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A Numerical Study of Cauchy Reaction-Diffusion Equation

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Abstract: In this paper, new algorithm of homotopy analysis method is successfully applied to obtain the approximate analytical solutions of the Cauchy reaction-diffusion equation. Reaction-diffusion equations have special importance in engineering and sciences and constitute a good model for many systems in various fields. Application of new algorithm of homotopy analysis method to this problem shows the rapid convergence of the sequence constructed by this method to the exact solution. The solutions of the problem for different generalized particular cases are presented graphically.

Keywords: Homotopy analysis method, Cauchy reaction-diffusion equation, External force, Reaction term.
MSC(2010) Classification code: 34G20; 35A10; 35K57.

1 Introduction

Reaction-diffusion systems (Grindrod, 1996; Cantrell and Cosner, 2003) are mathematical models which provide how the concentration of one or more substances dispersed in space changes under the influence of two processes. First one, local chemical reactions in which the substances are transformed into each other, and second one, diffusion which causes the substances to spread out over a surface in space. This explanation cleared that reaction-diffusion systems are naturally applied in chemical engineering. However, the system can also describe dynamical processes for example biology, geology and physics (Smoller, 1994) and ecology, of non-chemical nature.

Tello (2006) derived the stability of steady states of the Cauchy problem for the exponential reaction-diffusion equation. In this paper, the author has discussed about the study of the radial steady states of the equation and the many intersections distinguishing in four different particular cases. Guo and Wei (2007) evaluate the Cauchy problem for a reaction-diffusion equation with a singular nonlinearity and Mainge (2008) has studied a fully discretization for computing the positive and blowing-up (possibly) solution of the Cauchy problem. Roux and Roux (2008) compute the solution of the Cauchy problem by a numerical method when the initial condition is a nonnegative function with compact support. Here, the problem is split into two parts, (i) a hyperbolic term solved by using the Hopf and Lax formula, (ii) a parabolic term solved by a backward linearized Euler method in time and a finite element method in space. Recently, Bai et al. (2011) studies the Cauchy problem for the fast diffusion equation with the presence of localized reaction.

Bataineh et al. (2008) have solved the one dimensional Cauchy reaction diffusion equations by using HAM for both characteristic and non-characteristic cases. But to the best of authors' knowledge the one-dimensional, time dependent Cauchy reaction diffusion equation (Lesnic, 2007; Dehghan and Shakeri, 2008)

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t)u(x,t), \quad (x,t) \in \Omega \subset \Re^2 \tag{1}$$

where $u(x,t)$ is the concentration, $p(x,t)$ is the reaction parameter and $D > 0$ is the diffusion coefficient, subject to the initial condition and boundary conditions

$$u(x,0) = g(x), \quad x \in \Re \tag{2}$$

$$u(0,t) = f_0(t), \quad and \quad \frac{\partial u(0,t)}{\partial t} = f_1(t), \quad t \in \Re \tag{3}$$

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has not been solved for general values of $p(x,t)$. The problem given by equations (1) and (2), is called the characteristic Cauchy problem in the domain $\Omega = R\times R^{+}$ and the problem given by equations (1) and (3), is called the non-characteristic Cauchy problem in the domain $\Omega = R^{+}\times R$.

Approximate analytical solutions of the reaction-diffusion problems were calculated by Lesnic (2007) using Adomian decomposition method (ADM), Dehghan and Shakeri (2008), Khan et al. (2010) through variational iteration method (VIM). Yildirim (2009), Momani and Yilddrim (2010), Yildirim and Sezer (2010) have used the homotopy perturbation method (HPM) to find the solution of the Cauchy reaction-diffusion equation, fractional convection-diffusion equation and space-time fractional reaction-diffusion equations, respectively. Wang and He (2008) used the VIM for a nonlinear reaction-diffusion process. Khan et al. (2012) applied the homotopy analysis method (HAM) to reaction-diffusion equation of Fisher type. Recently, Das et al. (2011) have discussed about the study of reaction parameter ($p(x,t)$) in four different cases.

In recent years, HAM has been successfully employed to solve many types of linear and nonlinear problems in science and engineering (Liao, 1992; 2003; 2004; 2007; Hashim et al., 2009). The motivation of this paper is to extend the application of the new reliable algorithm of homotopy analysis method proposed by Odibat et al. (2010) and several authors (Zhang et al., 2011) apply in different mathematical/ physical/ biological problems to solve nonlinear differential equations. HAM contains a certain auxiliary parameter $\hbar$, which provides us with a simple way to adjust and control the convergence region and rate of convergence of the series solution. Moreover, by means of the so-called $\hbar$-curve, it is easy to find the valid regions of h to gain a convergent series solution.

2 The reliable algorithm

The homotopy analysis method which provides an analytical approximation solution is applied to various nonlinear problems. In this section, we have discussed a reliable approach of the homotopy analysis method. This new modification can be implemented for integer/fractional order nonlinear differential equations (Odibat et al., 2010; Zhang et al., 2011). To illustrate the basic idea of the new algorithm, we consider the following general nonlinear differential equation

$$D_t u(t) = N[u(t)] + g(t), \quad t > 0 \tag{4}$$

where $N[\bullet]$ is a nonlinear operator, $g(t)$ is a known analytic function. In view of the homotopy technique, we construct a homotopy $\varphi(t;q) : \Omega \times [0,1] \to \Re$, which satisfies

$$(1-q)\mathtt{L}[\varphi(t;q) - \varphi_0(t)] = q\hbar H(t)[D_t\varphi(t;q) - N[\varphi(t;q)] - g(t)], \tag{5}$$

where $q \in [0,1]$ is the embedding parameter, $\hbar \neq 0$ is a non zero auxiliary parameter, $H(t) \neq 0$ is an auxiliary function, $\varphi_0(t)$ is an initial guess of $u(t)$ and L is an auxiliary linear operator that may be defined as $L = \frac{d}{dt}$ When $q = 0$, equation (5) becomes,

$$\mathtt{L}[\varphi(t;0) - \varphi_0(t)] = 0, \tag{6}$$

It is obvious that when $q = 1$, equation (5) becomes the original nonlinear equation (4). Thus as q varies from 0 to 1, the solution $u(t;q)$ varies from the initial guess $\varphi_0(t)$ to the solution $\varphi(t;1)$. The basic assumption of the new approach is that the solution of equation (5) can be expressed as a power series in q,

$$\varphi(t) = \varphi_0(t) + q\varphi_1(t) + q^2\varphi_2(t) + \ldots\ldots, \tag{7}$$

