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Singular Perturbation Problems in Nonlinear Elliptic Partial Differential Equations: A Survey

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Abstract: The present paper deals with a survey of numerical techniques for solving nonlinear elliptic singular perturbation problems, developed by numerous researchers in a chronological order as the field developed year after year from 1985. A large amount of research material related to nonlinear elliptic singularly perturbed problems has been collected with the aim of introduction to some of the ideas and methods of solving these problems, which may be useful for many present researchers from the point of view of their numerical and computer realizations. Some nonlinear elliptic singular perturbation models arising in different branches of science and engineering are also given at the end of this review.

Keywords: nonlinear elliptic partial differential equations; singularly perturbed problems; elliptic reaction-diffusion equation; Navier-Stokes systems; semiconductor modelling; nerve impulse system

A.M.S. Subject Classification: 65N20

1 Introduction

An impressive number of results in nonlinear analysis are a consequence of various problems raised by mathematical physics, optimization, and economy. In the modelling of natural phenomena, a crucial role is played by the study of nonlinear partial differential equations of elliptic type, which arise almost in every field of science. Consequently, the desire to understand the solutions of these equations has always a prominent place in the efforts of mathematicians. It has inspired such diverse fields as functional analysis, variational calculus, potential theory, algebraic topology, differential geometry, and so forth. Singular perturbation is a maturing mathematical subject with a fairly long history and a strong promise for continued important applications throughout science and engineering. The singular perturbation problems has found place in many areas of engineering and applied mathematics, e.g. fluid mechanics, fluid dynamics, quantum mechanics, elasticity, chemical reactor theory, magneto hydrodynamics and reaction diffusion process, etc. With the development of science and technology, many critical problems, such as the mathematical boundary-layer theory or approximation of solutions of various problems described by differential equations involving large or small parameters, became more complex, and hence it is very natural to ask about asymptotic methods in their analysis. However, the asymptotic analysis for differential operators has a developed theory in case of regular perturbations, when the perturbations carry a subordinate character with respect to the unperturbed operator. In some problems the perturbations are operative over very narrow regions across which the dependent variable undergoes very rapid changes. These narrow regions frequently adjoin the boundaries of the domain of interest, owing to the fact that the small parameter multiplies the highest derivative. Consequently, they are usually referred to as boundary layers in fluid mechanics, edge layers in solid mechanics, skin layers in electrical applications, shock layers in fluid and solid mechanics, transition points in quantum mechanics and Stokes lines and surfaces in mathematics. These problems are very difficult to solve numerically also. While the singular perturbation problems can be traced back to 1904, when Prandtl [97] introduced the terminology boundary layer in third International Congress of Mathematicians in Heidelberg, it still constitutes a very active research area, as evidence by recent books

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such as [3, 42, 72, 101, 104, 111]. Even a large amount of work has been carried out and number of unsolved problems still remain to be solved, especially for two dimensional nonlinear problems [101]. During the past decade, many approximate methods for analysis of nonlinear singularly perturbed problem had been developed and refined, including the method of averaging, boundary layer method, methods of matched asymptotic expansion and multiple scales. Mo et al. considered the class of singularly perturbed nonlinear boundary value problems for the ordinary differential equations [56], the reaction diffusion equations [55, 58, 60], the initial boundary value problems of hyperbolic equations [32], the shock layer solution of nonlinear equations for singularly perturbed problems [61], the problems of atmospheric physics [62] and the boundary value problems of elliptic equations [59]. Numerical solution of singularly perturbed elliptic partial differential equations plays a very important role in computational fluid dynamics to simulate flow problems. Traditional finite difference discretization schemes such as second order central difference scheme, first order upwind scheme etc. are used generally, which have the drawbacks of either lack of stability (central difference) or lack of accuracy (upwind). Numerical solutions of nonlinear differential equations based on iterative methods (finite difference) become increasingly difficult, which may converge slowly or even diverge. There is considerably interest in developing improved discretization schemes for the elliptic partial differential equations [24]. Although considerable amount of work has been done in the past for singular perturbation problems (for detail, see Kadalbajoo and Patidar [63, 64], Kumar et al. [68-71] and the references therein) but still there is lack of computational discretization schemes that are computational convenient for all types of linear and non-linear singular elliptic equations.

Singular perturbation problems can be analyzed in two ways: numerically and asymptotically. Numerical analysis provides quantitative information about particular problem whereas asymptotic analysis provides insight into the quantitative behaviour of the family of the problems and gives only semi quantitative information about any particular member of the family. Thus the techniques studied for solving singular perturbation problems by various researchers in the different research papers and books included in this survey that can be classified as numerical analysis and asymptotic analysis. The present survey is mainly focused the research articles published in different journals on nonlinear elliptic singularly perturbed partial differential equations along with some nonlinear elliptic singular perturbation equations arising in different branches of science and engineering; however some nonlinear elliptic partial differential equations without singular perturbations is considered just to provide the completeness to the study of nonlinear elliptic problems. In this survey article, we presented crux of published research articles starting from 1985 and onwards chronologically, according to their appearance in different journals/conference proceedings. The references [1-119] are kept in alphabetical order according to first author's surname.

A brief outline of this survey paper is as follows:

Section 2: we present the summary of various research articles published in different journals for solving nonlinear elliptic singular perturbation problems in chronological order of their publication.

Section 3: describes some nonlinear elliptic equations (including singularly perturbed), which arise in various branches of science and engineering.

Section 4: includes the concluding remarks and further scopes for research in the field of nonlinear elliptic singular perturbation problems.

2 Nonlinear elliptic singular perturbation problems

In this section, we have summarized the research articles published in the field of nonlinear elliptic singular perturbation problems, in a chronological order according to their appearance in different journals/conference proceedings.

DeSANTI [39] considered the following singularly perturbed quadratically nonlinear elliptic Dirichlet problem:

$$\epsilon\Delta u = A(x,u)(\nabla u.\nabla u) + B(x,u)\nabla u + C(x,u), x \in \Omega;\ \ u(x,\epsilon) = f(x)\ \ for\ \ x\ \ on\ \ \Gamma, \tag{1}$$

where Ω is an open and bounded set in Euclidean n-space E^n endowed with the standard inner product, denoted by "." and the boundary Γ of Ω is a smooth $(n-1)$-dimensional manifold. Points in E^n are denoted by x. The scalar valued functions A, C and f, and the vector valued function B are assumed to be smooth. The parameter ϵ is assumed to be small and positive. Under explicit and easily checked conditions, solutions are shown to exist for ϵ sufficiently small and to exhibit specified behaviour as $\epsilon \to 0$. The results are obtained using a method based on the theory of partial differential inequalities.

In [52], Ishii and Yamada are concerned with the singular perturbations of obstacle problems with gradient constraint. For any $\varepsilon > 0$, the following singularly perturbed nonlinear elliptic partial differential equation has been considered:

$$\max\{L_\varepsilon u_\varepsilon - f, |Du_\varepsilon| - g\} = 0\ \ in\ \ \Omega;\ \ u_\varepsilon = 0\ \ on\ \ \partial\Omega, \tag{2}$$

where L_ε is a linear second order elliptic operator defined in a bounded domain $\Omega \subset R^n$; $L_\varepsilon u = -\varepsilon^2 a_{ij} u_{x_i x_j} + \varepsilon b_i u_{x_i} + cu$, Du denotes the gradient of u and f, g are nonnegative functions in Ω. The main purpose is to get an estimate on the rate of convergence of the solution u_ε of (2) to the solution u_0 of the first order partial differential equation:

$$\max\{cu_0 - f, |Du_0| - g\} = 0 \ \ in \ \ \Omega; \ \ u_0 = 0 \ \ on \ \ \partial\Omega. \tag{3}$$

Concerning the existence and uniqueness of the viscosity solutions of (2) and (3) satisfying Dirichlet boundary conditions, it has been shown that for each $\varepsilon > 0$, there exists a unique viscosity solution u_ε of (2) in $W^{1,\infty}(\bar{\Omega})$ satisfying $u_\varepsilon = 0$ on $\partial\Omega$ and also there exists a unique viscosity solution u_0 of (3) in $W^{1,\infty}(\bar{\Omega})$ satisfying $u_0 = 0$ on $\partial\Omega$. Further, the rate of convergence of solutions of the Hamilton-Jacobi-Bellman (HJB) equation with gradient constraint is considered and an elliptic equation with gradient constraint whose principal part is fully nonlinear operator has been treated.

Using the method of sub and super solutions, Kelly and Ko [65] considered the following semilinear elliptic singular perturbation problem:

$$\varepsilon^2 \Delta u + f(x,u) = 0 \ \ in \ \ \Omega; \ \ u = 0 \ \ on \ \ \partial\Omega, \tag{4}$$

where $\Omega \subset R^n$ is a bounded open set with a $C^{2+\alpha}, 0 < \alpha < 1$ boundary and $f \in C^1(\bar{\Omega} \times R)$. The function $f(x,y)$ is supposed to satisfy the following assumptions:

(i) there exist $z_i(x) \in C^2(\bar{\Omega})$ so that $z_1(x) < z_2(x) < z_3(x), 0 < z_2(x), f(x, z_i(x)) = 0, i = 1,2,3, f_u(x, z_i(x)) < 0, (i = 1,3)$ and $z_2(x)$ is the unique zero of $f(x, .)$ between $z_1(x)$ and $z_3(x)$,

(ii) $\int_0^{z_1(x)} f(x,y)\, du > 0$ for $x \in \partial\Omega$ and $0 < \theta < z_1$,

(iii) there is a nonempty open set $W \subset \Omega$ such that $\int_{z_1(x)}^{z_3(x)} f(x,y)\, du > 0$ for $x \in W$.

It has been established that the Dirichlet problem (4) has an intermediate solution $u(x,\varepsilon)$ satisfying $z_1(x) - \gamma < u(x,\varepsilon)$ if $dist(x, \partial) \gg \varepsilon; u(x,\varepsilon) < z_3(x) + \gamma$ for all $x \in \Omega$ and there exists $x \in B$ such that $u(x,\varepsilon) < u_0(x) + \gamma$, where ε and γ are sufficiently small positive constants and $u_0(x)$ is the unique solution of $\int_{z_1(x)}^{u_0(x)} f(x,y)\, du = 0, x \in W$. Moreover, it is also shown that there exists a $\delta > 0$ independent of ε such that $\max\{u(x,\varepsilon), x \in \Omega\} = u(x_0,\varepsilon) > z_2(x_0) + \delta$.

