + \frac{3}{4}b^2b_1 + \frac{3}{4}b^2a_1 - 2a^4a_1 - \frac{3}{2}a^3a_1^2 = 0 \tag{3.4.8}$$

$$F^{-2} : \frac{3}{2}a^3a_1b_1 - acb_1 + 2a^4b_1 - \frac{3}{4}b^2b_1 = 0 \tag{3.4.9}$$

$$F^4 : \frac{3}{2}a^3a_1b_1 + \frac{3}{2}a^4a_1 + \frac{3}{4}a^3b_1^2 + \frac{3}{2}a^4b_1 + \frac{3}{4}a^3a_1^2 = 0 \tag{3.4.10}$$

$$F^{-4} : \frac{3}{4}a^3b_1^2 - \frac{3}{2}a^4b_1 = 0 \tag{3.4.11}$$

Solving the set of algebraic equations for a_0, a_1, b_1, a, b, c and K by Maple, we get
Case 1:

$$a_1 = -2a, \qquad b_1 = 0, c = -\frac{1}{4}\frac{4a^4 + 3b^2}{a}, K = 0 \tag{3.4.12}$$

where a, b, a_0 are arbitrary.
Case 2:

$$a_1 = -2a, b_1 = 2a, c = -\frac{1}{4}\frac{4a^4 + 3b^2}{a}, K = 0 \tag{3.4.13}$$

where a, b, a_0 are arbitrary.
Case 3:

$$a_1 = -4a, b_1 = 2a, c = -\frac{1}{4}\frac{16a^4 + 3b^2}{a}, K = 0 \tag{3.4.14}$$

where a, b, a_0 are arbitrary.
Substituting the general solutions of Eq.(2.5) and Eq.(2.6) into Eq.(3.4.6), we get two types of travelling wave solutions of the (2+1) dimensional PKP equation (3.4.1) corresponding to above mentioned three cases. Furthermore, based on the two types of general solution of the governing equations Eq.(2.5) and Eq.(2.6), there will be two different classes of solutions for each of three types.

Type 1: Substituting Eq.(3.4.12) into Eq.(3.4.6) we have the solution of Eq.(3.4.1) as follows:

Class I: When $G(\xi) = \pm\,\text{sech}(\xi),\ F(\xi) = \tanh(\xi)$

$$u_{11}(x,t) = a_0 + 2a\tanh(ax + by - \frac{1}{4}\frac{4a^4 + 3b^2}{a}t) \tag{3.4.15}$$

where a, b, a_0 are arbitrary.
Class II: When $G(\xi) = \pm\,\text{csch}(\xi),\ F(\xi) = \coth(\xi)$

$$u_{12}(x,t) = a_0 + 2a\coth(ax + by - \frac{1}{4}\frac{4a^4 + 3b^2}{a}t) \tag{3.4.16}$$

where a, b, a_0 are arbitrary.

Type 2: Substituting Eq.(3.4.13) into Eq.(3.4.6) we have the solution of Eq.(3.4.1) as follows:

Class I: When $G(\xi)=\pm\,\mathrm{sech}(\xi),\ \ F(\xi)=\tanh(\xi)$

$$\begin{aligned}u_{21}(x,t)=a_0+2a\tanh(ax+by-\frac{1}{4}\frac{4a^4+3b^2}{a}t)+2a(\tanh^{-1}(ax+by-\frac{1}{4}\frac{4a^4+3b^2}{a}t)\\ -\tanh(ax+by-\frac{1}{4}\frac{4a^4+3b^2}{a}t))\end{aligned} \tag{3.4.17}$$

where a,b,a_0 are arbitrary.

Class II: When $G(\xi)=\pm\,\mathrm{csch}(\xi),\ \ F(\xi)=\coth(\xi)$

$$\begin{aligned}u_{22}(x,t)=a_0+2a\coth(ax+by-\frac{1}{4}\frac{4a^4+3b^2}{a}t)+2a(\coth^{-1}(ax+by-\frac{1}{4}\frac{4a^4+3b^2}{a}t)\\ -\coth(ax+by-\frac{1}{4}\frac{4a^4+3b^2}{a}t))\end{aligned} \tag{3.4.18}$$

where a,b,a_0 are arbitrary.

Type 3: Substituting Eq.(3.4.14) into Eq.(3.4.6) we have the solution of Eq.(3.4.1) as follows:

Class I: When $G(\xi)=\pm\,\mathrm{sech}(\xi),\ \ F(\xi)=\tanh(\xi)$

$$\begin{aligned}u_{31}(x,t)=a_0+4a\tanh(ax+by-\frac{1}{4}\frac{16a^4+3b^2}{a}t)+2a(\tanh^{-1}(ax+by-\frac{1}{4}\frac{16a^4+3b^2}{a}t)\\ -\tanh(ax+by-\frac{1}{4}\frac{16a^4+3b^2}{a}t))\end{aligned} \tag{3.4.19}$$

where a,b,a_0 are arbitrary.

Class II: When $G(\xi)=\pm\,\mathrm{csch}(\xi),\ \ F(\xi)=\coth(\xi)$

$$\begin{aligned}u_{32}(x,t)=a_0+4a\coth(ax+by-\frac{1}{4}\frac{16a^4+3b^2}{a}t)+2a(\coth^{-1}(ax+by-\frac{1}{4}\frac{16a^4+3b^2}{a}t)\\ -\coth(ax+by-\frac{1}{4}\frac{16a^4+3b^2}{a}t))\end{aligned} \tag{3.4.20}$$

where a,b,a_0 are arbitrary.