Substituting the series (7) into the homotopy (5) and then equating the coefficients of the like powers of q, we obtain the high-order deformation equations,

$$\begin{aligned}
\mathtt{L}[\varphi_1(t)] &= \hbar H(t)[D_t\varphi_0(t) - N_0[\varphi_0(t)] - g(t)],\\
\mathtt{L}[\varphi_2(t)] &= \mathtt{L}[\varphi_1(t)] + \hbar H(t)[D_t\varphi_1(t) - N_1[\varphi_0(t), \varphi_1(t)]],\\
\mathtt{L}[\varphi_3(t)] &= \mathtt{L}[\varphi_2(t)] + \hbar H(t)[D_t\varphi_2(t) - N_2[\varphi_0(t), \varphi_1(t), \varphi_2(t)]],\\
\mathtt{L}[\varphi_4(t)] &= \mathtt{L}[\varphi_3(t)] + \hbar H(t)[D_t\varphi_3(t) - N_3[\varphi_0(t), \varphi_1(t), \varphi_2(t), \varphi_3(t)]],\\
\vdots \quad & \quad \vdots \quad \vdots \quad \vdots \quad \vdots
\end{aligned} \tag{8}$$

where

$$N[\varphi_0 + q\varphi_1 + q^2\varphi_2 + \ldots\ldots] = N_0[\varphi_0] + qN_1[\varphi_0, \varphi_1] + q^2N_2[\varphi_0, \varphi_1, \varphi_2] + \ldots\ldots,$$

The approximate solution of equation (4), therefore, can be readily obtained,

$$u = lim_{q\to 1}\varphi(t) = \varphi_0 + \varphi_1 + \varphi_2 + \ldots\ldots \tag{9}$$

The accomplishment of the technique is based on the proper selection of the initial approximation φ_0. The main advantage of the new modification, as we will see in the next section, is that it's applicable to a wide class of nonlinear differential equations and the set of base functions will be easily constructed.

3 Solution of the problem

Consider the Cauchy reaction-diffusion equation

$$\frac{\partial u(x,t)}{\partial t} = D\frac{\partial^2 u(x,t)}{\partial x^2} + p(x,t)u(x,t), \quad (x,t) \in \Omega \subset \Re^2 \tag{10}$$

subject to the initial condition

$$u(x,0) = g(x), \tag{11}$$

In view of the algorithm presented in the previous section, if we select the auxiliary linear operator as $L = \frac{d}{dt}$, we can construct the homotopy,

$$(1-q)\mathtt{L}[\varphi(x,t;q) - u_0(x,t)] = q\hbar H(t)[D_t\varphi(x,t;q) - DD_{xx}\varphi(x,t;q) - p(x,t)\varphi(x,t;q)]. \tag{12}$$

According to HAM, we have the initial guess

$$\varphi_0(x,t) = g(x). \tag{13}$$

Taking $H(t) = 1$ and substituting (7) with the initial guess (13) into the homotopy equation (12), then equating the terms with identical powers of q, we obtain the following set of linear differential equation:

$$\begin{array}{rcrcl}
q^1 & : & D_t\varphi_1(x,t) & = & \hbar[D_t\varphi_0(x,t) - DD_{xx}\varphi_0(x,t) - p(x,t)\varphi_0(x,t)] \\
q^2 & : & D_t\varphi_2(x,t) & = & D_t\varphi_1(x,t) + \hbar[D_t\varphi_1(x,t) - DD_{xx}\varphi_1(x,t) - p(x,t)\varphi_1(x,t)] \\
q^3 & : & D_t\varphi_3(x,t) & = & D_t\varphi_2(x,t) + \hbar[D_t\varphi_2(x,t) - DD_{xx}\varphi_2(x,t) - p(x,t)\varphi_2(x,t)] \\
 & & \vdots & & \vdots \qquad \vdots \qquad \vdots \\
q^n & : & D_t\varphi_n(x,t) & = & D_t\varphi_{n-1}(x,t) + \hbar[D_t\varphi_{n-1}(x,t) - DD_{xx}\varphi_{n-1}(x,t) - p(x,t)\varphi_{n-1}(x,t)]
\end{array}$$

The approximate solution of equation (10), therefore, can be easily obtained,

$$u = lim_{q\to 1}\varphi(t) = \varphi_0 + \varphi_1 + \varphi_2 + \ldots\ldots \tag{14}$$

The convergence of the technique is based on the proper selection of the initial approximation $\varphi_0(x,t)$.

4 Test Examples

In this section, we check the consistency and reliability of the HAM for different general values of $p(x,t)$. Five examples have been selected so that their analytical solutions exist.

Example 1 *Case* $p(x,t) = k$ *and* $D = 1$

The equation (1) reduces in the form of the Kolmogorov-Petrovsky-Piskunov (KPP) equation:

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + ku(x,t), \quad (x,t) \in \Omega \tag{15}$$

subject to the initial condition

$$u(x,0) = x + e^{-x}, \quad x \in \Re. \tag{16}$$

The exact solution of the problem is $u(x,t) = e^{-x} + xe^{-t}$ (when $k = -1$).

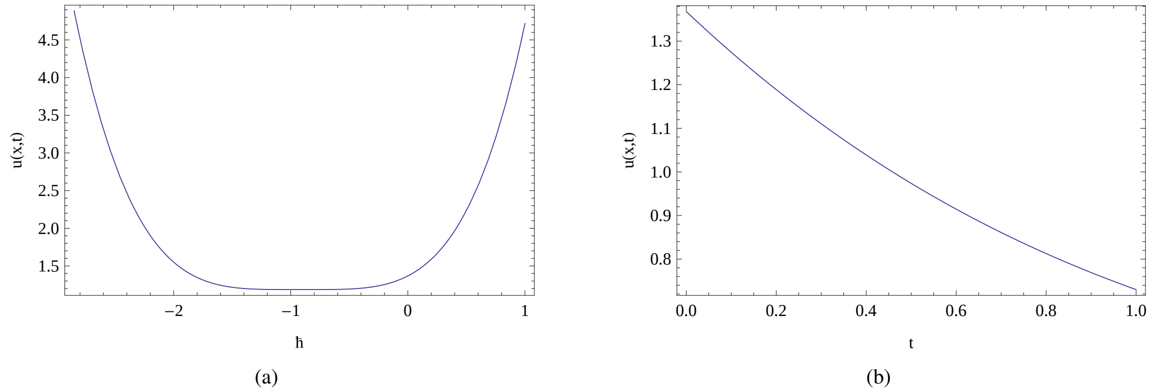


Figure 1: **(a)**Plot of $u(x,t)$ vs. $\hbar$ at $x = 1$ and $t = 0.2$ **(b)** Plot of $u(x,t)$ vs. t at $x = 1$ and $\hbar = -1$ for $k = -1$ and $g(x) = x + e^{-x}$