Let us consider a series of research papers authored by Blatov [11-13]. In [11], Blatov considered the nonlinear elliptic singular perturbation problem of the type:

$$-\varepsilon^2 \Delta u + q(x,u) = 0 \ \ in \ \ \Omega; \ \ u = 0 \ \ on \ \ \partial\Omega, \tag{5}$$

where $0 < \varepsilon \ll 1$ and Ω is either a bounded interval or a bounded plane domain with sufficiently smooth boundary $\partial\Omega$. An exponentially graded mesh, which is locally one-dimensionally fitted in the direction of normal to the boundary, has been used in finite element method. The present paper is concerned with the linear theory; in particular the relation $\|Pv\|_\infty \le C\|v\|_\infty$; P is a Ritz projection, is established and finally, it has been shown that this method is $O(h^2)$ accurate inL_∞-norm. The proof of this result is carried over in the next research paper [12]. In this research paper, he explicitly designed bi-orthogonal bases for the one dimensional and two dimensional bounded domains, respectively. Together with the corresponding approximation results, the proof of the main theorem of [11] is finished here. The problem of error estimations and multilevel iterative solution of the linear systems-both uniformly well behaved with respect to time step-can be solved simultaneously within the frame work of preconditioning [cf. 25]. He derived a multilevel nodal basis preconditioner, able to handle highly nonuniform meshes. A numerical example as an application of the method of the bio-heat transfer equation is also considered in this paper. In the third paper [13], of this series, the following nonlinear elliptic singular perturbation problem is considered:

$$-\varepsilon^2 \Delta u + q(x) u = f(x) \ \ in \ \ \Omega = (-1,1) \times (-1,1); \ \ u = 0 \ \ on \ \ \partial\Omega, \tag{6}$$

where $0 < \varepsilon \ll 1$; q, f are sufficiently smooth functions and the two constants p_0, p_1 are such that $0 < p_0^2 \le q(x) \le p_1^2$. Using the techniques developed in [11, 12], the solution of the problem (6) having corner boundary layers is obtained. The grid is of tensor product type and the corresponding finite element space consists of bilinear elements near the boundary and linear elements in the remaining part of Ω. He has shown that this method is $O(h^2 \mathrm{lnh})$ accurate in L_∞-norm under the assumption $\varepsilon|\ln\varepsilon| \le Ch, h > 0$, a grid parameter, where the dimensions of the space of basis functions is of order $O(h^{-2}|\mathrm{lnh}|)$.

With the aim of locating the maxima of least energy solutions and of studying the behaviour as $\varepsilon \to 0$, Ni and Wei [90] considered the following singularly perturbed nonlinear elliptic problem in a bounded domain $\Omega \subset R^n$:

$$\varepsilon^2 \Delta u - u + u^p = 0 \ \ in \ \ \Omega; \ \ u > 0 \ \ in \ \ \Omega \ \ and \ \ u = 0 \, on \ \ \partial\Omega, \tag{7}$$

where $\varepsilon > 0$ and for $n \in N, p \in (1, \frac{n+2}{n-2})$ if $n \ge 3$, $p \in (1,\infty)$ if $n = 2$. Using mountain pass lemma, they have shown the existence and characterization of the least positive critical value c_ε of the energy functional $J_\varepsilon(u) = \int_\Omega (\varepsilon^2 \Delta u^2 +$

$u^2) - \frac{1}{p+1}\int_\Omega u_+^{p+1}$. Further, it has been established that: the least energy solution $u_\varepsilon > 0$ of (7), which minimizes $J_\varepsilon(u)$ has only one local maximum over $\bar{\Omega}$ and it is attained at exactly one point $P_\varepsilon \in \Omega$. Moreover, $u_\varepsilon \to 0$ as $\varepsilon \to 0$, except at P_ε, thereby exhibiting a single spike layer and $d(P_\varepsilon, \partial\Omega) \to \max d(P, \partial\Omega), P \in \Omega$ as $\varepsilon \to 0$, where $d(P, \partial\Omega)$ is the distance from P on $\partial\Omega$. Finally, they have obtained the asymptotic profile in ε of u_ε, which provides detailed explanation of u_ε for sufficiently small ε.

Shishkin [105] considered the following Dirichlet problem for the quasilinear elliptic singularly perturbed equation:

$$L_\varepsilon u(x) = f(x), x = (x_1, x_2) \in D = \{x : -1 < x_1 < 1, x_2 \in R\}; u(x) = \Phi(x), x \in \Gamma = \Gamma(D) = \bar{D}_s D, \quad (8)$$

where

$$L_\varepsilon u(x) = \left\{\varepsilon \sum_{s,k=1,2} a_{sk}(x) \frac{\partial^2}{\partial x_s \partial x_k} - 2u(x)\frac{\partial}{\partial x_1} - d(x)\right\} u(x), d(x) \geq c_0 > 0, x \in \bar{D}, \varepsilon \in (0, 1]. \quad (9)$$

The coefficients $a_{sk}(x)$ satisfy the ellipticity condition and the boundary function satisfies the condition: $\Phi(x) = \Phi(-1, x_2) = \Phi^+(x_2), \Phi(x) = \Phi(1, x_2) = \Phi^-(x_2), x \in \Gamma$, where $\Phi^+(x_2) > 0$, $\Phi^-(x_2) > 0, x_2 \in R$. Moreover, he also discussed an auxiliary problem for equations (8) and (9) in the domain $\bar{D}$, with curvilinear boundary. The reduced problem corresponding to equations (8) and (9) is studied in the domain $\Gamma_0 = \{x : x_1 = s_2(x_2), x_2 \in R\}$ as follows: $L_0 u(x) \equiv \left\{-2u(x)\frac{\partial}{\partial x_1} - d(x)\right\} u(x) = f(x), x \in \Gamma_0$ and $u(s_2(x_2), x_2) = 0, x \in R$, where $s_2(x_2)$ is a sufficiently smooth function satisfying $s_2(x_2) \leq 2^{-4}$. The domain Γ_0 lies inside the domain D and the function $x_1 = s_2(x_2)$ is a right boundary of the domain $\bar{D}$. He gave special difference schemes for solving problems (8) and (9) with their proof of uniform convergence, whose grids are condensed in the boundary and interior layers which appear as $\varepsilon \to 0$.

Wei [114] constructed a family of single-peaked solutions of the following singularly perturbed semilinear elliptic problem in an n-dimensional bounded domain Ω:

$$\varepsilon^2 \Delta u - u + u^p = 0 \ \ in \ \ \Omega \ \ u > 0 \ \ in \ \ \Omega \ \ u = 0 \ \ on \ \ \partial\Omega, \quad (10)$$

where $\varepsilon > 0$ and for $n \in N, p \in (1, \frac{n+2}{n-2})$ if $n \geq 3, p = 0$ if $n = 2$. It has been established that there exists an $\varepsilon_0 > 0$ such that for $0 < \varepsilon < \varepsilon_0$, the problem (10) has a solution u_ε with exactly one local maximum point $P_\varepsilon \in \Omega$, where $P_\varepsilon \to P_0$, the strict local maximum point of the distance between a point $P \in \Omega$ and the boundary $\partial\Omega$ and $u_\varepsilon(P_\varepsilon) \to w(0)$ as $\varepsilon \to +0$, and to zero otherwise. Here $w(.)$ is the unique solution of the problem: $\Delta w - w + w^p = 0, w > 0$ in R^n,$w(z) \to 0$ as $z \to \infty, \max_{z \in R^n} w(z) = w(0)$. The result may be considered as a kind of converse of the result stated by Ni and Wei [90] about least energy solutions to the problem (10).

In the research paper [35], Dancer and Wei studied the following singularly perturbed nonlinear elliptic problem:

$$\varepsilon^2 \Delta u + (u - a_1)(u - a_2)(u - a_3) = 0, \ \ u > 0 \ \ in \ \ \Omega; \ \ u = 0 \ \ on \ \ \partial\Omega, \quad (11)$$

where Ω is a bounded smooth domain in $R^n, 0 < a_1 < a_2 < a_3$ and $\int_{a_1}^{a_3}(s - a_1)(s - a_2)(s - a_3)ds > 0$. Using the mountain pass argument, the existence of a solution u_ε between $\underline{u}_\varepsilon$ and $\bar{u}_\varepsilon$ has been shown. By considering two choices of the solution u_ε as $\bar{u}_\varepsilon + v_\varepsilon$ and $\underline{u}_\varepsilon + v_\varepsilon$, it is established that v_ε has only one local maximum point P_ε in Ω with $d(P_\varepsilon, \Omega) \geq \delta > 0$ for some positive constants and $v(P_\varepsilon)$ approaches to some positive constant as ε approaches to 0.

Sirotkin [107] described iterative domain decomposition algorithms for the solution of following semilinear singularly perturbed elliptic problem:

$$\mu^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2}\right) = f(P, u), P = (x, y) \in \Omega, u(P) = U(P), P \in \partial\Omega,$$

$$f_u(P, u) \geq \beta_0 = \text{constant} > 0, (\text{P}, u) \in \bar{\Omega} \times \{-C, C\}, \quad (12)$$

where μ is a positive parameter, C is a sufficiently large number, $\partial\Omega$ is the boundary of Ω and $f(P, u)$ and $U(P)$ are sufficiently smooth functions. For $\mu \leq 1$, the problem is singularly perturbed and has boundary layers near $\partial\Omega$. In the present paper, iterative algorithms based on two domain decomposition methods: the Schwarz alternating procedure [103] and the related method from [19] are considered. Unlike the standard Schwarz method, in the latter method the boundary conditions on subdomain interfaces are defined using the solution of small auxiliary problems. They have also established the convergence properties of these algorithms and have presented estimates of a convergence rate depending on the geometric characteristics for domain decomposition and the values of the perturbation parameter μ followed by numerical stability of the iterative algorithms. Finally, the results of some numerical experiments using the developed algorithms are presented. In the experiments for numerical solution of singularly perturbed problems, a finite difference method on special non uniform grids has been applied.

