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Modeling the Effect of Toxicant on Plant Biomass with Time Delay

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Abstract: In this paper, a nonlinear mathematical model is proposed and analysed to study the effect of toxicant on plant biomass with time delay. Plants are affected by toxicants released into the atmosphere from different sources such as industries, vehicular exhausts, etc. We assume that toxicant uptaken by plant biomass is converted into intermediate toxic product after a considerable time period and this intermediate toxic product then affects the intrinsic growth of plant biomass. It is further assumed that the carrying capacity of plant biomass is affected by environmental concentration of toxicant emitted directly into the atmosphere. The model is analysed using stability theory of differential equations and numerical simulation. It is found that as the emission rate of toxicant increases, the equilibrium level of plant biomass decreases. The density of plant biomass also decreases with increase in the depletion rate of toxicant uptaken by biomass and with the increase in the rate of formation of intermediate toxic product. Further, it is also shown that the positive equilibrium which is locally stable without delay remains locally stable under certain conditions when the time delay parameter is less than the threshold value, otherwise it may become unstable.

Keywords: Plant biomass density; time delay; intermediate toxic product; stability; simulation

1 Introduction

Due to rapid growth of industries and human population, a large amount of gaseous pollutants, particulate matters, toxic gases and contaminants enter into the atmosphere. They cause serious effect on human beings as well as on plant species when present in the concentration beyond their threshold value. Smog, dust, etc. reduce the amount of light reaching the plant and clogging the stomata reducing the carbon dioxide intake. The leaves of the plants are severely affected by the pollutants in the form of necrosis, chlorosis, etc. Gaseous pollutants such as CO_2, CO, SO_2, NO_x $\left(NO \text{ and } NO_2^-\right)$ and C_2H_4 are emitted into the atmosphere due to burning of hydrocarbons in motor vehicles, from industries, household discharges, etc. Dust pollution arise from roads, cement works and other industrial areas. These have adverse effect on plant and vegetation interfering with the plant growth and photosynthesis. In plant cells SO_2 dissolves to produce sulfite SO_3^{--} and bisulfate HSO_3^{--} ions in which sulfate is very toxic at high concentrations and damage the plasma lemma and other membranes, inhibit enzymes and disrupt metabolism. Disruptions of chloroplast membranes give rise to visible symptoms of injury such as bleaching and necrosis at the leaf margins and intercostals (between leaf veins) regions. Further, the ozone dissolves in the apoplastic water and rapidly decomposes to several high toxic radicals $\left(HO_2^-, O_2^-, O^-, HO^-\right)$. These radicals damage nucleic acids, give rise to mutations and cause lipid peroxidation, which breaks down lipids in membranes.

The effect of toxicants on plan t species have been studied by many investigators [1-4, 8, 9, 17-19, 21]. In particular, Crittenden and Read [3] compared the yield of Lolium multiflorum cv $S22$ and Dactylis glomerata cv $S143$ with ambient SO_2 polluted air in a sealed glass chamber and filtered air free from SO_2 and particulates and found that the yield decreases by 30 to 40% after 8 to 10 weeks in unfiltered air. The effect of ozone and hydrogen flouride on yield, leaf, necrosis, etc. have been investigated in [13, 14]. Fuhrer and Bungener [8] studied the effect of ozone on plants and found adverse effect on plants at given ozone concentrations.

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In recent years, some mathematical investigations have been made to study the effect of toxicants, emitted into the atmosphere from industries, household sources, road traffics, etc., on biological species [6, 7, 11, 16, 20, 22, 23]. In this direction, Freedman and Shukla [7] proposed models to study the effect of a toxicant on single species and predator-prey systems by assuming that the intrinsic growth rate of species decreases as the uptake concentration of the toxicant increases while its carrying capacity decreases with the environmental concentration of the toxicant. Liu et al. [11] investigated a single species model in a polluted closed environment with pulse toxicant input at fixed moment and showed that the population is extinct when the impulsive period is less than some critical value, otherwise the population is permanent. Shukla et al. [23] proposed a nonlinear mathematical model to study the effect of a resource dependent population assuming that resources are attacked simultaneously by a toxicant emitted into the environment from external source as well as formed by precursors of this population. They have shown that as the cumulative concentration of the toxicant in the environment increases, the density of population and its resource settle down to lower equilibria.

Some modeling studies have also been made to study the effect of time delay on biological species [5, 10, 12, 15]. In this regard, Dubey [5] presented a mathematical model to study the effect of time delay on the dynamics of a single species population living in a polluted environment and found that time delay has destabilizing effect on the system.

It may be noted here that the intrinsic growth of plant biomass is very often, affected by intermediate toxic product which is formed due to interaction of toxicant and plant biomass [16]. This aspect has not been taken in above mentioned studies. Therefore, in this paper, we propose a nonlinear mathematical model to study the effect of toxicant on plant biomass by taking into account the formation of intermediate toxic product with time delay.

2 Mathematical model

Consider a plant biomass with density $B(t)$ being affected by emission of a toxicant with cumulative emission rate $Q(t)$ in the environment. To model the phenomenon of effect of time delay, it is assumed that the toxicant uptaken by plant biomass interacts with bio-fluid present in it and forms an intermediate toxic product after time τ which then affects the biomass. It is further assumed that the intermediate toxic product decreases the intrinsic growth rate of biomass density while the environmental concentration of toxicant decreases its carrying capacity. The growth of plant species is assumed to be governed by logistic equation.

In view of the above, the model system is assumed to be governed by the following set of nonlinear ordinary differential equations,

$$\frac{dB(t)}{dt} = \left(r_0 - r_1 U_1(t) - r_2 \frac{B(t)}{K(t)}\right) B(t) \tag{1}$$

$$\frac{dT(t)}{dt} = Q(t) - \delta T(t) - \alpha B(t)T(t) + \pi v U(t)B(t) \tag{2}$$

$$\frac{dU(t)}{dt} = \alpha B(t)T(t) - \alpha U(t-\tau) - vU(t)B(t) - \delta_1 U(t) \tag{3}$$

$$\frac{dU_1(t)}{dt} = \alpha U(t-\tau) - \alpha_0 U_1(t) \tag{4}$$

$$B(0) = B_0 \geq 0,\ T(0) = T_0 \geq 0,\ U(0) = U_0 \geq 0,\ U_1(0) = U_{10} \geq 0$$

Here $B(t)$ is the biomass density, $T(t)$ is the concentration of toxicant in the environment, $U(t)$ is the concentration of toxicant uptaken by plant biomass and $U_1(t)$ is the concentration of intermediate toxic product, the constant $\delta > 0$ is the natural washout rate coefficient of $T(t)$, $\alpha > 0$ is the depletion rate coefficient of $T(t)$ due to its uptake by plant biomass, δ_1 is the natural removal rate coefficient of uptaken concentration of toxicant, $\alpha_1 > 0$ is rate of change of uptaken concentration of the toxicant into intermediate toxic product, $v > 0$ is the depletion rate coefficient of $U(t)$ due to decay of some numbers of plant biomass, $\alpha_0 > 0$ is the natural washout rate coefficient of $U_1(t)$, $r(U_1)$ is the intrinsic growth rate of $B(t)$ depending upon intermediate toxic product $U_1(t)$, r_0 is the maximum growth rate of $B(t)$ and $K(T)$ is the carrying capacity of plant biomass which depends upon $T(t)$. A fraction of depleted toxicants uptaken by decayed plant biomass (i.e. $\pi v U(t)B(t)$) may also re-enter the environment due to recycling phenomenon to increase the concentration of toxicants, where $0 \leq \pi \leq 1$ is reversible rate coefficient of toxicant due to recycling.