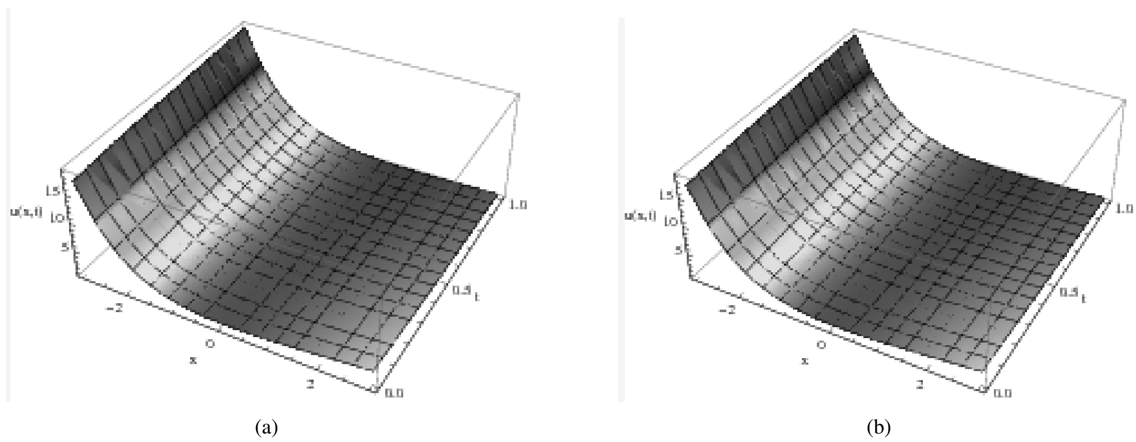


Figure 2: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $g(x) = x + e^{-x}$ **(a)** $\hbar = -0.6$ **(b)** $\hbar = -1$

According to the homotopy analysis procedures, we get the solutions

$$u_0(x,t) = x + e^{-x}, \tag{17}$$

$$u_1(x,t) = -\hbar[e^{-x} + k(x + e^{-x})]t, \tag{18}$$

$$u_2(x,t) = e^{-x}[-\hbar(\hbar+1)(1 + k + kxe^{x})t + \hbar^2(1 + 2k + k^2(1 + xe^{x}))\frac{t^2}{2!}], \tag{19}$$

$$u_3(x,t) = e^{-x}[-\hbar(\hbar+1)^2(1 + k + kxe^{x})t + \hbar^2(\hbar+1)(1 + 2k + k^2(1 + xe^{x}))t^2 -\hbar^3(1 + 3k + 3k^2 + k^3(1 + xe^{x}))\frac{t^3}{3!}], \tag{20}$$

and so on.

The approximate solution of equation (15), therefore, can be easily obtained,

$$u(x,t) = lim_{q\to 1} u_n(x,t). \tag{21}$$

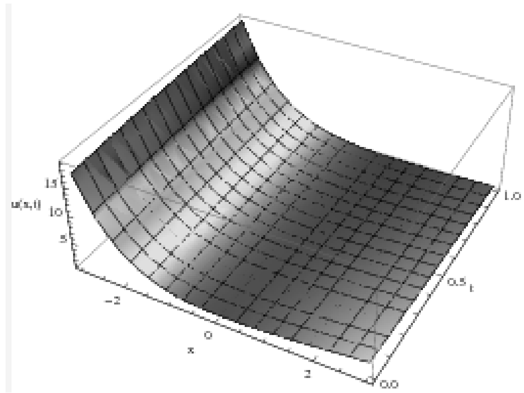


Figure 3: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $u(x,t)$ is exact solution at $k=1$

Example 2 *Case* $p(x,t) = a_1x^2 + b_1x + c_1$ *and* $D = 1$

The equation (1) reduces in the form of

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \left(a_1x^2 + b_1x + c_1\right)u(x,t), \quad (x,t) \in \Omega \subset \Re^2 \tag{22}$$