With the aim to study the effect of the properties of the domain Ω, such as geometry and topology on the solutions of nonlinear elliptic typical singular perturbation problems, Dancer and Wei [36] considered the following problem:

$$\varepsilon^2\Delta u - u + u^p = 0 \ in \ \Omega; \ \ u > 0 \ in \ \Omega \ and \ u = 0 \ on \ \partial\Omega. \tag{13}$$

Recently, the geometry of the domain on the solutions of (13) has been a subject of the study. Beginning in [90], Ni and Wei studied the least energy solutions of (13) and showed that for ε sufficiently small, the least energy solution has only one local maximum point P_ε and P_ε must lie in the most centered part of Ω, namely $d(P_\varepsilon, \partial\Omega) \to \max d\,(P, \partial\Omega)\,, P \in \Omega$, where $d\,(P, \partial\Omega)$ is the distance from Pon $\partial\Omega$. On the other hand, in [114], a kind of converse was proved. Namely, for each strictly local maximum point of the distance function $d\,(P, \partial\Omega)\,,$ there is a solution of (13) with only one local maximum point near that point. This shows that the geometry of the domain plays a very important role in the multiplicity of solutions of (13). The effect of the geometry of Ω on single peaked solutions has been studied in [115]. In particular, both necessary and sufficient conditions for the existence of single peaked solutions are established. These conditions depend highly on the geometry of the domain. For further studies in this direction, readers are suggested to see [31, 76, 96]. On the other hand, Benci and Cerami ([6], [7]) studied the effect of the topology of Ωon solutions of (13). More precisely they showed that there are at least $\mathrm{cat}\,(\Omega) + 1$ solutions for $\varepsilon \ll 1$. In fact, what they actually showed was there are at least $\mathrm{cat}\,(\Omega) + 1$ single-peaked solutions (that is, solutions with single maximum point), where $\mathrm{cat}\,(\Omega)$ denotes the category of Ω. In [36], Dancer and Wei studied the effect of domain topology on multiple-peak solutions (that is, solutions with more than one local maximum points), whose existence and multiplicity are highly related with geometry and topology of the domain Ω and proved that if reduced homology is nontrivial then for ε sufficiently small, there exists a two peak solution of the problem (13).

Li and Navon [75] considered the following quasi-linear singularly perturbed elliptic problem:

$$\varepsilon^2\Delta u = F\,(x, y, u) \ \ in \ \Omega = (0, 1) \times (0, 1); \ \ u = 0 \ on \ \partial\Omega, \tag{14}$$

which has already been discussed by Boglaev [14], where a nonlinear finite difference scheme is constructed with a uniform convergence rate at the nodal points $O(N^{(-1/2)})$; N is the total number of grid points. In this paper, using the techniques developed in [73, 74], they have constructed a bilinear finite element method for solving the problem (14) on a piecewise uniform mesh, where the quasi-optimal global uniform convergence rate $O(N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y)$, independent of the perturbation parameter, is obtained, where N_x and N_y are number of elements in the x and y directions. More preciously, it is established that under certain assumptions, $\|u - u_h\| \leq N_x^{-2}\ln^2 N_x + N_y^{-2}\ln^2 N_y$, where u_h is the finite element solution of $\varepsilon^2(\nabla u_h, \nabla v_h) + (F\,(u_h, x, y)\,, v_h) = 0, v_h \in S_h\,(\Omega)\,;$ here $S_h\,(\Omega)$ is tensor-product quadratic element space, and u is the solution of the problem (14).

Cao and Noussair [30] considered the following singularly perturbed nonlinear elliptic problem to establish the existence of single and multi-peak solutions:

$$-\varepsilon^2\Delta u + u = u^{p-1}, \ \ u > 0 \ \ in \ \ \Omega; \ \ u = 0 \ \ on \ \ \partial\Omega, \tag{15}$$

where $\Omega$is a smooth bounded domain in $R^N(N \geq 2), \varepsilon > 0$ is a small positive number and $p \in (1, \frac{2N}{N-2})$ if $n \geq 3, p \in (2, +\infty)$ for $N = 2$. Using a variational technique, it has been established that if Ω has a saddle point $P \in \Omega$, then there exists $\varepsilon_0 > 0$ such that for $\varepsilon \in (0, \varepsilon_0]\,,$ the problem (15) has a positive solution u_ε of the form $u_\varepsilon = \alpha_\varepsilon \wp_\varepsilon W_{\varepsilon, P_\varepsilon} + v_\varepsilon$ for suitably defined $\alpha_\varepsilon > 0, P_\varepsilon \in \Omega$ and $v_\varepsilon \in E^1_{\varepsilon, P_\varepsilon}$. Similar types of results have been established under the assumptions when Ω has k-saddle points, and when, in addition, the distance function $dist(., \partial\Omega)$ has m strict local maximum points satisfying certain properties. Thus they have constructed the positive solutions of the problem (15) concentrating near given "saddle points" of the distance function $dist(., \partial\Omega)$ with an explicit expression for the dominant parts of the solutions.

The following semilinear singularly perturbed Dirichlet boundary value problem for elliptic differential equation has been considered by Mo Jiaqui [57]:

$$\varepsilon^{m-k} L_{2m}\,[u] - L_{2k}\,[u] - f\,(x, u, \varepsilon) = 0, x \in \Omega \subset R^n; \frac{\partial^i u}{\partial n^j} = g_j\,(x)\,, x \in \partial\Omega, j = 1, 2, \ldots, m-1, \tag{16}$$

where ε is a positive small parameter, Ω signifies a bounded domain in R^n with smooth boundary $\partial\Omega$, $\frac{\partial}{\partial n}$ denotes the outward derivative on $\partial\Omega$ and C^∞ and

$$L_{2m} \equiv \sum_{1 \leq |\mu|, \sigma \leq m}^{n} (-1)^\mu\, D^\mu(a^{\mu\sigma}\,(x)\,D^\sigma), L_{2k} \equiv \sum_{1 \leq \mu, \sigma \leq k}^{n} (-1)^\mu\, D^\mu(b^{\mu\sigma}\,(x)\,D^\sigma),$$

$$D_j = \frac{\partial}{\partial x_j}, D^\alpha = D_1^{\alpha_1} D_2^{\alpha_2}, \ldots, D_n^{\alpha_n}, \alpha = \sum_{j=1}^{n} \alpha_j,$$

with $1 \leq k < m$ and the coefficients $a^{\mu\sigma}$ and $b^{\mu\sigma}$ are assumed real valued and of class $C^m(\Omega) \cap C^k(\Omega)$, f and g are sufficiently smooth functions with regard to their variables in correspondence ranges, L_{2m} and L_{2k} are uniformly strongly elliptic operators in $\bar{\Omega}$:

$$\sum_{1\leq\mu,\sigma\leq m}^{n} \xi^\mu (a^{\mu\sigma}(x)\,\xi^\sigma) \geq \lambda_m \xi^{2m} = \lambda_m (\sum_{i=1}^{n} \xi_i^2)^m, \xi \in R^n, x \in \bar{\Omega}, \lambda_m > 0,$$

$$\sum_{1\leq\mu,\sigma\leq k}^{n} \xi^\mu (b^{\mu\sigma}(x)\,\xi^\sigma) \geq \lambda_k \xi^{2k} = \lambda_k (\sum_{i=1}^{n} \xi_i^2)^k, \xi \in R^n, x \in \bar{\Omega}, \lambda_k > 0.$$

It is assumed that:
[H1] There is a positive constant δ such that $\frac{\partial f(x,\mu,\varepsilon)}{\partial u} \geq \delta, x \in \bar{\Omega}, u \in R$.
[H2] There exists unique solution of reduced problem:

$$L_{2k}[w(x)] - f(x, w, 0) = 0, x \in \Omega \subset R^n; \frac{\partial^i w}{\partial n^j} = g_j(x), x \in \partial\Omega, j = 1, 2, \ldots, k-1,$$

of the problem (16). Under the hypothesis [H1] and [H2] and under suitable conditions, using fixed point theorems, he established that there exists a solution u of the singularly perturbed boundary value problem (16) for the nonlinear reaction-diffusion equation having a uniformly valid asymptotic expansion and holds estimation $\mathrm{u}(\mathrm{x}, \varepsilon) = \mathrm{w}(\mathrm{x}) + \bar{\mathrm{v}}(\mathrm{x}) + O(\varepsilon)$, where $\mathrm{w}(\mathrm{x})$ is a solution of the reduced problem and $\bar{\mathrm{v}}(\mathrm{x})$ is a boundary layer correction.

A following class of singularly perturbed nonlocal problems for nonlinear elliptic equation of fourth order is considered by Yusen [119]:

$$Lu \equiv \varepsilon^2 \Delta^2 - A(x)\Delta u = f(x, u, \varepsilon), x \equiv (x_1, x_2, \ldots, x_n) \in \Omega, A(x) \geq A_0 > 0, \tag{17}$$

$$u = g_1(x, Tu, \varepsilon), x \in S, \tag{18}$$

$$\Delta u = \varepsilon^{-2} g_2(x, \varepsilon), x \in S, \tag{19}$$

$$Tu = \Psi(x) + \int_S K(x, y)\, u(y)\, dy, K(x, y) \geq 0, x, y \in S, \tag{20}$$

where ε $(0 < \varepsilon \ll 1)$ is a positive parameter, Δ is Laplacian, Ω denotes a boundary region in R^n, S signifies a smooth boundary of Ω. Under the following assumptions:
(L1) A, f, g_1, Ψ and K are sufficiently smooth functions with regard to their variables in corresponding ranges.
(L2) $f_y(x, y, \varepsilon) \geq 0$.
(L3) The reduced problem of (17) to (20): $A(x)\Delta(u) = -f(x, U, 0)$,$x \in \Omega$, $U = g_1(x, Tu, 0)$, $x \in S$ has a unique solution $U_0(x) \in C^4(\Omega) \cap C^2(\Omega + S)$, author has proved that there exists a solution $u(x, \varepsilon)$ of (17) to (20), which has uniformly valid asymptotic expansion: $u(x, \varepsilon) \sim \sum_{i=0}^{m} U_i e^i + \sum_{i=0}^{m+2} \bar{v}_i \varepsilon^i + O\left(\varepsilon^{m+1}\right), x \in \Omega + S.$

Let Ω be a bounded domain in R^n, $N \geq 3$. The following singular boundary value problem is considered by Sun and Wu [110]:

$$\Delta u + q(x)\, u^\alpha + p(x)\, u^{-\gamma} = 0 \;\; u > 0 \;\; in \;\; \Omega; \;\; u = 0 \;\; on \;\; \partial\Omega, \tag{21}$$

where $p(x)$ and $q(x)$ are two given Hölder continuous functions on $\bar{\Omega}$; $0 < \gamma < \frac{1}{N}$ and $0 < \alpha < 1$ are two constants. The aim of the present paper was to study the combined effect of sublinear and singular nonlinearities to learn how the changing of sign of $f'(u)$ gives rise to the existence of solutions for singular elliptic problems. They have shown the existence and uniqueness of positive classical solution to the problem (21) and established the iteration processes for approximating the unique solution based on the mixed monotone iterative method. In particular, under the assumptions, $q(x) \geq 0, p(x) \geq 0$ and $p(x) + q(x) \neq 0$, and certain smoothness assumptions, they have shown that there exists a unique classical positive solution $u^*(x) \in C^{2+\min(\alpha,\gamma)}(\Omega) \cap C^{1,r}\left(\bar{\Omega}\right), 0 < r < 1$.