To model the problem, we assume that the growth rate of uptake concentration $U(t)$ is increased by the same amount as the depletion rate of the toxicant in the environment due to its uptake by the plant species. It is further assumed that the growth rate of formation of intermediate product is increased by the same amount as the rate of conversion of uptaken concentration due to its interaction with bio - fluid.

The function $r(U_1) = (r_0 - r_1U_1(t))$ denotes the intrinsic growth rate of the plant biomass which decreases with increase in U_1 and hence we assume that,

$$r(0) = r_0,\ \frac{dr}{dU_1} < 0 \textit{ i.e. } r\prime(U_1) < 0 \text{ for } U_1 > 0$$

The function $K(T) = K_0 - bT$ denotes the carrying capacity of plant biomass which decreases with increase in T i.e.

$$K(0) = K_0,\ \frac{dK}{dT} < 0 \textit{ i.e. } K\prime(T) < 0 \text{ for } T > 0$$

Remark 1 *For $\tau = 0$ and $\delta_1 = 0$, the model system (1) - (4) reduces to Naresh et al. [16] wherein the non-trivial equilibrium $E^*\,(B^*,\, T^*,\, U^*,\, U_1^*)$ exists for the case when toxicant is emitted continuously with a constant rate. This equilibrium is locally as well as globally asymptotically stable under certain conditions as stated in [16].*

2.1 Positivity of solutions

Model (1) - (4) describes the dynamics of a plant biomass and therefore, it is important to prove that all quantities will be positive for all time. We prove that all solutions of the system (1) - (4) with positive initial data will remain positive for all time $t > 0$.

Theorem 2 *Let the initial data be $B(0) = B_0 > 0$, $T(0) = T_0u) \geq 0$ for all $u \in [-\tau, 0]$ with $T_0(0) > 0$, $U(0) = U_0 \geq 0$, $U_1(0) = U_{10} \geq 0$. Then, the solution $(B(t),\, T(t),\, U(t),\, U_1(t))$ of the model system remain positive for all time $t > 0$.*

Proof. From Eq. (2), we have
$\frac{dT(t)}{dt} \geq -\delta T(t)$ and by applying a theorem on differential inequalities, we obtain $T(t) \geq c_1 exp\,\{-\delta t\} > 0$
where c_1 is a constant of integration.

Hence, solutions of the model are always positive for $t > 0$. Similar reasoning holds for other variables of the model system. ■

3 Stability analysis

In this section, we analyse the model system (1) - (4) using stability theory of differential equations under the two situations of toxicant emission in the environment i.e., $Q(t) = 0$, the situation when toxicant is emitted by an instantaneous source and $Q(t) = Q_0$, representing the continuous emission of toxicant with a constant rate, for example stacks emitting continuously.

Case 1 $\mathbf{Q(t) \equiv 0}$ (Instantaneous source)

In this case, the model (1) - (4) has only one non-negative equilibrium namely $E_0\left(\frac{r_0K_0}{r_2}, 0, 0, 0\right)$. The Jacobian matrix corresponding to E_0 is given by,

$$M_0 = \begin{bmatrix} -r_0 & 0 & 0 & -\frac{r_1r_0K_0}{r_2} \\ 0 & -\left(\delta + \frac{\alpha r_0K_0}{r_2}\right) & \frac{\pi v r_0 K_0}{r_2} & 0 \\ 0 & \frac{\alpha r_0K_0}{r_2} & -\left(\alpha_1 e^{-\lambda\tau} + \frac{v r_0K_0}{r_2} + \delta_1\right) & 0 \\ 0 & 0 & \alpha_1 e^{-\lambda\tau} & -\alpha_0 \end{bmatrix}$$

It can be easily seen that two eigenvalues are negative, i.e. $-r_0$ and $-\alpha_0$. Thus, the stability of E_0 depends on the roots of the equation,

$$\lambda^2 + \lambda A + B + (\alpha_1\lambda + C)\, e^{-\lambda\tau} = 0 \tag{5}$$

where

$$A = \left(\delta_1 + \delta + \frac{\alpha r_0 K_0}{r_2} + \frac{v r_0 K_0}{r_2}\right)$$
$$B = \left(\delta\delta_1 + \delta_1 \frac{\alpha r_0 K_0}{r_2} + \delta \frac{v r_0 K_0}{r_2}\right) + \frac{v\alpha r_0^2 K_0^2 (1-\pi)}{r_2^2}$$
$$C = \alpha_1 \left(\delta + \frac{\alpha r_0 K_0}{r_2}\right)$$

When $\tau = 0$, Eq. (5) becomes

$$\lambda^2 + \lambda A + B + (\alpha_1 \lambda + C) = 0$$

We note that $A > 0, B > 0$ and $C > 0$, are all positive as all the model parameters are assumed to be non-negative. Thus, this steady state is locally asymptotically stable without any condition.

When $\tau \neq 0$, we want to determine if the real part of some eigenvalues increases to reach zero and eventually becomes positive as τ increases

Let $\tau = i\omega$ is the root of Eq. (5), then by separating real and imaginary parts, it follows that

$$\omega^2 - B = C \cos\omega\tau + \alpha_1 \sin\omega\tau \tag{6}$$

$$-A\omega = \alpha_1 \omega \cos\omega\tau - C \sin\omega\tau \tag{7}$$

Thus, we get

$$\omega^4 + \left(A - 2B - \alpha_1^2\right)\omega^2 - C^2 = 0 \tag{8}$$

Clearly, Eq. (8) has a positive root ω_0^2, substituting ω_0^2 in Eqs. (6) - (7) and solving for τ, we get

$$\tau_n = \frac{1}{\omega_0}\left\{\cos^{-1}\left(\frac{\left(w_0^2 - B\right)C - A\alpha_1\omega_0^2}{C^2 + \alpha_1\omega_0^2}\right) + 2n\pi\right\}, \qquad n = 0, 1, 2, 3, \ldots$$