subject to the initial condition

$$u(x,0) = e^{x^2}, \quad x \in \Re. \tag{23}$$

The exact solution of the problem (when $a_1 = -4, b_1 = 0, c_1 = -1$*) is*

$$u(x,t) = e^{x^2+t} \tag{24}$$

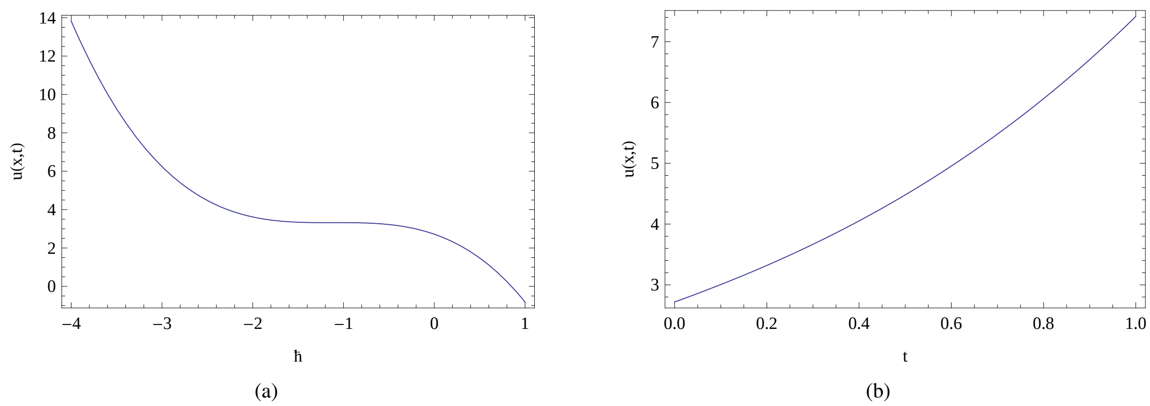
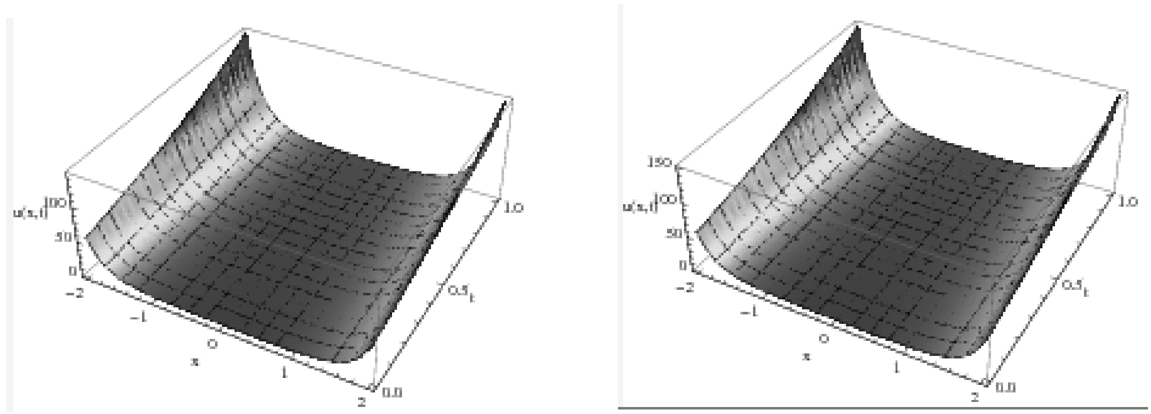


Figure 4: **(a)**Plot of $u(x,t)$ vs. $\hbar$ at $x=1$ and $t=0.2$ **(b)** Plot of $u(x,t)$ vs. t at $x=1$ and $\hbar=-1$ for $a_1=-4$, $b_1=0$, $c_1=-1$ and $g(x)=e^{x^2}$



(a) (b)

Figure 5: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $g(x) = e^{x^2}$ **(a)** $\hbar = -1$ **(b)** $\hbar = -1.2$ at $a_1 = -4$, $b_1 = 0$, $c_1 = -1$

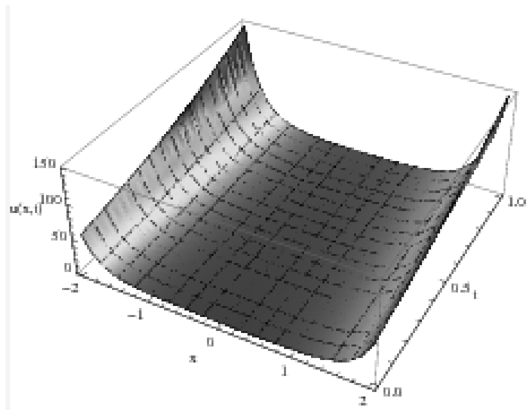


Figure 6: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $u(x,t)$ is exact solution at $a_1 = -4$, $b_1 = 0$, $c_1 = -1$

According to the homotopy analysis procedures, we get the solutions

$$u_0(x,t) = e^{x^2}, \tag{25}$$

$$u_1(x,t) = -e^{x^2}\hbar[2 + 4x^2 + a_1x^2 + b_1x + c_1]t, \tag{26}$$

$$\begin{aligned} u_2(x,t) = \; & e^{x^2}[-\hbar(\hbar+1)(2 + 4x^2 + a_1x^2 + b_1x + c_1)t + \hbar^2(12 + 48x^2 + 16x^4 + a_1^2x^4 + b_1^2x^2 + 4c_1 \\ & +8c_1x^2 + c_1^2 + 2b_1x(4x^2 + c_1 + 4) + 2a_1(4x^4 + b_1x^3 + c_1x^2 + 6x^2 + 1))\frac{t^2}{2!}], \end{aligned} \tag{27}$$

$$\begin{aligned} u_3(x,t) = \; & e^{x^2}[-\hbar(\hbar+1)^2(2 + 4x^2 + a_1x^2 + b_1x + c_1)t + 2\hbar^2(12 + 12\hbar + 48x^2 + 16x^4 + 64\hbar x^4 \\ & +(\hbar+1)a_1^2x^4 + (\hbar+1)b_1^2x^2 + 4(\hbar+1)c_1 + c_1^2(\hbar+1) + 2(\hbar+1)b_1x(4x^2 + c_1 + 4) \\ & +8(\hbar+1)c_1x^2 + 2(\hbar+1)a_1(4x^4 + b_1x^3 + c_1x^2 + 6x^2 + 1))t^2 - \hbar^3(120 + 720x^2 + 480x^4 \\ & +64x^6 + a_1^3x^6 + b_1^3x^3 + 36c_1 + 144c_1x^2 + 48c_1x^4 + 6c_1^2 + 12c_1^2x^2 + c_1^3 + b_1^2(2 + 18x^2 \\ & +12x^4 + 3c_1x^2) + a_1^2x^2(14 + 30x^2 + 12x^4 + 3b_1x^3 + 3c_1x^2) + 3b_1x(4(4x^4 + 16x^2 + 9) \\ & +8c_1(x^2 + 1) + c_1^2) + a_1(4(12x^6 + 60x^4 + 59x^2 + 7) + 3b_1^2x^4 + 6c_1(4x^4 + 6x^2 + 1) \\ & +3c_1^2x^2 + 2b_1x(12x^4 + 24x^2 + 3c_1x^2 + 7))))\frac{t^3}{3!}], \end{aligned} \tag{28}$$

and so on.