Molle [87] considered a class of singularly perturbed nonlinear elliptic problems (P_ε) with subcritical nonlinearity. The coefficient of the linear part is assumed to concentrate in a point of the domain, as $\varepsilon \to 0$ and the domain is supposed to be unbounded and with unbounded boundary. Domains that enlarge at infinity, and whose boundary flattens or shrinks at infinity, are considered. It is also proved that in such domains problem (P_ε) has at least two solutions.

In [26], Byeon considered the following singularly perturbed nonlinear elliptic problem: Let Ω a bounded domain in R^n with $\partial\Omega \in C^2$:

$$\varepsilon^2 \Delta u - u + f(u) = 0,\ \ u > 0\ \ in\ \ \Omega;\ \ u = 0\ \ on\ \ \partial\Omega. \tag{22}$$

The limiting equation to the equation (22) is given by:

$$\Delta u - u + f(u) = 0,\ \ u > 0\ \ in\ \ R^n;\ \ \lim_{|x|\to 0} u(x) = 0. \tag{23}$$

Under the conditions (f1): for $f \in C(R,R)$, $f(t) = 0$ for $t \le 0$ and $\lim_{t\to 0} \frac{f(t)}{t} = 0$; (f2): there exists $p \in (1, \frac{n+2}{n-2})$ such that $\limsup_{t\to\infty} \frac{f(t)}{t^p} < \infty$; and (f3): for $F(t) = \int_0^t f(s)ds$, there exist $\mu > 2$, such that $0 < \mu F(t) < f(t)\,t$ for $t > 0$, there exists a mountain pass solution $u_\varepsilon > 0$ of problem (23). Using comparison principle and energy estimate, the author deduced that for a maximum point x_ε of u_ε, there exists constants $C, c > 0$, independent of $\varepsilon > 0$ satisfying $u_\varepsilon(x) \le C \exp\left(-\frac{c\mathrm{dist}(x,x_\varepsilon)}{\varepsilon}\right), x \in \Omega$. Thus u_ε exhibits a spike layer as $\varepsilon \to 0$. Ni and Wei [2] proposed the following asymptotic behaviour under some additional conditions than (f1), (f2) and (f3) about the location of maximum point x_ε of the solution

$$u_\varepsilon\ \ (\varepsilon > 0):\ \lim_{\varepsilon\to 0} \mathrm{dist}(x_\varepsilon, \partial\Omega) = \max \mathrm{dist}\left(x, \partial\Omega\right), x \in \partial\Omega. \tag{24}$$

In [94], del Pino and Felmer showed that the asymptotic behaviour (24) can be obtained by the monotonicity condition: $\frac{f(t)}{t}$ is non-decreasing on (0,∞). Extending the ideas of [94], the author has proved that the asymptotic behaviour (24) can be obtained without the monotonicity condition. Thus, only under the conditions (f1), (f2) and (f3), it has been proved that for sufficiently small $\varepsilon > 0$, there exists a mountain pass solution u_ε of (22) and for a transformed function $w_\varepsilon(x) = u_\varepsilon(\varepsilon(x - x_\varepsilon))$ and for any $\varepsilon_m > 0$ with $\lim_{m\to\infty} \varepsilon_m = 0$, the sequence $\{w_{\varepsilon_m}\}$ converges up to a subsequence, uniformly to a least energy solution of (23).

Let M be a connected compact smooth Riemannian manifold of dimension $n \ge 3$ with or without smooth boundary ∂M. Byeon and Park [28] considered the following singularly perturbed nonlinear elliptic problem on M :

$$\varepsilon^2 \Delta_M u - u + f(u) = 0,\ \ u > 0\ \ in\ \ M;\ \ \frac{\partial u}{\partial v} = 0\ \ on\ \ \partial M, \tag{25}$$

where Δ_M is the Laplace-Beltrami operator on M, a typical form of $f(u)$ is $|u|^{p-1}u, p \in (1, \frac{n+2}{n-2})$ and ν is an exterior normal to ∂M. For certain f, there exists a mountain pass solution u_ε of above problem which exhibits a spike layer. They are interested in the asymptotic behaviour of the spike layer. Without any non-degeneracy condition and monotonicity of $\frac{f(t)}{t}$, they have shown that if $\partial M = \phi$ ($\partial M \ne \phi$), the peak point x_ε of the solution u_ε converges to a maximum point of the scalar curvature S on M (the mean curvature H on ∂M) as $\varepsilon \to 0$, respectively. Thus, the authors have studied the similar problem (33) on a smooth Riemannian manifold, and characterized a concentration point with respect to a local geometric quantity of the manifold. Thus, the problem (25) under more general conditions than (f1), (f2) and (f3) has been studied. For $n = 2$, their approach in this paper does not work well. The problem (33) for $n = 1$ is rather simple and does not reflect the geometry of M.

In the research paper [17], Boglaev considered the following semilinear singularly perturbed problems which correspond to the reaction-diffusion and the convection-diffusion problems of the elliptic type:

$$\begin{gathered} Lu = -f(x,y,u), \\ Lu = L_\mu u \equiv -\mu^2 (u_{xx} + u_{yy}), \text{ or } Lu = L_\varepsilon u \equiv -\varepsilon (u_{xx} + u_{yy}) + b_1 u_x + b_2 u_y, \\ (x,y) \in w, w = w^x \times w^y = \{0 < x < 1\} \times \{0 < y < 1\}, \\ f_u \ge c_* > 0, (x,y,u) \in \bar{w} \times (-\infty,\infty), f_u \equiv \frac{\partial f}{\partial u}, \\ b_1 \ge \beta_1 \ge 0, b_2 \ge \beta_2 \ge 0\ \ on\ \ \bar{w},\ \ u = g\ \ on\ \ \partial w, \end{gathered} \tag{26}$$

where μ and ε are small positive parameters, c_* and $\beta_{1,2}$ are constants. If f, g and $b_{1,2}$ are sufficiently smooth, then under suitable continuity and compatibility conditions on the data, a unique solution u of (26) exists (see [72] for details). For $\mu \ll 1$, the reaction-diffusion problem (26) with $L = L_\mu$ is singularly perturbed and characterized by the boundary layers of width $O(\mu|ln\mu|)$ near ∂w [15]. For $\varepsilon \ll 1$, the convection-diffusion problem (26) with $L = L_\varepsilon$ is singularly perturbed and characterized by the regular boundary layers of width $O(\varepsilon|ln\varepsilon|)$ at $x = 1$ and $y = 1$ [81]. In the study of numerical methods of nonlinear singularly perturbed problems, the two major points have to be developed: (i) constructing parameter uniform difference schemes (ii) obtaining reliable and efficient computing algorithms for computing nonlinear discrete problems. Keeping this in mind, the author has presented difference schemes which approximate the nonlinear problem (26), and also constructed a monotone iterative method for solving these nonlinear difference schemes. The convergence properties of these methods have also been studied. For the reaction-diffusion problem (26) with $L = L_\mu, f(u) = \frac{(u-4)}{(5-u)}$ and $g = 1$, the author has given the number of iterations required to satisfy the stopping criterion for the monotone method on the piecewise uniform mesh $\varphi(\xi) = 4\xi$ and on the long-mesh $\varphi(\xi) = \frac{\ln[1-4(1-\mu)]}{\ln \mu}$, respectively. From the data, it can be concluded that the numbers of iterations are independent of the perturbation parameter μ and numerical results confirm the theoretical results. Similarly, for the convection-diffusion problem (26) with $L = L_\varepsilon, f(u) = \frac{(u-4)}{(5-u)}$ and $g = 1$, the number of iterations required to satisfy the stopping criterion for the monotone method on the piecewise uniform mesh $\varphi(\xi) = 1 - 2\xi$ and on the long-mesh $\varphi(\xi) = \frac{\ln[1-(1-\varepsilon)(1-2\xi)]}{\ln \varepsilon}$ are obtained. From the data, it can be concluded here also that the numbers of iterations are independent of the perturbation parameter ε. Again the numerical results confirm the theoretical results.

Hardy and Boglaev [50] described the parallel implementation of a monotone decomposition algorithm for the following nonlinear singularly perturbed reaction-diffusion problem of the elliptic type:

$$-\mu^2(u_{xx} + u_{yy}) + f(x, y, u) = 0, (x, y) \in w; \;\; w = w^x \times w^y = (0,1) \times (0,1), u = g \;\; on \;\; \partial w, \tag{27}$$

where $\mu \ll 1$ is the perturbation parameter and ∂w is the boundary of w. They also assumed that $c^* \geq f_u \geq c_*, (x, y, u) \in \bar{w} \times R,$ where c^* and c_* are positive constants. The solution is characterized by boundary layers of width $O(\mu|ln\mu|)$. Discrete approximation of (27) leads to an algebraic system of nonlinear difference equations whose solution converges with mesh refinement to that of the continuous problem. The algebraic system is typically solved by Newton's method or some other iterative technique. One drawback of Newton's method is its sensitivity to the initial guess. On the other hand, the method of upper and lower solutions generates a monotonically convergent sequence from any one of a wide class of initial iterates [16, 93]. No knowledge of the solution is necessary to implement the algorithm. The advent of the Beowulf cluster has brought high-performance computing within reach of academe and thus fostered renewed interest in alternating Schwarz-type domain decomposition algorithms. In [20], the domain is partitioned into non-overlapping boxes and the monotone iterative method is applied on each subdomain. At each horizontal and vertical boundary, interfacial subdomains are introduced and corresponding linear problems generate boundary Dirichlet data for the non-overlapping subdomains. As shown theoretically and confirmed by serial computations [20], the algorithm retains global monotonicity under such decomposition. This paper describes a parallel implementation of the Box-domain decomposition algorithm with numerical experiment for a model problem [20]. They have applied the algorithm to the reaction-diffusion problem: $-\mu^2(u_{xx} + u_{yy}) + \frac{(u-4)}{(5-u)}, (x, y) \in w; w = w^x \times w^y = (0,1) \times (0,1), u = 1$ on ∂w. The solution to the reduced problem $(\mu = 0)$ is $\mu_r = 4$ and the solution increases sharply for $u = 1$ on ∂w to $u = 4$ on the interior. The nonlinear scheme:

$$L^h + f(p, u) = 0, p \in w^h \;\; and \;\; w = g \;\; on \;\; \partial w^h, \;\; where \;\; L^h = -\mu^2(D_x^2 + D_y^2), \tag{28}$$

is solved by the domain decomposition algorithm. They started the algorithm with the mesh function $V^{(0)}(w^h) = 0, V^{(0)}(\partial w^h) = 1$, which guarantees monotonic convergence to the solution of (28). The convergence improves as the interfacial subdomains are enlarged. Between any two processors, the necessary inter-subdomain transfers are buffered and sent as one MPI message. The parallel scale-up of the implementation improves with increasing mesh size N. On a 1024×1024 mesh, the optimal decomposition for any number of processors is 32×32, which is a minimal interfacial subdomain decomposition. On 16 processors, they observed a parallel speed up of 13.04 which translate to a computational efficiency of 81.5%.