Further, as τ increases through τ_0, E_0 bifurcates into small periodic solutions, where

$$\tau_0 = \frac{1}{\omega_0}\left\{\cos^{-1}\left(\frac{\left(w_0^2 - B\right)C - A\alpha_1\omega_0^2}{C^2 + \alpha_1\omega_0^2}\right)\right\}$$

Theorem 3 *If $B(0) > 0$, then E_0 is globally asymptotically stable with respect to the non-negative octant.*

Proof. We note that

$$\frac{dB(t)}{dt} = \left(r_0 - r_1 U_1(t) - r_2 \frac{B(t)}{K(t)}\right) B(t)$$
$$\frac{dB(t)}{dt} \leq \left(r_0 - r_2 \frac{B(t)}{K_0)}\right) B(t)$$

Thus,

$$\lim_{x\to\infty} sup B(t) \leq \frac{K_0 r_0}{r_2}$$

Now from Eqs. (2) - (4), we obtain

$$\begin{aligned}\frac{d(T+U+U_1)}{dt} &= -\delta t-\delta_1 U-\alpha_0 U_1-(1-\pi)vUB\\ &\leq -\delta t-\delta_1 U-\alpha_0 U_1\\ &-\delta_2(T+U+U_1),\ \text{where } \min(\delta,\delta_1,\alpha_0)=\delta_2\end{aligned}$$

Thus, $T(t)+U(t)+U_1(t)\leq\{T(0)+U(0)+U_1(0)\}\exp(-\delta_2 t)$

Hence the system is dissipative and, therefore, it follows that

$\lim_{t\to\infty} supT(t)=\lim_{t\to\infty} supU(t)=\lim_{t\to\infty} supU_1(t)=0$. Hence the limit $B(t)$ tends to K_0r_0/r_2 and since $B(0)>0$, the theorem follows. ■

Case 2: $\mathbf{Q(t)=Q_0>0}$ (Continuous source)

In this case, the model has two equilibria namely $E(0,Q_0/\delta,0,0)$ and $E^*(B^*,T^*,U^*,U_1^*)$. The existence of E is obvious. To prove the existence of non-trivial equilibrium E^*, we show that the positive solution of (B^*,T^*,U^*,U_1^*) is given by,

$$T=\frac{Q_0(\alpha_1+\delta_1+vB)}{f(B)}=g(B) \tag{9}$$

$$U=\frac{\alpha Q_0 B}{f(B)} \tag{10}$$

$$U_1=\frac{\alpha_1\alpha B}{\alpha_0 f(B)}=h(B) \tag{11}$$

where $f(B)=(\alpha_1+\delta_1)\delta+[\alpha(\alpha_1+\delta_1)+\delta v]B+(1-\pi)\alpha vB^2$

Let

$$F(B)=r_2B-r_0K\{g(B)\}+r_1h(B)K\{g(B)\}=0 \tag{12}$$

when $\quad B=0, \quad g(0)\neq 0, \quad f(0)\neq 0, \quad h(0)=0$

Then, $F(0)=-r_0K\{g(0)\}<0$

and

$$F\left(\frac{K_0r_0}{r_2}\right)=r_0K_0-r_0K\left\{g\left(\frac{K_0r_0}{r_2}\right)\right\}+r_1h\left(\frac{K_0r_0}{r_2}\right)K\left\{g\left(\frac{K_0r_0}{r_2}\right)\right\}>0$$

This guarantees the existence of a root B^* of $F(B)=0$ in $0<B<K_0r_0/r_2$. Further, this root will be unique provided

$$F(B)=r_2-r_0K\{g(B)\}g(B)+r_1h(B)K\{g(B)\}+r_1K\{g(B)\}g(B)>0.$$

Knowing the value of B^*, the values of T^*, U^* and U_1^*, can be computed from Eqs. (9) - (11), respectively.

3.1 Local stability of the non-trivial equilibrium

In this section, we analyze the local stability of the non-trivial equilibrium E^*. The Jacobian matrix corresponding to E^* is obtained as follows:

$$\begin{bmatrix}-a_1 & a_2 & 0 & -a_3\\ -a_4 & -a_5 & a_6 & 0\\ a_7 & a_8 & -(\alpha_1 e^{-\lambda\pi}+a_9) & 0\\ 0 & 0 & \alpha_1 e^{-\lambda\pi} & -\alpha_0\end{bmatrix}$$

where

$a_1=\dfrac{r_2B^*}{K(T^*)}$, $\quad a_2=\dfrac{r_0B^{*2}K\prime(T^*)}{[K(T^*)]^2}$, $\quad a_3=r_1B^*$, $\quad a_4=(\alpha T^*-\pi vU^*)$

$a_5=(\delta+\alpha B^*)$, $\quad a_6=(\pi vB^*)$, $\quad a_7=(\alpha T^*-vU^*)$, $\quad a_8=\alpha B^*$, $\quad a_9=vB^*+\delta_1$

The Jacobin matrix leads to the characteristic equation,

$$\lambda^4 + m_3\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + \left(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0\right)e^{-\lambda\tau} = 0 \tag{13}$$

where

$m_3 = \alpha_0 + \alpha_1 + \alpha_4 + \alpha_9$
$m_2 = \alpha_0 a_1 + a_2 a_4 + (\alpha_0 + a_1)(a_5 + a_9) + (a_5 a_9 - a_6 a_8)$
$m_1 = \alpha_0 a_1 (a_5 + a_9) + (\alpha_0 + a_1)(a_5 a_9 - a_4 a_8) + a_2 a_4 (\alpha_0 + a_9) - a_2 a_6 a_7$
$m_0 = \alpha_0 a_1 (a_5 a_9 - a_6 a_8) + \alpha_0 a_2 (a_4 a_9 - a_6 a_7)$
$n_3 = \alpha_1$
$n_2 = \alpha_1(\alpha_0 + a_1 + a_5)$
$n_1 = \alpha_1(\alpha_0 a_1 + a_5(\alpha_0 + a_1) + a_3 a_7 + a_2 a_4)$
$n_0 = \alpha_1(\alpha_0 a_1 a_5 + \alpha_0 a_2 a_4 - a_3 a_4 a_8 + a_3 a_5 a_7)$

We note that $m_3 > 0, m_2 > 0, m_1 > 0, m_0 > 0, n_3 > 0, n_2 > 0, n_1 > 0$ and $n_0 > 0$ are all positive. For $\tau = 0$, we may state the following result that follow directly from Eq. (13) and the Routh-Hurwitz criteria.