The approximate solution of equation (22), therefore, can be easily obtained,

$$u(x,t) = lim_{q\to 1}u_n(x,t). \tag{29}$$

Example 3 *Case* $p(x,t) = a_2t^2 + b_2t + c_2$ *and* $D = 1$

The equation (1) reduces in the form of

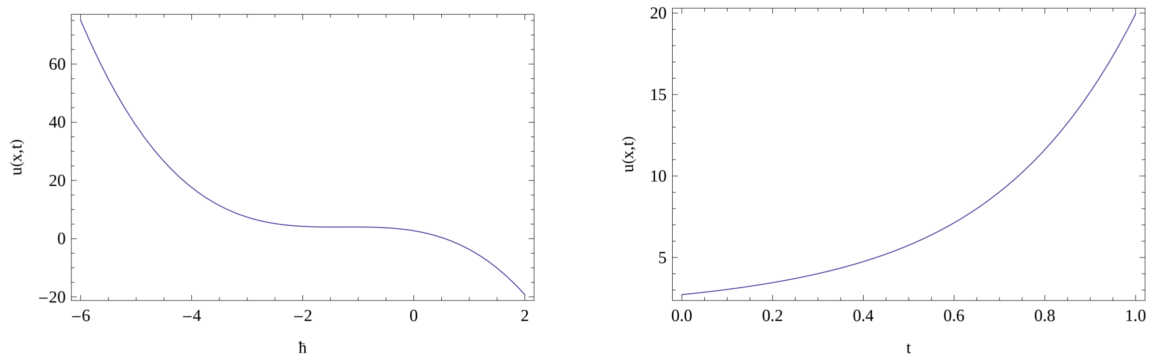
$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \left(a_2t^2 + b_2t + c_2\right)u(x,t), \quad (x,t) \in \Omega \subset \Re^2 \tag{30}$$

subject to the initial condition

$$u(x,0) = e^x, \quad x \in \Re. \tag{31}$$

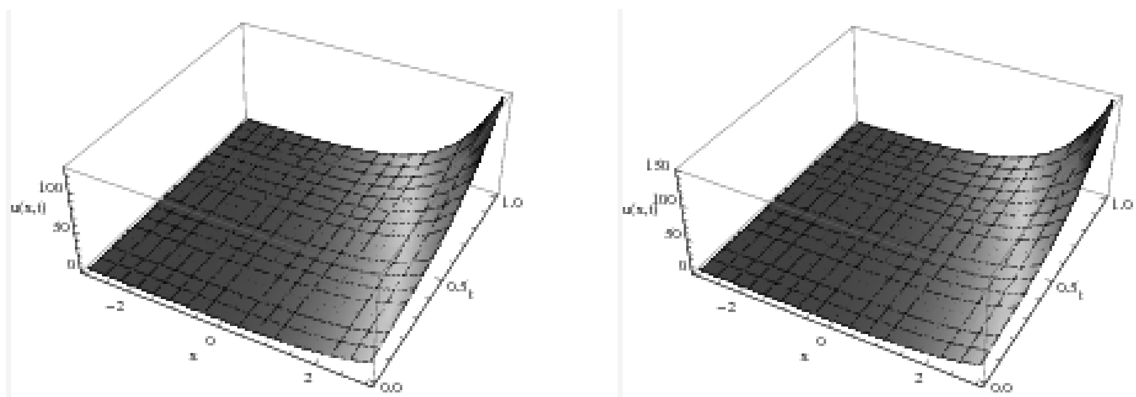
The exact solution of the problem (when $a_2 = 0, b_2 = 2, c_2 = 0$*) is*

$$u(x,t) = e^{x+t+t^2} \tag{32}$$



(a) (b)

Figure 7: **(a)**Plot of $u(x,t)$ vs. $\hbar$ at $x = 1$ and $t = 0.2$ **(b)** Plot of $u(x,t)$ vs. t at $x = 1$ and $\hbar = -1.2$ for $a_2 = 0$, $b_2 = 2$, $c_2 = 0$ and $g(x) = e^x$



(a) (b)

Figure 8: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $g(x) = e^x$ **(a)** $\hbar = -1$ **(b)** $\hbar = -1.3$ at $a_2 = 0$, $b_2 = 2$, $c_2 = 0$

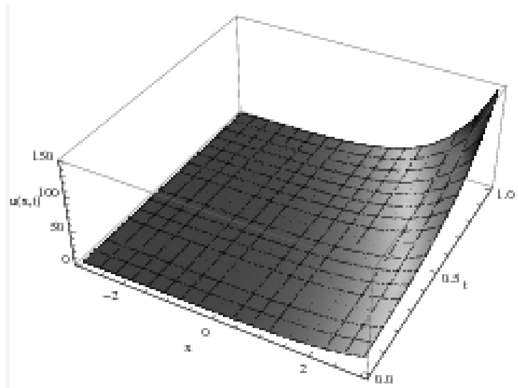


Figure 9: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $u(x,t)$ is exact solution at $a_2 = 0, b_2 = 2, c_2 = 0$

According to the homotopy analysis procedures, we get the solutions

$$u_0(x,t) = e^x, \quad (33)$$

$$u_1(x,t) = -e^x\hbar[(c_2+1)t + b_2\frac{t^2}{2!} + 3a_2\frac{t^3}{3!}], \quad (34)$$

$$u_2(x,t) = e^x\hbar[-(\hbar+1)(c_2+1)t - (b_2(\hbar+1) - (c_2+1)^2)\frac{t^2}{2!} - (2a_2(\hbar+1) - 3b_2\hbar(c_2+1)) \frac{t^3}{3!} + \hbar(3b_2^2 + 8a_2(c_2+1))\frac{t^4}{4!} + \hbar a_2 b_2\frac{t^5}{6} + \hbar a_2^2\frac{t^6}{18}], \quad (35)$$

$$u_3(x,t) = e^x[-\hbar(\hbar+1)^2(c_2+1)t - \hbar(\hbar+1)(b_2(\hbar+1) - 2\hbar(c_2+1)^2)\frac{t^2}{2!} - \hbar(2a_2(\hbar+1)^2 +\hbar(c_2+1)(-6b_2(\hbar+1) + \hbar(c_2+1)^2))\frac{t^3}{3!} + 2\hbar^2(8a_2(c_2+1)(\hbar+1) + 3b_2(b_2(\hbar+1) -\hbar(c_2+1)^2))\frac{t^4}{4!} + 5\hbar^2(-3b_2^2(c_2+1)\hbar + a_2(8b_2(\hbar+1) - 4(c_2+1)^2\hbar))\frac{t^5}{5!} +5\hbar^2(16a_2^2(\hbar+1) - 3b_2^3\hbar - 24a_2b_2\hbar(c_2+1))\frac{t^6}{6!} - 70a_2\hbar^3(3b_2^2 + 4a_2(c_2+1))\frac{t^7}{7!} -1120a_2^2b_2\hbar^3\frac{t^8}{8!} - 2240a_2^3\hbar^3\frac{t^9}{9!}], \quad (36)$$