On the vertical strip $\bar{D} = D \cup \Gamma$, $D = \{x : x_1 \in (0, d), x_2 \in R\}$, Tselishcheva and Shishkin [112] considered the following semilinear singularly perturbed elliptic equation of reaction-diffusion type:

$$L_{(1,1)}(u(x)) \equiv L^2_{(11)}u(x) - f(x, u(x)) = 0, x \in D, \tag{29}$$

$$lu(x) \equiv \left\{\varepsilon\alpha(x)\frac{\partial}{\partial n} + \beta(x)\right\} u(x) = \Psi(x), x \in \Gamma. \tag{30}$$

Here $L^2_{(1,1)} \equiv \varepsilon^2 \sum\limits_{s=1,2} a_s(x) \frac{\partial^2}{\partial x_2^2} - c(x), \varepsilon \in (0,1]$ is the singular perturbation parameter and n denotes the outward normal to the boundary Γ. The functions $a_s(x), c(x)$ and $f(x, u)$ are assumed to be sufficiently smooth on $\bar{D}$ and $\bar{D} \times R$, respectively,$\alpha(x), \beta(x)$ and $\Psi(x)$ are sufficiently smooth on Γ, and also

$$0 < a_0 \leq a_s(x) \ll a^0, s = 1, 2, 0 < c_0 < c(x), x \in \bar{D};$$

$$|f(x,u)| \le M, 0 < c_0 \le c(x) + \frac{\partial f(x,u)}{\partial n} \le c^1, (x,y) \in \bar{D} \times R;$$
$$0 \le \alpha(x), \beta(x) \le M, \alpha(x) + \beta(x) \ge m, |\Psi(x)| \le M, x \in \Gamma. \quad (31)$$

They have Dirichlet problem if $\alpha(x) = 0$ on Γ and the Neumann problem if $\beta(x) = 0$ on Γ. As $\varepsilon \to 0$, the boundary layer arises in a neighbourhood of Γ. Thus they have considered the semilinear singularly perturbed elliptic reaction diffusion problem (29) and (30) on a strip with Robin boundary conditions with the requirement to construct a base difference scheme and a scheme based on successive approximations and, for such schemes, to develop domain decomposition schemes. It is necessary that these schemes converge ε-uniformly and the number of iterations required for convergence is independent of ε. They introduced a rectangular grid $\bar{D}_h = \bar{w}_1 \times w_2$, where $\bar{w}_1$ and w_2 are generally arbitrary nonuniform meshes on $[0, d]$ and the axis x_2, respectively and approximated the problem by the difference scheme [102]:

$$\Lambda_{(2.2)}(z(x)) \equiv \Lambda^2_{(2.2)} z(x) - f(x, z(x)) = 0, x \in D; \lambda^*_{(2.2)} z(x) = \Psi^*(x; z(x)), x \in \Gamma_h, \quad (32)$$

here $D_h = D \cap \bar{D}_h, \Gamma_h = \Gamma \cap \bar{D}_h, \Lambda^2_{2.2} = \varepsilon^2 \sum_{s=1,2} a_s(x) \partial_{\widehat{\bar{x}s}} - c(x), \varepsilon \in (0,1]$. For scheme (32), an overlapping domain decomposition method [106] is described and it is proved that under the condition (31), the difference schemes (32) converges uniformly having an error bound. To linearize the scheme (32), they constructed monotone linearized schemes of the ε-uniform accuracy and applied the technique of upper and lower solutions to find a posteriori the number of iterations in the linearized scheme under which the accuracy of its solutions is the same as for the base scheme. The number of required iterations is independent of ε. With respect to total computational costs, the iterative method is close to a method of obtaining the solution for linear problems, since the number of iterations is only weakly depending on the number of mesh points used.

Byeon [27] considered the following nonlinear elliptic singularly perturbed problem on Ω, a bounded domain in R^n, $n \ge 3$:

$$\varepsilon^2 \Delta u - u + f(u) = 0, \;\; u > 0 \;\; in \;\; \Omega; \;\; \frac{\partial u}{\partial v} = 0 \;\; on \;\; \partial\Omega, \quad (33)$$

where $f(u)$ has the typical form $|u|^{p-1}u, p \in (1, \frac{n+2}{n-2})$. [2] assumed the following conditions for a continuous function $f : R \to R$:

(C1) $f(t) = 0$ for $t \le 0$ and $\lim\limits_{t \to 0} \frac{f(t)}{t} = 0$;

(C2) there exists $p \in (1, \frac{n+2}{n-2})$ such that $\limsup\limits_{t \to \infty} \frac{f(t)}{t^p} < \infty$;

(C3) there exist $\mu > 2$ and $t_0 > 0$ such that $\mu \int\limits_0^t f(s)\,ds < f(t)\,t$ for $t > t_0$,

and shown the existence of mountain pass solution $u_\varepsilon > 0$ for the problem (33) for all $\varepsilon > 0$. In papers [77, 89], authors proved under rather stronger conditions than (C1), (C2) and (C3), that for a sufficiently small $\varepsilon > 0$, there exists a unique maximum point $x_\varepsilon \in \partial\Omega$ of u_ε and constants $C, c > 0$ satisfying:

(i) $\liminf\limits_{\varepsilon \to 0} u_\varepsilon(x_\varepsilon) > 0, u_\varepsilon(x) \le C \exp(-c \frac{|x - x_\varepsilon|}{\varepsilon})$;

(ii) for a diffeomorphism Ψ from R^n_+ to a neighborhood *B* of x_ε in Ω satisfying $\Psi\left(\partial R^n_+\right) = \bar{B} \cap \partial\Omega$, a transformed solution $u_\varepsilon(x) \equiv u_\varepsilon o\ \Psi(\varepsilon x)$ converges uniformly to a radially symmetric least energy solution *U* of the limiting problem of (33):

$$\Delta u - u + f(u) = 0, u > 0 \;\; in \;\; R^n_+; \;\; \frac{\partial u(x)}{\partial x_n} = 0 \;\; on \;\; \partial R^n_+ \;\; and \;\; \lim_{|x| \to 0} u(x) = 0; \quad (34)$$

(iii) for the mean curvature function H on $\partial\Omega$, $\lim\limits_{\varepsilon \to 0} H(x_\varepsilon) = \max\limits_{x \in \partial\Omega} H(x)$.

On the other hand, Berestycki and Lions had shown in [9] that equation (34) has a radially symmetric least energy solution U satisfying $|D^\alpha U(x)| \le C \exp(-\delta |x|), x \in R^n$, for some $C, \delta > 0$ and for any $\alpha \le 2$, under the conditions (C1), (C2) and condition **(C3)**: there exists $T > 0$ satisfying $F(T) = \int\limits_0^t f(t)\,dt > T^2/2$. They believed that the conditions (C1), (C2) and **(C3)** (called Berestycki-Lions conditions) on f are almost optimal for the existence of solution of (34). In [54], Jeanjean and Tanaka have shown that a least energy solution of (34) is a mountain pass solution. Since a mountain pass solution is structurally stable, it is natural to expect that there exists a corresponding solution of a singularly perturbed problem (33) for small $\varepsilon > 0$ whenever a limiting problem (34) has a least energy solution. In this paper, the author has proved that this is true when $f \in C^1$ and $n \ge 3$. In fact, it is shown that for any isolated set of local maximum points of the mean curvature H on $\partial\Omega$, there exists a localized solution concentrating around the local minimum points if $\varepsilon > 0$ is small. If f is just continuous, it is not certain whether any least energy solution of (34) is radially symmetric up to a translation. If there exists a nonradial least energy solution of (34) up to a translation, the limiting behaviour of the spike layer may not depend only on the geometry of $\partial\Omega$ [94]. Thus, it seems that the C^1-condition for f is not so restrictive. On the other hand, it is proved in [29] that if f is continuous and satisfies (C1), (C2) and **(C3)**, this expectation are true for singularly perturbed problems on R^n with a non constant potential. In conclusion, author [27] has pointed out that it would be interesting to construct multiple boundary layer or interior spike layer solutions just under Berestycki-Lions conditions.

Mohanty and Singh [85] considered the following two dimensional singularly perturbed nonlinear elliptic partial differential equation:

$$\varepsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)=f\left(x,y,u,u_x,u_y\right),(x,y)\in\Omega, \tag{35}$$

defined in the domain $\Omega=\{(x,y):0<x,y<1\}$ with the boundary $\partial\Omega$, where $\varepsilon>0$. The Dirichlet boundary conditions are given by $u\left(x,y\right)=g\left(x,y\right),(x,y)\in\Omega$. It is assumed that

(i) $f\left(x,y,u,u_x,u_y\right)$ is continuous,

(ii) $(\frac{\partial f}{\partial u}),(\frac{\partial f}{\partial u_x})$ and $(\frac{\partial f}{\partial u_y})$ exist and continuous,

(iii) $\frac{\partial f}{\partial u}>0,|\frac{\partial f}{\partial u_x}|\le w_1$ and $|\frac{\partial f}{\partial u_y}|\le w_2$,

where w_1 and w_2 are positive constants. In addition, they assumed that $u\in C^6\left(\Omega\right)$, where $C^6\left(\Omega\right)$ stands for the set of all functions x and y whose partial derivatives up to the order six are continuous in Ω. Thus they assumed that the boundary value problem (35) has a unique solution and tried to find it. A new difference method of $O(h^4)$ is discussed here, called as arithmetic average discretization for the solution of two dimensional nonlinear singularly perturbed elliptic partial differential equation of the form: $\varepsilon\left(u_{xx}+u_{yy}\right)=f\left(x,y,u,u_x,u_y\right),0<x,y<1$, subject to the appropriate Dirichlet boundary conditions where $\varepsilon>0$ is a small parameter. A new methods of $O(h^4)$ for the estimates of $\frac{\partial u}{\partial n}$ is also derived which are quite often of interest in many physical problems. The proposed methods are directly applicable to singular elliptic problems. They did not require any fictitious points to discretize the differential equation (35) near the boundary. [85] presents a new fourth order compact finite difference discretization strategy based on arithmetic average discretization (Chawla and Shivakumar [32]) for the solution of two dimensional singularly perturbed nonlinear elliptic partial differential equations, which is directly applicable to problems in polar coordinates while the fourth order numerical methods developed in [83, 84], are not directly applicable to these elliptic problems and a special technique is required to solve these problems in polar coordinates. Numerical experiments shows that the new fourth order discretization strategy is advantageous over five point discretization of $O(h^2)$. It is demonstrated by numerical experiments that although the order for the proposed nine point fourth order compact scheme drops to lower order but numerical oscillations do not appear throughout the computations and method converges fast for the small parameter value of $\varepsilon>0$, whereas the five point second order method becomes unstable. They also discussed the convergence analysis and outlined the advantage to solve the relevant sparse systems. From numerical experiments the necessity of high accuracy solutions has been shown.