Theorem 4 *When $\tau = 0$, the characteristic equation (13) yields*

$$\lambda^4 + (m_3 + n_3)\lambda^3 + (m_2 + n_2)\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0$$

which can be written as,

$$\lambda^4 + b_3\lambda^3 + b_2\lambda^2 + b_1\lambda + b_0 = 0$$

where

$b_3 = m_3 + n_3 = \alpha_0 + \alpha_1 + a_1 + a_4 + a_8$
$b_2 = m_2 + n_2 = \alpha_0 a_1 + (\alpha_0 + a_1)(a_4 + a_8) + (a_4 a_8 - a_5 a_7) + \alpha_1(\alpha_0 + a_1 + a_4)$
$b_1 = m_1 + n_1 = \alpha_0 a_1 (a_4 + a_8) + (\alpha_0 + a_1)(a_4 a_8 - a_5 a_7) + \alpha_1(\alpha_0 a_1 + a_4(\alpha_0 + a_1) + a_2 a_6)$
$b_0 = m_0 + n_0 = \alpha_0 a_1 (a_4 a_8 - a_5 a_7) + \alpha_1(\alpha_0 a_1 a_4 - a_2 a_3 a_7 + a_2 a_4 a_6))$

The equilibrium E^* is locally asymptotically stable if and only if the following inequalities hold:

$$b_1 b_2 > b_0,\ \ b_3\left(b_1 b_2 - b_3\right) > b_1^2 b_4$$

The main purpose of this article is to study the stability behaviour of E^* in the case $\tau \neq 0$. For this, we determine the sign of real parts of the zeros of Eq. (13) that characterizes the stability behaviour of E^*. Obviously, $i\eta(\eta > 0)$ is the root of Eq. (13) if and only if η satisfies

$$\eta^4 - im_3\eta^3 - m_2\eta^2 + m_1 i\eta + m_0 = -(-n_3 i\eta^3 - n_2\eta^2 + n_1 i\eta + n_0)(\cos\eta\tau - i\sin\eta\ tau)$$

Separating the real and imaginary parts, we have

$$\eta^4 - m_2\eta^2 + m_0 = -(-n_2\eta^2 + n_0)\cos\eta\tau - (-n_3\eta^3 + n_1\eta)\sin\eta\tau \tag{14}$$
$$-m_3\eta^3 + m_1\eta = -(-n_3\eta^3 + n_1\eta)\cos\eta\tau + (-n_2\eta^2 + n_0)\sin\eta\tau \tag{15}$$

Eliminating τ by squaring and adding Eqs. (14) and (14), we get the equation determining forη as;

$$\eta^8 + p\eta^6 + q\eta^4 + r\eta^2 + s = 0 \tag{16}$$

where

$$p = (m_3^2 - 2m_2 - n_3^2)$$
$$q = (m_2^2 - 2m_1m_3 + 2n_1n_3 + 2m_0 - n_2^2)$$
$$r = (m_1^2 - 2m_0m_2 + 2n_0n_2 - n_1^2)$$
$$s = m_0^4 - n_0^4$$

Substituting $\eta^2 = z$ in Eq. (16), we define a polynomial

$$P(z) = z^4 + pz^3 + qz^2 + rz + s = 0 \tag{17}$$

Now, we obtain the following results on the distribution of roots of equation,

$$p_1 = \frac{1}{2}q - \frac{3}{16}p^2, \qquad q_1 = \frac{1}{32}p^3 - \frac{1}{8}pq + r, \qquad \triangle = \left(\frac{q_1}{2}\right)^3 + \left(\frac{p_1}{3}\right)^3, \qquad \omega = \frac{-1 + i\sqrt{3}}{2})$$
$$y_1 = \sqrt[3]{-\frac{q_1}{2} + \sqrt{\triangle}}$$
$$y_2 = \omega\sqrt[3]{-\frac{q_1}{2} + \sqrt{\triangle}} + \omega^2\sqrt[3]{-\frac{q_1}{2} - \sqrt{\triangle}}$$
$$y_3 = \omega^2\sqrt[3]{-\frac{q_1}{2} + \sqrt{\triangle}} + \omega\sqrt[3]{-\frac{q_1}{2} - \sqrt{\triangle}}$$
$$z_i = y_i - \frac{3p}{4}$$

Corollary 5 *For the polynomial equation (17)*

(i) If $s < 0$, then Eq. (17) has at least one positive root;

(ii) If $s \geq 0$ and $\triangle \geq 0$, then Eq. (17) has at least one positive roots if and only if $z_1 > 0$ and $P(z_1) < 0$

(iii) If $s \geq 0$ and $\triangle < 0$, then Eq. (17) has at least one positive roots if there exists at least one $z^* \in \{z_1, z_2, z_3\}$ such that $z^* > 0$ and $P(z^*) \leq 0$, where $P(z) = z^4 + pz^3 + qz^2 + rz + s$.

Corollary 6 *Suppose that*

$$r_2 v\left(\frac{\delta}{\alpha} + \frac{1}{r_1}\right) + \frac{\pi v}{r_1}r_0 > \frac{r_0}{r_1}(\alpha_1 + \delta_1)$$

$$\frac{\alpha_0\delta r_2 v}{\alpha\alpha_1} + Q_0 > \frac{\alpha_0}{\alpha_1}r_0(\alpha_1 + \delta_1)\left(\frac{\delta}{\alpha} + \frac{1}{r_1}\right)$$

(i) If one of the followings holds: (a) $s < 0$; (b) $s \geq 0, \triangle < 0, z_1 > 0$ and $P(z_1) < 0$; (c) $s \geq 0, \triangle < 0$; there exists a $z^* \in \{z_1, z_2, z_3, z_4\}$ such that $z^* > 0$ and $P(z^*) \leq 0$, then all roots of Eq. (17) have negative real parts for $\tau \in [0, \tau_0)$. Here, τ_0 is a certain positive constant.

(ii) If the conditions (a) - (c) of (i) are not satisfied, then all roots of Eq. (17) have negative real parts for all $\tau \geq 0$.

Suppose that equation (17) has positive roots. Without loss of generality, we assume that it has four positive roots, defined by z_1, z_2, z_3 and z_4 respectively. Then equation (17) has four positive roots [12]

$$\eta_1 = \sqrt{z_1}, \qquad \eta_2 = \sqrt{z_2}, \qquad \eta_3 = \sqrt{z_3}, \qquad \eta_4 = \sqrt{z_4}$$

From Eqs. (14) - (15), we have

$$\cos\eta\tau = \frac{(\eta^4 - m_2\eta^2 + m_0)(n_2\eta^2 - n_0) + (-n_3\eta^3 + n_1\eta)(m_3\eta^3 - m_1\eta)}{(n_2\eta^2 - n_0)^2 + (-n_3\eta^3 + n_1\eta)^2}$$

Thus, if we denote

$$\tau_k^{(j)} = \frac{1}{\eta_k}\left\{\cos^{-1}\left(\frac{(\eta_k^4 - m_2\eta_k^2 + m_0)(n_2\eta_k^2 - n_0) + (-n_3\eta_k^3 + n_1\eta_k)(m_3\eta_k^3 - m_1\eta_k)}{(n_2\eta_k^2 - n_0)^2 + (-n_3\eta_k^3 + n_1\eta_k)^2}\right) + 2j\pi\right\}$$

where $k = 1, 2, 3, 4; j = 0, 1, \ldots,$ *then* $\pm i\eta_k$ *is a pair of purely imaginary roots of equation (13) with* $\tau_k^{(j)}$. *Define*

$$\tau_0 = \tau_{k_0}^{(0)} = \min_{k\in\{1,2,3,4\}}\left\{\tau_k^{(0)}\right\}, \eta_0 = \eta_{k_0}$$