and so on.
The approximate solution of equation (30), therefore, can be easily obtained,

$$u(x,t) = lim_{q\to 1}u_n(x,t). \quad (37)$$

Example 4 *Case* $p(x,t) = a_3x^2 + a_4t^2 + b_3x + b_4t + c_3$ *and* $D = 1$

The equation (1) reduces in the form of

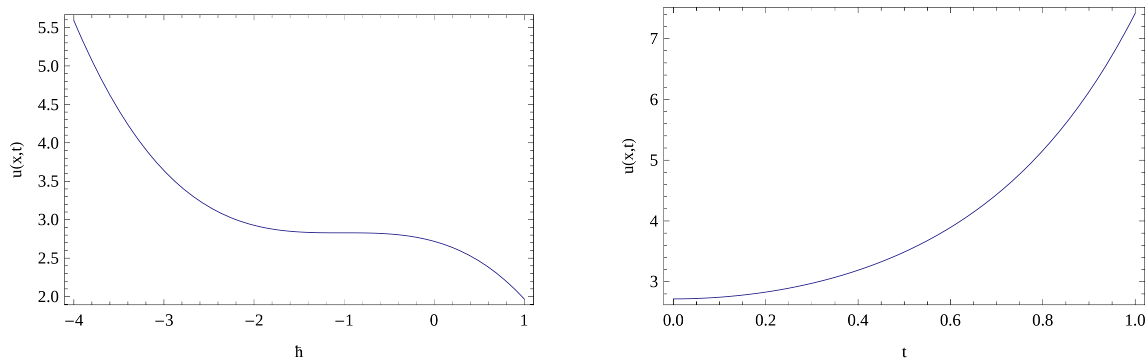
$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \left(a_3x^2 + a_4t^2 + b_3x + b_4t + c_3\right)u(x,t), \quad (x,t)\in\Omega\subset\Re^2 \quad (38)$$

subject to the initial condition

$$u(x,0) = e^{x^2}, \quad x\in\Re. \quad (39)$$

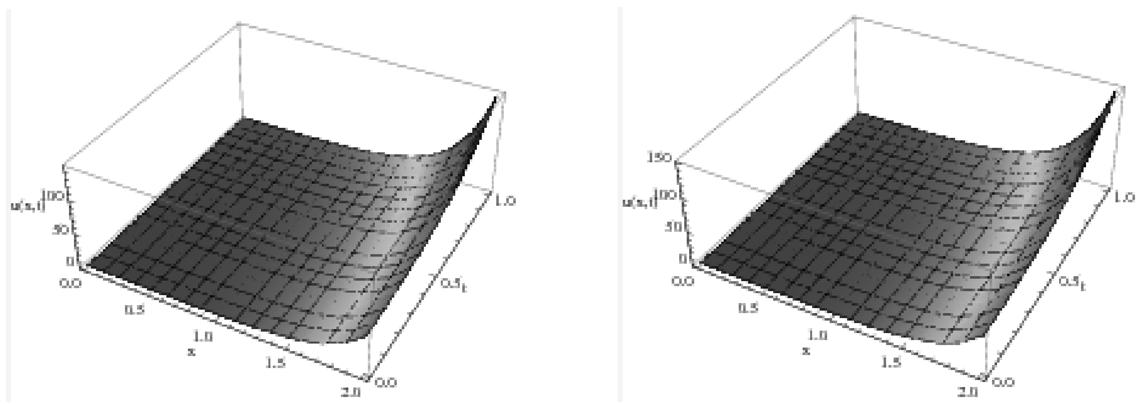
The exact solution of the problem (when $a_3 = -4, a_4 = 0, b_3 = 0, b_4 = 2, c_3 = -2$*) is*

$$u(x,t) = e^{x^2+t^2} \quad (40)$$



(a) (b)

Figure 10: **(a)**Plot of $u(x,t)$ vs. $\hbar$ at $x=1$ and $t=0.2$ **(b)** Plot of $u(x,t)$ vs. t at $x=1$ and $\hbar=-1.2$ for $a_3=-4$, $b_3=0$, $c_3=-2$, $a_4=0$, $b_4=2$ and $g(x)=e^{x^2}$



(a) (b)

Figure 11: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $g(x)=e^{x^2}$ **(a)** $\hbar=-1$ **(b)** $\hbar=-1.3$ at $a_3=-4$, $b_3=0$, $c_3=-2$, $a_4=0$, $b_4=2$

According to the homotopy analysis procedures, we get the solutions

$$u_0(x,t) = e^{x^2}, \tag{41}$$

$$u_1(x,t) = -e^{x^2}\hbar[(a_3x^2+4x^2+b_3x+c_3+2)t+b_4\frac{t^2}{2!}+2a_4\frac{t^3}{3!}], \tag{42}$$

$$\begin{aligned} u_2(x,t) = {} & e^{x^2}[-\hbar(\hbar+1)(a_3x^2+4x^2+b_3x+c_3+2)t+\hbar(a_3^2\hbar x^4+16\hbar x^4+b_3^2\hbar x^2+48\hbar x^2 \\ & +8c_3\hbar x^2+c_3^2\hbar+4c_3\hbar-b_4\hbar-b_4+12\hbar+2b_3\hbar x(4x^2+c_3+4)+2a_3\hbar(4x^4+b_3x^3 \\ & +c_3x^2+6x^2+1))\frac{t^2}{2!}-\hbar(2a_4(\hbar+1)-3b_4\hbar(a_3x^2+4x^2+b_3x+c_3+2))\frac{t^3}{3!} \\ & +\hbar^2(3b_4^2+8a_4(a_3x^2+4x^2+b_3x+c_3+2))\frac{t^4}{4!}+a_4b_4\hbar^2\frac{t^5}{6}+a_4^2\hbar^2\frac{t^6}{18}] \end{aligned} \tag{43}$$