Mohanty and Singh [86] employed a new fourth order compact finite difference formula based on arithmetic average discretization to solve the following three dimensional nonlinear singularly perturbed elliptic partial differential equation:

$$\varepsilon\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)=f\left(x,y,z,u,u_x,u_y,u_z\right),0<x,y,z<1, \tag{36}$$

subject to appropriate Dirichlet boundary conditions prescribed on the boundary, where $\varepsilon>0$ is a small parameter. They have also described new fourth order methods for the estimates of u_x,u_y and u_z, which are quite often of interest in physical problems. In all cases, they required only a single computational cell with 19 grid points. The proposed methods are directly applicable to solve singular problems without any modification. They have solved three test problems numerically to validate the proposed fourth order methods. They have compared the advantages and implementation of the proposed methods with the standard central difference approximations in the context of basic iterative methods. Numerical examples are also given to verify the fourth order convergence rate of the methods.

The following nonlinear singularly perturbed problems of elliptic type have been considered by Boglaev [18] for its numerical solution:

$$\begin{aligned}-\mu^2u^{''}+f\left(x,u\right)=0,x\in w=\left(0,1\right),u\left(0\right)=0,u\left(1\right)=0,\\ f_u\ge c_*=\text{constant}>0,(x,u)\in\bar{w}\times\left(-\infty,\infty\right),\end{aligned} \tag{37}$$

where μ is a positive parameter and f is sufficiently smooth function. For $\mu\ll 1$, this problem is singularly perturbed and the solution has boundary layer near $x=0$ and $x=1$. A major point about nonlinear difference schemes is to obtain reliable and efficient computational methods for computing the solution. The reliability of iterative techniques for solving nonlinear difference schemes can be essentially improved by using component wise monotone globally convergent iterations. Such methods can be controlled every time. A fruitful method for the treatment of these nonlinear schemes is the method of upper and lower solutions and its associated monotone iterations [91]. Since an initial iteration in the monotone iterative method is either an upper or lower solution, which can be constructed directly from the difference equation without any knowledge of the exact solution, this method simplifies the search for initial iteration as is often required in Newton's method. In the context of solving systems of nonlinear equations, the monotone iterative method belongs to the class of methods based on convergence under partial ordering [91]. In the present paper the author has constructed a μ-uniform numerical method for solving the problem (37), that is a numerical method which generates μ-uniformly convergent numerical approximations to the solution of problem (37). Also a Robust monotone iterative method for solving nonlinear difference scheme is constructed. There is a use of numerical method based on classical difference scheme and the piece wise uniform mesh of Shishkin-type [82]. Finally, numerical results obtained for $f(u) = \frac{(u-3)}{(4-u)}$ confirm the theoretical results.

For a given (M, g) a smooth compact Riemannian N-manifold and for any fixed positive integer K, Dancer, Micheletti and Pistoia [38] considered the following nonlinear elliptic singularly perturbed problem:

$$-\varepsilon^2 \Delta_g u + u = u^{p-1} \ \ in \ \ M; \ \ u > 0 \ \ in \ \ M, \ \ u \in H_g^1(M), \tag{38}$$

where $p > 2$ if $N = 2, 2 < p < 2* = \frac{2N}{N} - 2$ if $N \geq 3$ and ε is a positive parameter. They have shown that the problem (38) has a K-peaks solution, whose peaks collapse, as ε goes to zero, to an isolated local minimum point of the scalar curvature.

With the aim of study of existence, multiplicity and shape of positive solutions, Ramos [98] considered the following strongly coupled Hamiltonian systems of the form:

$$-\varepsilon^2 \Delta u + V(x)u = K(x)g(v); \ \ -\varepsilon^2 \Delta v + V(x)v = H(x)f(u), u, v \in R^n, \tag{39}$$

with $u, \ v > 0, \ u(x) \to 0$ and $v(x) \to 0$ as $|x| \to \infty$, the functions f and g are power-like nonlinearities with super-linear and subcritical growth at infinity, and $V, \ H$ and K are positive and locally Hölder continuous. Here $c(\xi)$ is the ground-state critical level of the autonomous problem in R^N given by: $-\Delta u + V(\xi)u = K(\xi)g(v), -\Delta v + V(\xi)v = H(\xi)f(u)$, whose associated energy functional is (formally) given by: $I(u, v) : \int_{R^n} \langle \nabla u, \nabla v\rangle + V(\xi)uv - H(\xi)F(u) - K(\xi)G(v))$. He merely mentioned that, as it is well known for the scalar equation: $-\varepsilon^2 \Delta u + V(x)u = f(u)$ in R^n with $u > 0, u(x) \to 0$ and $v(x) \to 0$ as $|x| \to \infty$, where $N \geq 3, V$ is continuous and f has super-linear and subcritical behaviour, or the gradient systems of the form: $-\varepsilon^2 \Delta u + V(x)u = \partial F(u, v)/\partial u, -\varepsilon^2 \Delta v + W(x)v = \partial F(u, v)/\partial v$ with model potential energy $F(u, v)$, the main issue here concerns the lack of compactness, which is in turn prevented by a condition such as: $0 < \inf_{\mathrm{R^n}} V < V_\infty := \lim_{\mathrm{x} \to \infty} \inf V(x)$, thanking to the monotonicity property of the ground-state (or mountain-pass) critical levels with respect to the values of the potential V. Due to the strongly indefinite character of the functional I, it is not straightforward to deduce such a monotonicity property for strongly coupled Hamiltonian systems (39). On the other hand, in [99] the latter property was indeed proved to hold provided the Palais-Smale condition holds. He considered this feature by developing the arguments in [99], which consist, roughly speaking, in working with suitable truncated problems for which the analysis of the compactness is a simple matter. He was not able to obtain the full extension of the multiplicity results in [4, 34, 41] to system (39). However, he has shown that to obtain the full extension of the multiplicity results is not a problem in the case where either $\inf\{c(\xi) : \xi \in R^N\}$ is attained at a finite number of points only, or else $H \equiv K \equiv 1$. Finally, he considered the problem of finding solutions concentrating around a prescribed critical point of V, which is not necessarily a minimum point and given a partial answer to the question raised in [100], by extending to system (39) the result in [95]. Here, however, he restricted himself to lower dimensions $(3 \leq N \leq 6)$ and leaved as an open problem to the question of finding multi-peak solutions in the spirit of [95].

Let (M, g) be a smooth connected compact Riemannian manifold of finite dimension $n \geq 2$ with a smooth boundary ∂M with Riemannian metric tensor g. Ghimenti and Micheletti [45] tried to find out the solution $u \in H_g^1(M)$ of the following singularly perturbed nonlinear elliptic problem:

$$-\varepsilon^2 \Delta_g u + u = |u|^{p-2}u, u > 0 \ \ on \ \ M; \ \ \frac{\partial u}{\partial v} = 0 \ \ on \ \ \partial M, \tag{40}$$

for $2 < p < 2^* = \frac{2N}{N-2}$, where ν is the external normal to ∂M. Here $H_g^1(M) = \{u : M \to R : \int_M |\triangle_g u|^2 + u^2 d\mu_g < \infty\}$; μ_g denotes the volume form on M associated with g. In [77, 89], Lin, Ni and Takagi established the least-energy solution of (40) and shown that for ε small enough, the least energy solution has a boundary spike. Later, it was established that for any stable critical point of the mean curvature of the boundary, it is possible to construct single boundary spike layer solutions (for details, see [116]), while in [49, 117], the authors constructed multiple boundary spike solutions. Finally, the authors [37] have been proved that for any integer K, there exists boundary K-peaks solutions. Recently, several efforts have been made to study the effect of the topology of the manifold M on the number of solutions of the equation: $-\varepsilon^2\Delta_g u + u = u^{p-2}u, u > 0$ on M without boundary. In [8], the authors have been shown that this equation has at least $\operatorname{cat}(M)+1$ positive nontrivial solutions for ε small enough, where $\operatorname{cat}(M)$ Lusternik-Schnirelman category of M. [113] considered with the same result with more general nonlinearity. In this research paper, the authors are concerned about the problem (40) on a manifold M with $\partial M \neq \Phi$ and they have shown that for ε small enough, the problem (41) has at least $\operatorname{cat}(\partial M)+1$ non-constant distinct solutions.

Further in the research paper [44], for a given (M,g), Ghimenti and Micheletti considered the following nonlinear elliptic singularly perturbed problem:

$$-\varepsilon^2 \bigtriangleup_{g_{0+h}} u + u = (u^+)^{p-1}, (\varepsilon, h) \in (0, \bar{\varepsilon}) \times \mathcal{B}_\rho, \tag{41}$$

where $\mathcal{B}_\rho$ is a ball centered at 0 with radius ρ in the Banach space of all C^k symmetric covariant 2-tensors on a smooth compact connected Riemannian manifold M of dimension $n \geq 2$ endowed with the metric tensor g_0. Using Poincaré polynomial of M(in [8], authors considered about Poincaré polynomial by assuming that all the solutions of the problem (41) are non-degenerate and proved that the problem (41) have at least $2P_1(M)-1$ solutions), they have obtained an estimate for the number of non-constant solutions with low energy for (ε, h) belonging to an residual subset.