4 Elliptic solution for ZKBBM equation

Now we consider the generalized governing equations leading to Jacobi elliptic functions discussed earlier in section 2. The method is implemented to the ZKBBM equation namely,

$$u_t+u_x-2auu_x-bu_{xxt}=0 \tag{4.1}$$

where a,b are nonzero constants.
From the previous calculation, we get

$$(1+V)u-au^2-bVu''+C=0 \tag{4.2}$$

where C is the integration constant to be determined later.
Balancing the order of u'' and u^2 in Eq.(4.2), we get

$$2+m=2m\Rightarrow m=2 \tag{4.3}$$

Hence the solution of Eq.(4.2) as described in step 3, can be expressed as,

$$u(\xi)=a_0+a_1\left(G'/G\right)+a_2\left(G'/G\right)^2+b_1\left(F'/F\right)+b_2\left(G'/G\right)\left(F'/F\right) \tag{4.4}$$

where a_0, a_1, a_2, b_1, b_2 all are constants to be determined later.
Substituting Eq.(4.4) into Eq.(4.2) and collecting all the terms with the same powers of G^i or $G'G^iF^j$ or F^i or $F'F^iG^j$ $(i, j = 0, \pm1, \pm2, \pm3)$ together and equating each coefficient to zero, yields a set of simultaneous algebraic equations for $a_0, a_1, a_2, b_1, b_2, C$ and V.
Solving the set of algebraic equations we get,

$$a_0 = -\frac{1}{12}\frac{aa_2 + 8k^2aa_2b - 8k'^2aa_2b - 6b}{ab}, \qquad a_1 = 0, \qquad b_1 = 0, \qquad b_2 = 0,$$

$$C = \frac{1}{144}\frac{-224a^2a_2^2k'^2k^2b^2 - 36b^2 + 12baa_2 + 16a^2a_2^2k'^2b^2 + 16a^2a_2^2k^4b^2 - a^2a_2^2}{ab^2},$$

$$V = -\frac{1}{6}\frac{aa_2}{b} \tag{4.5}$$

where a_2 is arbitrary.
Substituting the general solutions (2.11) and (2.12) into (4.4), we get the exact travelling wave solution of the ZKBBM equation (4.1) in terms of Jacobi elliptic functions as,

$$u(x,t) = -\frac{1}{12}\frac{aa_2 + 8k^2aa_2b - 8k'^2aa_2b - 6b}{ab} + a_2\, sn^2(x + \frac{1}{6}\frac{aa_2}{b}\, t)\, dc^2(x + \frac{1}{6}\frac{aa_2}{b}\, t) \tag{4.6}$$

where a_2 is arbitrary.

We stress that solution (4.6) is more general travelling wave solution of the ZKBBM equation than those solutions (3.1.16), (3.1.17) obtained in section 3.

It is instructive to consider following properties of Jacobi elliptic functions:

$$sn(\xi,k) \xrightarrow[k\to0]{k\to1} \begin{cases} \tanh(\xi) \\ \sin(\xi) \end{cases}, \quad cn(\xi,k) \xrightarrow[k\to0]{k\to1} \begin{cases} \operatorname{sech}(\xi) \\ \cos(\xi) \end{cases}, \quad dn(\xi,k) \xrightarrow[k\to0]{k\to1} \begin{cases} \operatorname{sech}(\xi) \\ 1 \end{cases} \tag{4.7}$$

It is not very difficult to notice that the above solutions can be reduced to previously obtained solutions in section 3, for certain values of arbitrary terms. Indeed followings are the two extreme limits.
Thus when $k \longrightarrow 1$, the solution tends to a hyperbolic solution,

$$u_1(x,t) = -\frac{1}{12}\frac{aa_2 + 8k^2aa_2b - 8k'^2aa_2b - 6b}{ab} + a_2 \tanh^2(x + \frac{1}{6}\frac{aa_2}{b}\, t) \tag{4.8}$$

where a_2 is arbitrary.

and when $k \longrightarrow 0$, the solution tends to a trigonometric solution,

$$u_2(x,t) = -\frac{1}{12}\frac{aa_2 + 8k^2aa_2b - 8k'^2aa_2b - 6b}{ab} + a_2 \tan^2(x + \frac{1}{6}\frac{aa_2}{b}\, t) \tag{4.9}$$

where a_2 is arbitrary.
The method can also be applied to other nonlinear equations.

5 Conclusions

In this paper, we have developed a variation of the (G'/G)-expansion method, in which we have used the full advantage of the well known solution of the coupled Riccati equation. The presented method is used to find new exact travelling wave solutions of the Zakharov - Kuznetsov - BBM (ZKBBM) equation, the Boussinesq equation, the modified Camassa - Holm (mCH) equation and the (2+1) dimensional Potential Kadomstev-Petviashvili (PKP) equation. We have also used the method to find travelling wave solutions in terms of the Jacobi elliptic functions. The obtained results shows that the method is a powerful mathematical tool to solve a huge variety of nonlinear partial differential equations in mathematical physics.

It is to be noticed that if we choose various type of auxiliary equations, we can obtain new types of exact travelling wave solutions, which can be seen in our further study. We shall also apply the generalization of the method on various nonlinear equations in our further study.

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