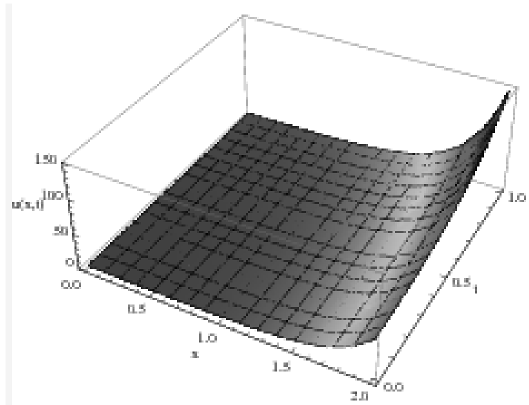


Figure 12: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $u(x,t)$ is exact solution at $a_3 = -4$, $b_3 = 0$, $c_3 = -2$, $a_4 = 0$, $b_4 = 2$

$$\begin{aligned}
u_3(x,t) \;=\; & e^{x^2}[-\hbar(\hbar+1)^2(a_3x^2+4x^2+b_3x+c_3+2)t+\hbar(\hbar+1)(2a_3^2\hbar x^4+32\hbar x^4+2b_3^2\hbar x^2 \\
&+16c_3\hbar x^2+96\hbar x^2-b_4\hbar+2c_3^2\hbar+8c_3\hbar-b_4+24\hbar+4b_3\hbar x(4x^2+c_3+4) \\
&+4a_3\hbar(4x^4+b_3x^3+c_3x^2+6x^2+1))\frac{t^2}{2!}-\hbar(a_3^3\hbar^2x^6+2a_4(\hbar+1)^2+a_3^2\hbar^2x(12x^4 \\
&+3b_3x^3+3c_3x^2+30x^2+14)+a_3\hbar(3b_3^2\hbar x^4-6b_4(\hbar+1)x^2+2b_3\hbar x(12x^4+3c_3x^2 \\
&+24x^2+7)+\hbar(4(12x^6+60x^4+59x^2+7)+6c_3(4x^4+6x^2+1)+3c_3^2x^2)) \\
&+\hbar(b_3^3\hbar x^3-6b_4(\hbar+1)(c_3+4x^2+1)+b_3^2\hbar(12x^4+3c_3x^2+18x^2+2) \\
&+\hbar(8(8x^6+60x^4+90x^2+15)+12c_3(4x^4+12x^2+3)+6c_3^2(2x^2+1)+c_3^3) \\
&+3b_3x(-2b_4(\hbar+1)+\hbar(4(4x^4+16x^2+9)+8c_3(x^2+1)+c_3^2))))\frac{t^3}{3!} \\
&+2\hbar^2(8a_4(\hbar+1)(a_3x^2+4x^2+b_3x+c_3+2)-3b_4(a_3^2\hbar x^4+16\hbar x^4+b_3^2\hbar x^2+8c_3\hbar x^2 \\
&+48\hbar x^2-b_4\hbar+c_3^2\hbar+4c_3\hbar-b_4+12\hbar+2b_3\hbar x(4x^2+c_3+4)+2a_3\hbar(4x^4+b_3x^3 \\
&+c_3x^2+6x^2+1)))\frac{t^4}{4!}-5\hbar^2(3b_4^2\hbar(a_3x^2+4x^2+b_3x+c_3+2)+4a_4(a_3^2\hbar x^4+16\hbar x^4 \\
&+b_3^2\hbar x^2+8c_3\hbar x^2+48\hbar x^2-2b_4\hbar+4c_3\hbar+c_3^2\hbar-2b_4+12\hbar+2b_3\hbar x(4x^2+c_3+4) \\
&+2a_3\hbar(4x^4+b_3x^3+c_3x^2+6x^2+1)))\frac{t^5}{5!}-a_4b_3b_4\hbar^3x\frac{t^6}{6}-a_4b_4^2\hbar^3\frac{t^7}{24}]
\end{aligned} \tag{44}$$

and so on.
The approximate solution of equation (38), therefore, can be easily obtained,

$$u(x,t) = lim_{q\to 1}u_n(x,t). \tag{45}$$

Example 5 *Case* $p(x,t) = -1 + cosx - sin^2x$ *and* $D = 1$ *[Dehghan and Shakeri, 2008]*

The equation (1) reduces in the form of

$$\frac{\partial u(x,t)}{\partial t} = \frac{\partial^2 u(x,t)}{\partial x^2} + \left(-1+cosx-sin^2x\right)u(x,t), \quad (x,t)\in\Omega\subset\Re^2 \tag{46}$$

subject to the initial condition

$$u(x,0) = \frac{1}{10}e^{cosx-11}, \quad x\in\Re. \tag{47}$$

The exact solution of the problem is

$$u(x,t) = \frac{1}{10}e^{cosx-t-11} \tag{48}$$

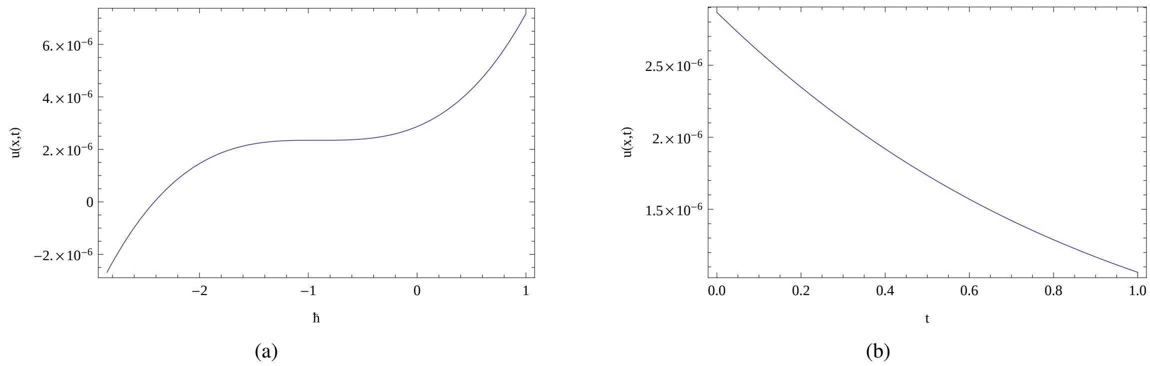


Figure 13: **(a)**Plot of $u(x,t)$ vs. $\hbar$ at $x = 1$ and $t = 0.2$ **(b)** Plot of $u(x,t)$ vs. t at $x = 1$ and $\hbar = -0.8$ for $g(x) = \frac{1}{10}e^{cosx-11}$