Micheletti and Pistoia [80] considered the following nonlinear elliptic singular perturbation problem for its anti-symmetric sign changing solutions:

$$-\varepsilon^2\Delta_g u + u = u^{p-2}u \ \ in \ \ M, \ \ u \in H_g^1(M), \tag{42}$$

where (M,g) is a smooth compact connected Riemannian manifold without boundary, of dimension $n \geq 2, p > 2$ if $n=2,\ 2<p<\frac{2n}{n-2}$ if $n \geq 3$ and ε is a positive parameter. Here $H_g^1(M)$ is completion of $C^\infty(M)$ with respect to $\|u\|_g^2 = \int_M | \bigtriangleup_g u|^2 d\mu_g + \int_M u^2 d\mu_g$. It is well known that any critical point of the energy functional J_ε constrained to Nehari manifold N_ε is a solution of (42). In [28], authors have shown that the least energy solution of (42), that is the minimum of J_ε on N_ε is a positive solution with spike layer, whose peak converges to the maximum point of the scalar curvature S_g of (M,g) as $\varepsilon \to 0$. Successively, in [51, 113] the authors have been pointed out that the topology of the manifold M influences the multiplicity of positive solutions of (42). Recently, in [79], it has been proved that the existence of positive solutions is strongly related to the geometry of M, that is stable critical points of the scalar curvature S_g, generate positive solutions with one or more peaks as $\varepsilon \to 0$. The first result regarding the existence of sign changing solutions to (42) has been proved by [79], where it has been constructed solutions with one positive peak and one negative peak, which approaches to the minimum and maximum point of S_g as $\varepsilon \to 0$ and if the scalar curvature is not constant. In [46], the author proved that the problem (42) has at least $G_\tau - \operatorname{cat}(M - M_\tau)$ sign changing solutions, which change sign exactly one, where $G_\tau - \operatorname{cat}(M - M_\tau)$ denotes the G_τ-equivalent Lusternik-Schnirelman category for the group $G_\tau = \{1, \tau\}$ and $M_\tau = \{x \in M : \tau x = x\}$. The authors [80] established that under the certain conditions and for ε small enough, all the solutions of the problem (42) are non-degenerate and there are at least $P_1(\frac{M}{G})$ pairs $u_t - u$ of nontrivial solutions to (42), where $P_1(\frac{M}{G})$ is a Poincaré polynomial $P_t(\frac{M}{G})$ when $t = 1$.

Kumar et al. [71] considered an application to nonconvex variational problem Ginzburg-Landau equation. The Dirichlet problem on a rectangle for a nonlinear singularly perturbed elliptic partial differential equation with convection in the case when the domain boundary has no characteristic parts; the highest derivatives in the equation are multiplied by a parameter $\varepsilon \in (0,1]$, takes the form:

$$\varepsilon\Delta u = f(x,u) \tag{43}$$

In finite element schemes several authors [24, 53, 83-85] used standard Newton's method, that is, $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$ to linearize the nonlinear problems, while in [71], authors introduced a new idea of using modified version of Newton's method with higher order convergence. In fact this is a third order convergent method with $x_{n+1} = x_n - \frac{f\left(x_n + \frac{f(x_n)}{f'(x_n)}\right) - f(x_n)}{f'(x_n)}$ to linearize the nonlinear problems. This modified method is in general better than standard Newton method and cheaper than existing third order methods used by several authors in [43, 92, 118]. This method requires the evaluation of second derivatives and is very interesting from the practical point of view. The modified Newton's method of third order convergence has not been previously used in finite element method. Numerical results using MATLAB are also provided to demonstrate the usefulness of the method with the comparison. Using this approach cost of computation is reduced in comparison to [22] and requires less number of iterations with lower order of continuity of $u(x,y)$ in comparison to [83], it requires only third order continuous functions in place of $u \in C^6(\Omega)$. In [1], Alberty et al. used the Matlab code only for particular values of discretization but in this paper they have generalized the Matlab code for any value of discretization.

3 Nonlinear elliptic equations arising in science and engineering

It is well known that many phenomena in science and engineering can be described by boundary value problems associated with various types of partial differential equations or systems. When we associate a mathematical model with phenomena, we generally try to capture what is essential, retaining the important quantities and omitting the negligible ones which involve small parameters. The equation that would be obtained by maintaining the small parameters is called the singularly perturbed equation. It is astonishing that in many excellent books and research papers, the nonlinear phenomenon in elliptic problems is either omitted or only treated with one or two examples. Although, the present paper is concentrated over the review of nonlinear elliptic singular perturbation problems, but to provide a comprehensive overview, the present section is divided into two parts. The first part of this section contains those nonlinear elliptic equations, which are not singularly perturbed in nature. In the second part of this section, we give some nonlinear elliptic singular perturbation equations arising in various field of applied mathematics as fluid mechanics, quantum mechanics, reaction-diffusion processes, mathematical biology, and population genetics. We do not go into details, but only give the corresponding equations and the meaning of the occurring functions and usually omit the techniques of solving these equations but for more details, readers can approach to the references cited along these equations.

(a) Nonlinear elliptic equations without singular perturbation

(1) ***Chladny sound figures***: Let us consider uniformly distributed tiny particles, for example, Sand, on a thin flexible square plate, horizontally fixed at its centre. Above the plate, there is fix a source of sound waves with variable frequency related with λ in the following model of Chladny sound figures [23]. The pressure of the sound waves acts as a load on the surface of the plate. In the case of resonance, a vibration with (unmoved) nodal lines is induced on the plate and the tiny particles are collected. Thus, the distribution of these particles with respect to varying λ illustrates the vibration of the plate. Let $\Omega := [0, L] \times [0, L]$ represent the plate and $u = u(x, y)$ denote the maximal deviation of the vibration from the trivial flat position of the plate at the point (x, y). Then $u(x, y)$ satisfies the strongly simplified mathematical model equation:

$$G(u, \lambda) := \Delta u + \lambda \sin u = 0 \ \ in \ \ \Omega = [0, L] \times [0, L] \ \ on \ \ C_b^2(\Omega) := \{u \in C^2(\Omega) : \frac{\partial u}{\partial n} = 0 \ \ on \ \ \partial\Omega\}. \qquad (44)$$

Here Δ is the Laplacian operator and $\frac{\partial u}{\partial n}$ the normal derivative in the outer direction of $\partial\Omega$. This Chladny problem (44) is a typical example of multiple bifurcations in nonlinear problems with symmetries.

(2) ***Buckling of a plate:*** Consider a plate P under compression of the form $\Omega \subset R^2$. It is defined by the Airy stress function $w(x, y)$ of the plate P at the point (x, y) and the deviation or deflection $u(x, y)$ for P from its flat trivial state. They are modelled by the following von Kármán equations given by the equation (45). A mathematical justification for this model is given by Berger and Five [10]; see also Ciarlet [33]:

$$Gs(u, w) := \begin{pmatrix} \Delta^2 u & -[u, w] \\ \Delta^2 w & \frac{1}{2}[u, w] \end{pmatrix} = \begin{pmatrix} f \\ 0 \end{pmatrix} \ \ in \ \ \Omega \subset R^2. \qquad (45)$$

Here λ is the external load, $\Delta^2 = \Delta\Delta$ is the two-dimensional biharmonic operator, Ω is the shape of the plate, and the bracket operator [., .] is defined by $[u, v] = u_{xx}v_{yy} - 2u_{xy}v_{xy} + u_{yy}v_{xx}$.

(3) ***The Monge-Ampére equation:*** For $n = 2$, it has the form [cf. 48]:

$$0 = G(u) := \det D^2 u - f(x, u, Du) = u_{xx}u_{yy} - u_{xy}^2 - f(x, u, Du). \qquad (46)$$

Here G is uniformly elliptic in $u_1 \in C^2(\Omega)$ if $D^2 u_1$ is strictly positive definite and symmetric in Ω. If $K = K(x)$ at $x \in \Omega$ be the prescribed Gauss curvature, then (46) has the special form:

$$G(u) := \det D^2 u - K\left(1 + |Du|^2\right)^{\frac{n+2}{2}} = 0, \ K(x) > 0. \qquad (47)$$

(b) Nonlinear elliptic equations with singular perturbation

(1) To study nerve impulses, the following nonlinear elliptic singular perturbation equation is considered in [42, 88]:

$$-\varepsilon^2 \Delta u + f(u) + \varepsilon\gamma \mathrm{v} = 0 \ \ in \ \ \Omega; \ \ -\Delta v + \mathrm{v} - u = 0 \ \ in \ \ \Omega; \ \ \partial_v u = \partial_v \mathrm{v} \ \ on \ \ \partial\Omega, \qquad (48)$$

on smooth bounded domain Ω. The perturbation parameter ε is positive and small. The outward normal derivatives of u and v on the boundary of Ω are given by $\partial_v u$ and $\partial_v \mathrm{v}$, respectively. The nonlinearity in the system (48) is of the FitzHugh-Nagumo type. The modelled phenomenon is the control of the electrical potential across cell membrane. This control is done by the change of flow of the ionic channels of the cell membrane. This results in the change in potential which is used to send electrical signals between cells. This is readily observed in muscle and other excitable cells. The two variables in the system are the excitable variable u and the recovery variable v. The dynamics of two variables are described by the reaction-diffusion system: $u_t = \varepsilon^2 \Delta u - f(u) - \varepsilon\gamma \mathrm{v}$; $k\mathrm{v_t} = \Delta v - \mathrm{v} + u$; $\partial_v u = \partial_v \mathrm{v}$ on $\partial\Omega$.

(2) Semilinear systems of reaction–diffusion equations are typical models in nuclear reactor physics, and in chemical and biological reactions. In particular, many chemical reactions, ecological systems and mechanisms of pattern formation are described mathematically as systems of semilinear differential equations. The following semilinear singularly perturbed problems which correspond to the reaction-diffusion and the convection-diffusion problems of the elliptic type, have been considered by Ladyženskaja and Ural'ceva [72]:

$$Lu = -f(x, y, u),$$

$$Lu = L_\mu u \equiv -\mu^2 (u_{xx} + u_{yy}), \text{ or } Lu = L_\varepsilon u \equiv -\varepsilon (u_{xx} + u_{yy}) + b_1 u_x + b_2 u_y,$$

$$(x, y) \in w, w = w^x \times w^y = \{0 < x < 1\} \times \{0 < y < 1\},$$

$$f_u \geq c_* > 0, (x, y, u) \in \bar{w} \times (-\infty, \infty), f_u \equiv \frac{\partial f}{\partial u},$$

$$b_1 \geq \beta_1 \geq 0, b_2 \geq \beta_2 \geq 0 \;\; on \;\; \bar{w}, u = g \;\; on \;\; \partial w, \tag{49}$$

Where μ and ε are small positive parameters, c_* and $\beta_{1,2}$ are constants. If f, g and $b_{1,2}$ are sufficiently smooth, then under suitable continuity and compatibility conditions on the data, a unique solution u of (49) exists (see [72] for details). For $\mu \ll 1$, the reaction-diffusion problem (49) with $L = L_\mu$ is singularly perturbed and characterized by the boundary layers of width $O(\mu|\ln\mu|)$ near ∂w (see [15] for details). For $\varepsilon \ll 1$, the convection-diffusion problem (49) with $L = L_\varepsilon$ is singularly perturbed and characterized by the regular boundary layers of width $O(\varepsilon|ln\varepsilon|)$ at $x = 1$ and $y = 1$(see [81] for details).