Next we turn to show

$$\frac{d(Re\lambda)}{d\tau} \mid \tau = \tau_k > 0$$

This will signify that there exists at least one eigenvalue value with positive real part for $\tau > \tau_k$. *Substituting* $\lambda(\tau)$ *into equation (13) and taking the derivative with respective to* τ *, we obtain*

$$\left(4\lambda^3 + 3m_3\lambda^2 + 2m_2\lambda + m_1\right)\frac{d\lambda}{d\tau} + \left(-\tau\frac{d\lambda}{d\tau} - \lambda\right)\left(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0\right)e^{-\lambda\tau} + \frac{d\lambda}{d\tau}\left(3n_3\lambda^2 + 2n_2\lambda + n_1\right)e^{-\lambda\tau} = 0$$

$$\left\{(4\lambda^4 + 3m_3\lambda^2 + 2m_2\lambda + m_1) - \tau(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau} + (3n_3\lambda^3 + 2n_2\lambda + n_1)e^{-\lambda\tau}\right\}\frac{d\lambda}{d\tau} = \lambda\left(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0\right)e^{-\lambda\tau}$$

This gives

$$\left\{\frac{d\lambda}{d\tau}\right\}^{-1} = \left\{\frac{(4\lambda^3 + 3m_3\lambda^2 + 2m_2\lambda + m_1) + (3n_3\lambda^2 + 2n_2\lambda + n_1)e^{-\lambda\tau}}{\lambda(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau}} - \frac{\tau}{\lambda}\right\}$$

$$\left\{\frac{d\lambda}{d\tau}\right\}^{-1} = \left\{\frac{(4\lambda^3 + 3m_3\lambda^2 + 2m_2\lambda + m_1)}{\lambda(n_3\lambda^+ n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau}} + \frac{(3n_3\lambda^2 + 2n_2\lambda + n_1)e^{-\lambda\tau}}{\lambda(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau}} - \frac{\tau}{\lambda}\right\}$$

Therefore,

$$Sign\left\{\left(\frac{d(Re\lambda)}{d\tau}\right)\right\} = Sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}$$

$$= Sign\left\{Re\left[\frac{(4\lambda^3 + 3m_3\lambda^2 + 2m_2\lambda + m_1)}{\lambda(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau}}\right]_{\lambda=i\eta} + Re\left[\frac{(3n_3\lambda^2 + 2n_2\lambda + n_1)}{\lambda(n_3\lambda^3 + n_2\lambda^2 + n_1\lambda + n_0)}\right]_{\lambda=i\eta}\right\}$$

$$= \frac{\eta_k^2\left(4\eta_k^6 + \left(3m_3^2 - 6m_2\right)\eta_k^4 + \left(2m_2^2 + 4m_0 - m_3m_1\right)\eta_k^2 + m_1^2 - 2m_2m_0\right)}{\bigwedge} + \frac{\eta_k^2\left(-3n_3^2\eta_k^4 + \left(-2n_2^2 + 4n_1n_3\right)\eta_k^2\right) + 2n_2n_0 - n_1^2}{\bigwedge}$$

$$= \frac{\eta_k^2 + \left(4\eta_k^6 + 3\left(m_3^2 - 2m_2 - n_3^2\right)\eta_k^4 + 2\left(m_2^2 + 2m_0 - 2m_3m_1 - n_2^2 + 2n_3n_1\right)\eta_k^2 + m_1^2 - 2m_2m_0 + 2n_2n_0 - n_1^2\right)}{\bigwedge}$$

$$= \frac{\eta_k^2 P(z_k)}{\bigwedge} > 0$$

where $\bigwedge = \eta_k^2\lfloor\left(n_0 - n_2\eta_k^2\right)^2 + \left(n_1\eta - n_3\eta_k^3\right)^2\rfloor$
Thus, we have

$$Sign\left\{\left(\frac{d(Re\lambda)}{d\tau}\right)\right\}_{\tau=\tau_k^{j0}} = Sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\tau=\tau_k^{j0}} = Sign\left[\frac{\eta_k^2 P\prime(z_k)}{\bigwedge}\right]$$

Furthermore, since $z_k > 0$ and $\bigwedge > 0$, we conclude that $\left(\frac{d(Re\lambda(\tau))}{d\tau}\right)_{\tau=\tau_k^{(j)}}$ and $P\prime(z_k)$ have the same sign. Thus if $P\prime(z_k)$, then for $\tau > \tau_k$, there is at least one zero of Eq. (13) which has positive real part, and hence E^* is unstable for $\tau > \tau_k$.

Lemma 7 *The set $\Omega = \left\{0 \le B \le \frac{K_0 r_0}{r_2}, 0 \le T + U + U_1 \le \frac{Q_0}{\delta_m}\right\}$ attracts all solutions initiating in the interior of non-negative octant, where* $\min(\delta, \delta_1, \alpha_0) = \delta_m$

Proof. From Eq. (1), we have

$$\frac{dB(t)}{dt} = \left(r_0 - r_1 U_1(t) - r_2 \frac{B(t)}{K(T)}\right) B(t)$$
$$\frac{dB(t)}{dt} \le \left(r_0 - r_2 \frac{B(t)}{K_0}\right) B(t)$$

Thus, we get $\lim_{t\to\infty} sup B(t) \le \frac{K_0 r_0}{r_2}$
Now consider

$$\begin{aligned}\frac{d(T+U+U_1)}{dt} &= Q_0 - \delta T - \delta_1 U - \alpha_0 U_1 - (1-\pi) vUB \\ &\le Q_0 - \delta T - \delta_1 U - \alpha_0 U_1 \\ &\le Q_0 - \delta_m (T+U+U_1) \text{ where } \min(\delta, \delta_1, \alpha_0) = \delta_m\end{aligned}$$

Therefore, $\lim_{t\to\infty} sup(T + U + U_1) \le \frac{Q_0}{\delta_m}$ ■

Theorem 8 *Let the function $K(T)$ satisfy in Ω such that $K_m \le K(T) \le K_0$, $0 \le -K\prime(T) \le q$ for some positive constants K_m and q, then if the following inequalities hold in Ω:*

$$\left[(\alpha + \pi v)\frac{Q_0}{\delta_m} + r_0 K_0 \frac{q}{K_m^2}\right] < \frac{2}{3}\frac{r_2}{K(T^*)}(\delta + \alpha B^*)$$

$$(\alpha + v)^2 \left(\frac{Q_0}{\delta_m}\right)^2 < \frac{2}{3}\frac{r_2}{K(T^*)}\left[(vB^* + \delta_1 - \alpha_1) - \frac{\alpha_1}{2}\right]$$

$$[\pi v + \alpha]^2 B^{*2} < (\delta + \alpha B^*)\left[(vB^* + \delta_1 - \alpha_1) - \frac{\alpha_1}{2}\right]$$

$$r_1^2 < \frac{4}{3}\left(\alpha_0 - \frac{\alpha_1}{2}\right)\frac{r_2}{K(T^*)}$$

the equilibrium E^* is non linearly asymptotically stable.