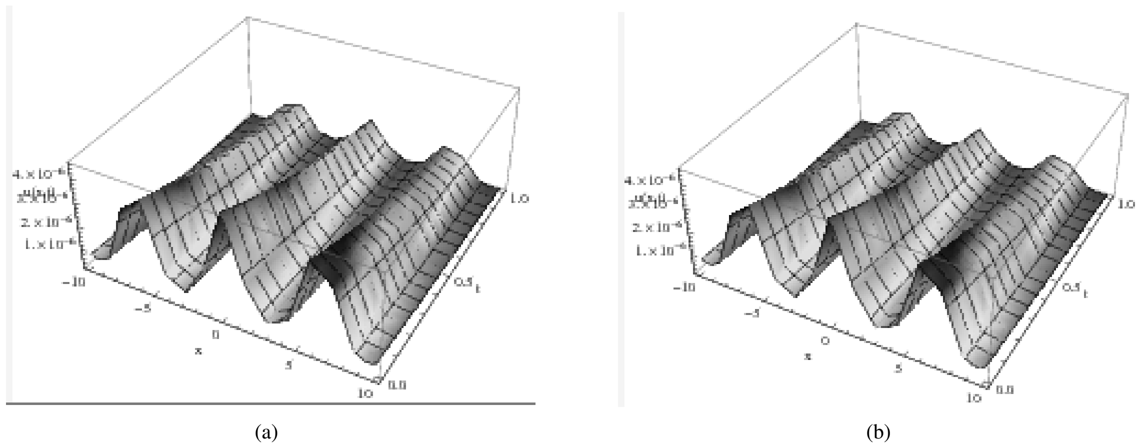


Figure 14: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $g(x) = \frac{1}{10}e^{cosx-11}$ **(a)** $\hbar = -0.8$ **(b)** $\hbar = -1$

According to the homotopy analysis procedures, we get the solutions

$$u_0(x,t) = \frac{1}{10}e^{cosx-11}, \quad (49)$$

$$u_1(x,t) = \hbar t\frac{1}{10}e^{cosx-11}, \quad (50)$$

$$u_2(x,t) = \frac{1}{10}e^{cosx-11}[\hbar(\hbar+1)t + \hbar^2\frac{t^2}{2!}], \quad (51)$$

$$u_3(x,t) = \frac{1}{10}e^{cosx-11}[\hbar(\hbar+1)^2t + \hbar^2(\hbar+1)t^2 + \hbar^3\frac{t^3}{3!}], \quad (52)$$

and so on.
The approximate solution of equation (46), therefore, can be easily obtained,

$$u(x,t) = lim_{q\to1}u_n(x,t). \quad (53)$$

5 Numerical results and discussion

In this section, the numerical results of the displacement $u(x,t)$ for the initial conditions $g(x) = x + e^{-x}, e^{x^2}, e^x, e^{x^2}$, $\frac{1}{10}e^{cosx-11}$ for various values of t and x are obtained with the proper choice of $\hbar$ and the results are depicted through Figs. 1-15. Here, the authors considered the fourth order term approximation of the series solution during the numerical computation. From $\hbar$-curve, it is seen that when $\hbar = -1$ the result vibrate (Liao, 2004) with the result obtained by another

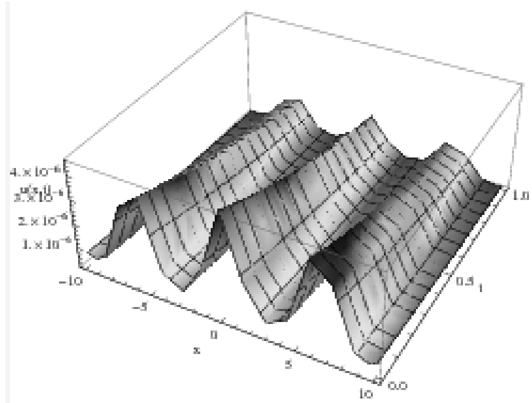


Figure 15: The behaviour of the $u(x,t)$ w.r.to x and t are obtained when $u(x,t)$ is exact solution

mathematical tool homotopy perturbation method. It is observed in all text examples that the result is quite similar to exact solution for a particular choice of $\hbar$, and it is also shown that the HPM solution ($\hbar = -1$) is slow convergence with respect to HAM.

Figs. 1(a), 4(a), 7(a), 10(a) and 13(a) represents the plots of the displacement $u(x,t)$ (which is a function of auxiliary parameter $\hbar$) against $\hbar$ at $x = 1$ and $t = 1$. Since converges to the exact values for different values of $\hbar$, there exists horizontal line segments shown in the figures, which are usually called $\hbar$-curves (Liao, 1992) and this shows the validity of the region of convergence of the series solutions of equations. The results justify the statement of Liao (2004) that by means of HAM, the convergence region and the rate of series solution can be adjusted and controlled by plotting $\hbar$-curves.

From the Figs. 1(b), 4(b), 7(b), 10(b) and 13(b) show that the displacement $u(x,t)$ has changed with the increase of t at $x = 1$ and various converging value of $\hbar$. The Fig. 2 shows that $u(x,t)$ decreases with the increase in x and t. It is seen from the Figs. 5, 8 and 11 that $u(x,t)$ increases with the increase in both x and t. While, the Fig. 14 shows that the displacement $u(x,t)$ vibrate with the increase in x, since the initial approximation is cosine function.

So, the text examples in the paper have proved that if specific values are assigned to the auxiliary parameters in the homotopy analysis method, then the approximate homotopy results successfully converge to the exact solution (see Figs. 3, 6, 9, 12 and 15). The obtained results have a good agreement with those obtained using an Adomian decomposition method (Lesnic, 2007) and Variational iteration method (Dehghan and Shakeri, 2008) for particular constant values.

6 Conclusion

In this paper, the new algorithm of the homotopy analysis method proposed by Odibat et al. (2010), has been analyzed with an aim to investigate the conditions which result in the convergence of the generated homotopy solutions of the nonlinear partial differential equations. In order to verify numerically whether the proposed methodology lead to higher accuracy, we can evaluate the numerical solutions using nth approximations show the high degree of accuracy and in most cases $u_n(x,t)$, the n^{th} approximation is accurate for quite low of $n(n = 3)$. The obtained numerical results are shown in Figs. 1-15.

It has been well demonstrated that in applying the powerful mathematical tool like HAM, the differential equation for characteristic Cauchy reaction diffusion equation. The present study shows that the method HAM gives quantitatively reliable results with less computational work and seems to be easily extended to solve more complicated model equations having more mechanical, physical and biophysical effects like convection, dispersion, diffusion for both linear and nonlinear cases.

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