Proof. Let us consider the following positive definite function about E^*,

$$V = \left(B - B^* - B^* \ln \frac{B}{B^*}\right) + \frac{1}{2}k_1 (T - T^*)^2 + \frac{1}{2}k_2 (U - U^*)^2 + \frac{1}{2}k_3 (U_1 - U_1^*)^2 \tag{18}$$

where k_1, k_2 and k_3 are arbitrary real constants to be chosen appropriately.
The derivative of V along solutions is given by,

$$\dot{V} = (B - B^*)\frac{1}{B}\dot{B} + k_1(T - T^*)\dot{T} + k_2(U - U^*)\dot{U} + K_3(U_1 - U_1^*)\dot{U_1}$$

$$\begin{aligned}\dot{V} = &-\frac{r_2}{K(T^*)}(B - B^*)^2 - k_1(\delta + \alpha B^*)(T - T^*)^2 - k_2(\delta_1 + vB^*)(U - U^*)^2 - k_3\alpha_0(U_1 - U_1^*)^2 - \\ &r_1(U_1 - U_1^*)(B - B^*) - [k_1(\alpha T - \pi vU) + r_2 B\eta(T)](B - B^*)(T - T^*) + (k_1\pi v + k_2\alpha)B^*(T - T^*)(U - U^*) + \\ &k_2(\alpha T - vU)(B - B^*)(U - U^*) - k_2\alpha_1(U(t - \tau) - U^*)(U - U^*) + k_3\alpha_1(U(t - \tau) - U^*)(U_1 - U_1^*)\end{aligned}$$

where

$$\eta(T) = \begin{cases} \frac{1}{K(T)} - \frac{1}{K(T^*)}, & T \neq T^* \\ \frac{K\prime(T^*)}{K^2(T^*)}, & T = T^* \end{cases}$$

$$\begin{aligned}\dot{V} = &-\frac{r_2}{K(T^*)}(B - B^*)^2 - k_1(\delta + \alpha B^*)(T - T^*)^2 - k_2(\delta_1 + vB^*)(U - U^*)^2 - k_3\alpha_0(U_1 - U_1^*)^2 - \\ &r_1(U_1 - U_1^*)(B - B^*) - [k_1(\alpha T - \pi vU) + r_2 B\eta(T)](B - B^*)(T - T^*) + (k_1\pi v + k_2\alpha)B^*(T - T^*)(U - U^*) + \\ &k_2(\alpha T - vU)(B - B^*)(U - U^*) + \frac{k_2\alpha_1}{2}(U(t - \tau) - U^*)^2 + \frac{k_2\alpha_1}{2}(U - U^*)^2 + \\ &\frac{k_3\alpha_1}{2}(U(t - \tau) - U^*)^2 + \frac{k_3\alpha_1}{2}(U_1 - U_1^*)^2\end{aligned}$$

$$\begin{aligned}\dot{V} = &-\frac{r_2}{K(T^*)}(B - B^*)^2 - k_1(\delta + \alpha B^*)(T - T^*)^2 - k_2\left(\delta_1 + vB^* - \frac{\alpha_1}{2}\right)(U - U^*)^2 - \\ &k_3\left(\alpha_0 - \frac{\alpha_1}{2}\right)(U_1 - U_1^*)^2 - r_1(U_1 - U_1^*)^2(B - B^*) - [k_1(\alpha T - \pi vU) + r_2 B\eta(T)](B - B^*)(T - T^*) + \\ &(k_1\pi v + k_2\alpha)B^*(T - T^*)(U - U^*) + k_2(\alpha T - \pi U)(B - B^*)(U - U^*) + \frac{(k_2 + k_3)\alpha_1}{2}(U(t - \tau) - U^*)^2\end{aligned}$$

We choose Lyapunov functional of the form,

$$W(t) = V(t) + \frac{(k_2 + k_3)\alpha_1}{2}\int_{t-\tau}^{t}(U(t) - U^*)^2\,dt \tag{19}$$

from which we get,

$$\dot{W} = \dot{V} + \frac{(k_2 + k_3)\alpha_1}{2}(U(t) - U^*)^2 - \frac{(k_2 + k_3)\alpha_1}{2}(U(t - \tau) - U^*)^2$$

$$\begin{aligned}\dot{W} = &-\frac{r_2}{K(T^*)}(B - B^*)^2 - k_1(\delta + \alpha B^*)(T - T^*)^2 - k_2\left(\delta_1 + vB^* - \frac{\alpha_1}{2}\right)(U - U^*)^2 - \\ &k_3\left(\alpha_0 - \frac{\alpha_1}{2}\right)(U_1 - U_1^*)^2 - r_1(U_1 - U_1^*)^2(B - B^*) - [k_1(\alpha T - \pi vU) + r_2 B\eta(T)](B - B^*)(T - T^*) + \\ &(k_1\pi v + k_2\alpha)B^*(T - T^*)(U - U^*) + k_2(\alpha T - \pi U)(B - B^*)(U - U^*) + \frac{(k_2 + k_3)\alpha_1}{2}(U(t - \tau) - U^*)^2 + \\ &\frac{(k_2 + k_3)\alpha_1}{2}(U(t) - U^*)^2 - \frac{(k_2 + k_3)\alpha_1}{2}(U(t - \tau) - U^*)^2\end{aligned}$$

$$\begin{aligned}\dot{W} = &-\frac{r_2}{K(T^*)}(B - B^*)^2 - k_1(\delta + \alpha B^*)(T - T^*)^2 - \left[k_2(\delta_1 + vB^* - \alpha_1) - \frac{k_3\alpha_1}{2}\right](U - U^*)^2 - \\ &k_3\left(\alpha_0 - \frac{\alpha_1}{2}\right)(U_1 - U_1^*)^2 - r_1(U_1 - U_1^*)(B - B^*) - [k_1(\alpha T - \pi vU) + r_2 B\eta(T)](B - B^*)(T - T^*) + \\ &(k_1\pi v + k_2\alpha)B^*(T - T^*)(U - U^*) + k_2(\alpha T - vB)(B - B^*)(T - T^*)\end{aligned}$$

Now, $\dot{W}$ will be negative definite, provided,

$$[k_1(\alpha T - \pi vU) + r_2 B\eta(T)]^2 < \frac{2k_1}{3}\frac{r_2}{K(T^*)}(\delta + \alpha B^*) \tag{20}$$

$$k_2^2\left(\alpha T - vU\right)^2 < \frac{2}{3}\frac{r_2}{K(T^*)}\left[k_2(vB^* + \delta_1 - \alpha_1) - \frac{k_3\alpha_1}{2}\right] \tag{21}$$

$$\left[k_1\pi v + k_2\alpha\right]^2 B^{*2} < k_1\left(\delta + \alpha B^*\right)\left[k_2(vB^* + \delta_1 - \alpha_1) - \frac{k_3\alpha_1}{2}\right] \tag{22}$$

$$r_1^2 < \frac{4}{3}k_3\left(\alpha_0 - \frac{\alpha_1}{2}\right)\frac{r_2}{K(T^*)} \tag{23}$$

Now choosing $k_1 = k_2 = k_3 = 1$ and on maximizing LHS and minimizing RHS, we get from above equations,

$$\left[(\alpha + \pi v)\frac{Q_0}{\delta_m} + r_0 K_0 \frac{q}{K_m^2}\right]^2 < \frac{2}{3}\frac{r_2}{K(T^*)}(\delta + \alpha B^*) \tag{24}$$

$$(\alpha + v)^2\left(\frac{Q_0}{\delta_m}\right)^2 < \frac{2}{3}\frac{r_2}{K(T^*)}\left[(vB^* + \delta_1 - \alpha_1) - \frac{\alpha_1}{2}\right] \tag{25}$$

$$\left[\pi v + \alpha\right]^2 B^{*2} < (\delta + \alpha B^*)\left[(vB^* + \delta_1 - \alpha_1) - \frac{\alpha_1}{2}\right] \tag{26}$$

$$r_1^2 < \frac{4}{3}\left(\alpha_0 - \frac{\alpha_1}{2}\right)\frac{r_2}{K(T^*)} \tag{27}$$

Under the conditions Eqs. (24) - (27), $\dot{W}$ will be negative definite showing that W is a Lyapunovs function and hence the theorem. ■

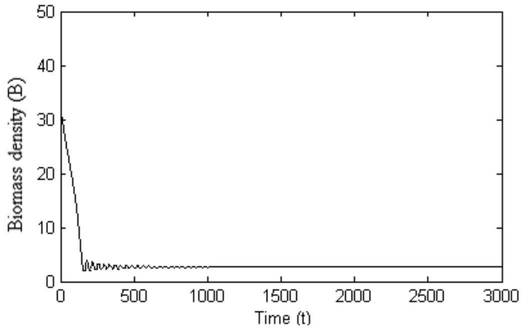

Figure 1: Variation of plant biomass density with time for $\tau \in [0, \tau_0)$.

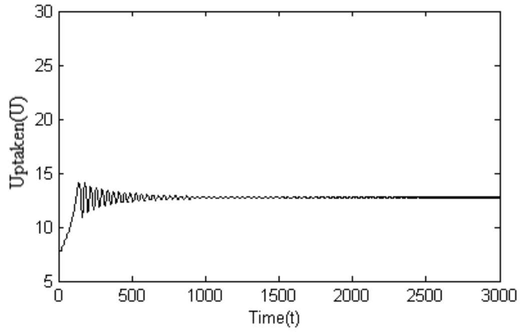

Figure 2: Variation of concentration of toxicant uptaken by biomass with time for $\tau \in [0, \tau_0)$.

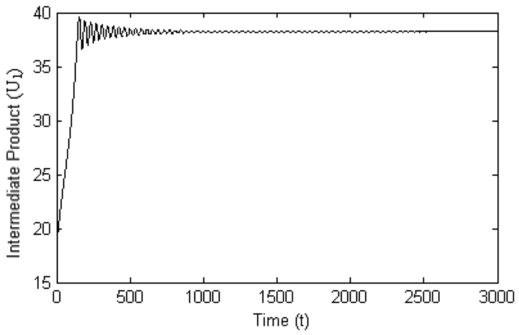

Figure 3: Variation of concentration of intermediate toxic product with time for $\tau \in [0, \tau_0)$.

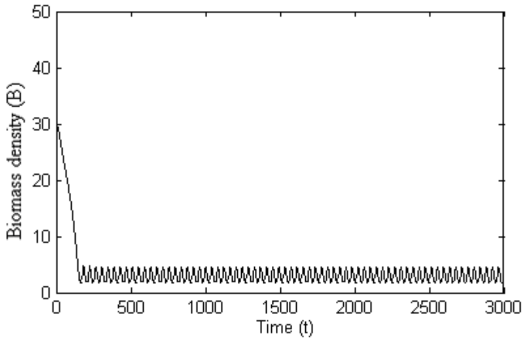

Figure 4: Variation of plant biomass density with time for $\tau > \tau_0$.

4 Numerical simulation and discussion

In this section, we present numerical simulation of system (1) - (4) for different values of parameters to study the dynamical behavior of the model system and to support the analytical results. For that the system(1) - (4) is integrated numerically with the help of MATLAB 7.1 using the following set of parameter values,

$r_0 = 2,\ r_1 = 0.05,\ r_2 = 0.05,\ \pi = 0.02,\ \upsilon = 0.03,\ \delta_1 = 0.15,\ \tau = 3.75,\ K_0 = 1.5,$

$b = 0.001,\ Q_0 = 10,\ \delta = 2,\ \alpha = 0.7,\ \alpha_1 = 0.15,\ \alpha_0 = 0.05$
with initial values $B(0) = 50,\ T(0) = 20,\ U(0) = 10$ and $U_1(0) = 15$.

The equilibrium values for different variables in E^* are computed as $B^* = 2.516157,\ T^* = 2.663663,\ U^* = 12.494614,\ U_1^* = 37.483842$.

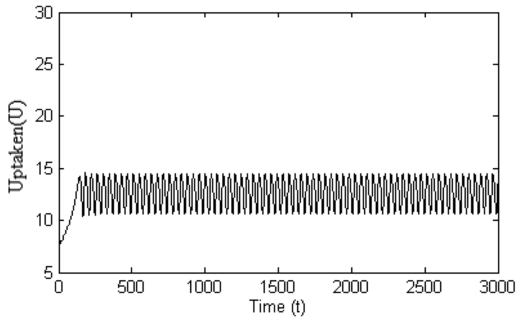

Figure 5: Variation of concentration of toxicant uptaken by biomass with time for $\tau > \tau_0$.

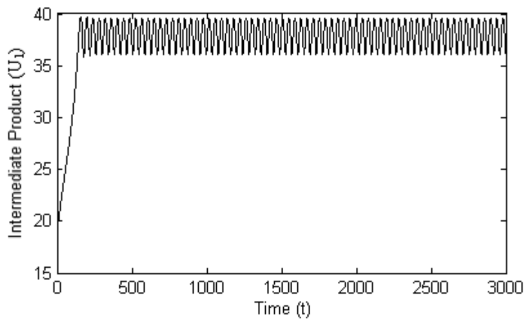

Figure 6: Variation of concentration of intermediate toxic product with time for $\tau > \tau_0$.

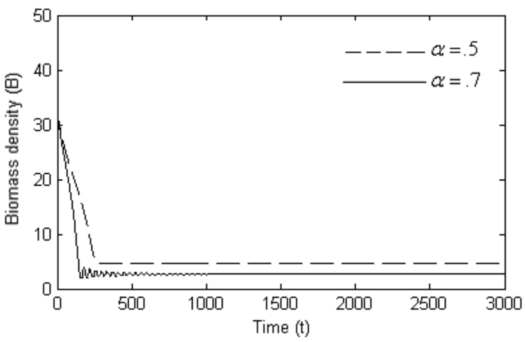

Figure 7: Variation of plant biomass density with time for different values of α

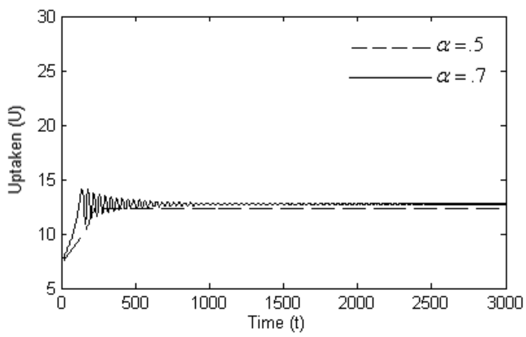

Figure 8: Variation of concentration of toxicant uptaken by biomass with time for different values of α.

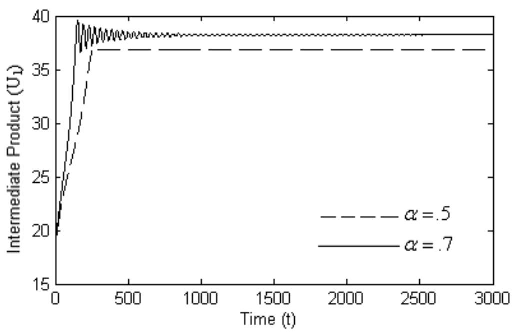

Figure 9: Variation of concentration of intermediate toxic product with time for different values of α.

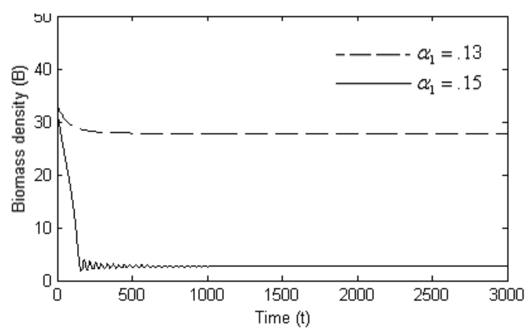

Figure 10: Variation of plant biomass density with time for different values of α_1.

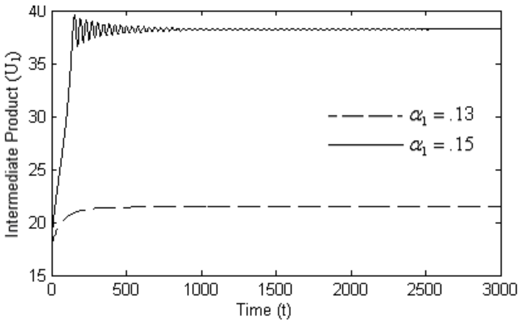

Figure 11: Variation of concentration of intermediate toxic product with time for different values of α_1.

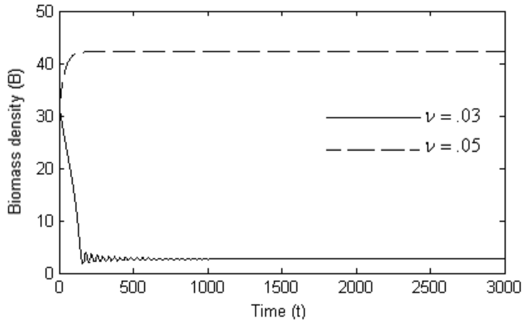

Figure 12: Variation of plant biomass density with time for different values of υ.

The variation of plant biomass density $B(t)$, concentration of toxicant uptaken by biomass (U) and the concentration of intermediate toxic product (U_1) is shown in figs. (1) - (6) respectively with time to see the effect of time delay τ. It is observed that when $\tau_0 > \tau \in [0, \tau = 4.5)$ the steady state is asymptotically stable [see figs. (1) - (3)]. When τ crosses the critical value i.e. when $5 = \tau > \tau_0$ the steady state becomes unstable and has the branch of periodic solutions bifurcating from the positive equilibrium E^* [see figs. (4) - (6)]. This observation supports the analytical results obtained. In figs. (7) - (9), we have shown the effect of depletion rate coefficient α of toxicant due to uptake by plant biomass on the variation of plant biomass density, concentration of toxicant uptaken by plant biomass and the concentration of intermediate toxic product with time respectively. It is found that the density of plant biomass decreases with increase in depletion rate of toxicant uptaken by biomass. This is due to the fact that with increase in α, the concentration of toxicant uptaken by plant biomass increases and as a consequence, the concentration of intermediate toxic product also increases leading to decline in the density of plant biomass. Figs. (10) - (11) depict the variation of plant biomass density and the concentration of intermediate toxic product, respectively with time for different values of α_1 the rate of formation of intermediate toxic product. It is seen that the density of plant biomass decreases with increase in the rate of formation of intermediate toxic product whereas the concentration of intermediate toxic product increases. However, the density of plant biomass increases with increase in the depletion rate coefficient of toxicant uptaken by biomass due to decay of some numbers of plant biomass. This is because of higher depletion of toxicant uptaken by decayed biomass which results in low formation of intermediate toxic product leading to increase in plant biomass density, (see fig. (12)).

5 Conclusion

In this paper, a nonlinear mathematical model is proposed and analyzed to study the effect of toxicant on plant biomass with time delay. It is assumed that toxicant uptaken by plant biomass is converted into intermediate toxic product after a considerable time period and this intermediate product affects the plant biomass. It is further assumed that the carrying capacity of plant biomass decreases with increase in the environmental concentration of toxicant, whereas its intrinsic growth rate decreases with increase in the concentration of intermediate toxic product. The model is analyzed by using the stability theory of differential equations and numerical simulation. It is seen that the equilibrium level of plant biomass decreases as the emission rate of toxicant increases. The plant biomass density further decreases as the rate of depletion of toxicant uptaken by plant biomass as well as the rate of formation of intermediate toxic product increase. It is shown that the positive equilibrium which is locally stable without delay remains locally stable under certain conditions when the time delay parameter is less than the threshold value, otherwise it may become unstable.

Acknowledgments

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