(3) In the discretization of the non-stationary Navier-Stokes systems and the non-stationary oscillation model, the following nonlinear elliptic singular perturbation problem is often arises [78, 109]:

$$-\varepsilon^2 \Delta^2 u - \Delta u = f \;\; in \;\; \Omega; \;\; u = \frac{\partial^2 u}{\partial n^2} = 0 \;\; on \;\; \partial\Omega, \tag{50}$$

where $f \in L^2(\Omega), \Omega \subset R^2$ is a bouneded rectangular domain, Δ is the Laplace operator, $\partial\Omega$ is the boundary of Ω and $\partial/\partial n$ denotes the outer normal derivative on $\partial\Omega$.

(4) Ko [66] studied the existence of multiple nontrivial solutions and interior transition layers for a class of semilinear elliptic singular perturbation problems with homogeneous Neumann boundary conditions in a regular bounded domain of R^n :

$$\varepsilon\Delta u + f(x, u) = 0 \;\; in \;\; \Omega; \;\; \partial u/\partial n = 0 \;\; on \;\; \partial\Omega, \tag{51}$$

where $\partial u/\partial n$ denotes the outward normal derivative of u on $\partial\Omega$. By considering the case $f(x, u) = g(x)h(u)$, it can be interpreted as a model in population genetics with two alleles A_1 and A_2 corresponding to three possible genotypes A_1A_1, A_1A_2 and A_2A_2. The population lives in the region Ω in R^n and $u(x, t)$ denotes the frequency of allele A_1 at time t at the point x in Ω. Changes in gene frequency are assumed to be caused only by the flow of genes within Ω and selective advantage for certain genotypes in certain sub-regions of Ω.

(5) One of the most stimulating equations for research in mathematics and physics is the Navier-Stokes equation. It models, for example, the flow of water around a moving vessel, and the air around the wing of an airplane. The following Navier-Stokes equations are considered by Dorfler [40] with appropriate boundary conditions:

$$-\varepsilon\Delta v + v\,.\nabla v + \nabla p = f, \nabla.u = 0, \tag{52}$$

where $v : \Omega \subset R^3 \to R^3$ and $p : \Omega \to R$ are maps. Maps v and p, respectively, are velocity and pressure of a fluid in a stationary motion enclosed in a container Ω and driven by a given volume forces f.

(6) Semilinear elliptic problems of the following form with a small parameter affecting the highest derivative is a constituent of the drift diffusion model (as Poisson's equation for the electrostatic potential), describes the behaviour of the semiconductor devices and is extensively used in semiconductor modelling [104]:

$$\mathcal{L}u(P) = f(P, u), P = (x, y) \in \Omega; u(P) = U(P), P \in \partial\Omega, \tag{53}$$

posed on a bounded domain $\Omega \subset R^2$, where $L \equiv \mu^2(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2})$ with μ being a positive parameter, and $\partial\Omega$ denotes the boundary of Ω. The functions $f(P,u)$ and $U(P)$ are supposed to be sufficiently smooth with f satisfying $f_u(P,u) \geq \beta^2$, $\beta = \text{constant} > 0$, $(P,u) \in \bar{\Omega} \times [-C_1, C_1]$; $f_{uu}(P,u) \leq \gamma = \text{constant}$, $(P,u) \in \bar{\Omega} \times [-C_2, C_2]$, where C_1 and C_2 are sufficiently large positive numbers. If the problem (53) is considered with the following input data:

$$\Omega = (-1,1) \times (-1,1), \{(x,y) : x^2 + y^2 \leq R^2, R = 0.4\},$$

$$g(x,y) = 0, x^2 + y^2 = R^2, -R \leq x \leq R, -R \leq y \leq R,$$

$$g(x,-1) = g(x,1) = 1, -1 \leq x \leq 1; g(-1,y) = g(1,y) = 1, -1 \leq y \leq 1,$$

$$f(P,u) = \exp(-1) - \exp(-U(P)), \tag{54}$$

then problem (53) and (54) can be considered as a model problem, which describes the distribution of the electrostatic potential in a two dimensional section of reverse biased semiconductor diode with cylindrical Schottky contact (for more details, see [21, 111]).

(7) A basic model is the following system, due to Gierer and Meinhardt [47], which models the densities of a chemical activator $\mathcal{U}$ and an inhibitor $\mathcal{V}$ and is used to describe experiments of regeneration of *hydra:*
$\mathcal{U}_t = d_1\Delta\mathcal{U} - \mathcal{U} + \frac{\mathcal{U}^p}{\mathcal{V}^q}$ in $\Omega \times (0,+\infty)$; $\mathcal{V}_t = d_2\Delta\mathcal{V} - \mathcal{V} + \frac{\mathcal{U}^r}{\mathcal{V}^s}$ in $\Omega \times (0,+\infty)$,

$$\frac{\partial\mathcal{U}}{\partial\vartheta} = \frac{\partial\mathcal{V}}{\partial\vartheta} \ \ on \ \ \partial\Omega \times (0,+\infty), \tag{55}$$

where $d_1, d_2, p, q, r, s > 0$, with the constraints $0 < \frac{p-1}{q} < \frac{r}{s+1}$.

(8) Ambrosetti and Malchiodi [3] considered the following problem arising in the study of reaction-diffusion systems with chemical of biological motivation:

$$-\varepsilon^2\Delta u + u = u^p, u > 0 \ \ in \ \ \Omega; \ \ \vartheta \ \frac{\partial u}{\partial\vartheta} = 0 \ \ on \ \ \partial\Omega, \tag{56}$$

where Ω is a bounded domain in R^n with smooth boundary $\partial\Omega$, ϑ denotes the unit outer normal at $\partial\Omega$ and $p \in (1, \frac{n+2}{n-2})$.

(9) The standing waves of the Nonlinear Schrodinger Equation is considered by Ambrosetti et al. [4]:

$$-\varepsilon^2\Delta u + V(x)u = u^p, \ \ u > 0 \ \ in \ \ R^n; \ \ u > 0, u \in W^{1,2}(R^n), \tag{57}$$

where $p > 1$ is subcritical and V is a smooth bounded potential.

4 Concluding remarks and further scope for research

A wide variety of important problems in science and engineering has been formulated in terms of nonlinear elliptic singularly perturbed partial differential equations, which model nonlinear waves, arise in gas dynamics, water waves, chemical reactions, transport of pollutants, flood waves in rivers, chromatography, traffic flow and a wide range of biological and ecological systems etc. The present paper includes asymptotic and numerical analysis for solving singularly perturbed problems in nonlinear (quasilinear, semilinear and superlinear etc.) elliptic partial differential equations and analyze a huge amount of literature related to these problems of different orders together with their final results in some of the research papers just to give a comprehensive insight about the quantitative behaviour of the solutions in that paper. The most of the references used in this paper are of great practical importance and from this survey, it can be observed that we have particularly motivated the researchers for solving nonlinear elliptic partial differential equations, which have hardly been examined so far in the literature dedicated to singular perturbations. There are several computational methods that have been proposed by several authors for analyzing singularly perturbed problems. The conventional discretization technique applied to singularly perturbed problems leads to system of linear or nonlinear equations containing a large number of unknowns, which is then solved using iterative methods. This technique may useless if the perturbation parameter is close to some critical value or the care is not taken in analyzing the dependence of this parameter on constants. Numerical grid generation is an important technique for handling nonlinear partial differential equations, where the availability of sharp error estimates ensures that the adaptive algorithm is efficient due to not producing a grid that is over refined for a specified error tolerance. Fitted mesh methods are also useful for tackling for many classes of singularly perturbed problems but it can be computationally expansive to implement, especially for nonlinear problems and are not easily extendable to multi-dimensions. Standard discretization on a nonuniform grid method can be used, but it requires significant a priori information about the presence, location, height and width of the layers (corner, interior, shock, etc.).

The finite element method now has become a useful tool in every conceivable area of science and engineering that can make use of models of nature characterized by partial differential equations. This is proposed in our future research work to develop some simple and efficient finite element approximations for solving nonlinear elliptic singular perturbation problems and computer implementation, which are easy to implement and are not costly in terms of computer time also.

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References

[1] J. Alberty, C. Carstensen and S. A. Funken. Remarks around 50 lines of Matlab: short finite element implementation. *Numerical Algorithms*, 20(1999): 117-137.

[2] A. Ambrosetti and P. H. Rabinowitz. Dual variational methods in critical point theory and applications. *Journal of Funcional Analysis*, 14(1973): 349-381.

[3] A. Ambrosetti and A. Malchiodi. Perturbation methods and semilinear elliptic problems. Progresses in Mathematics, Volume 240. 2006.

[4] A. Ambrosetti, A. Malchiodi and S. Secchi. Multiplicity results for some nonlinear Schrdinger equations with potentials. *Archive for Rational Mechanics and Analysis*, 159(2001): 253-271.

[5] A. Bahari, Y. Li and O. Rey, On the variational problem with lack of compactness: the topological effects of the critical points at infinity. *Calculus of Variation and Partial Differential Equations*, 3(1995): 67-93.

[6] V. Benci and G. Cerami. The effect of the domain topology on the number of positive solutions of nonlinear elliptic problems. *Archive for Rational Mechanics and Analysis*, 114(1991): 79-93.

[7] V. Benci and G. Cerami. Multiple positive solutions of some elliptic problems via the Morse theory and the domain topology. *Calculus of Variations and Partial Differential Equations*, 2(1994): 29-48.

[8] V. Benci, C. Bonanno and A. M. Micheletti. On the multiplicity of solution of a nonlinear elliptic problem on Riemannian manifolds. *Journal of Functional Analysis*, 252(2)(2007): 464-489.

[9] H. Berestycki and P.-L. Lions, Nonlinear scalar field equations: existence of ground states. *Archive for Rational Mechanics and Analysis*, 82(1983): 313-345.

[10] M. S. Berger and P. C. Five. On Von Karman's equations and the buckling of a thin elastic plate II. Plate with general edge conditions. *Communication in Pure and Applied Mathematics*, 21(1968): 227-241.

[11] I. A. Blatov. On the Galerkin finite-element method for elliptic quasilinear singularly perturbed boundary value problems I. *Differential Equations*, 28(7)(1992): 931-940.

[12] I. A. Blatov. On the Galerkin finite-element method for elliptic quasilinear singularly perturbed boundary value problems II. *Differential Equations*, 28(10)(1992): 1469-1477.

[13] I. A. Blatov. On the Galerkin finite-element method for elliptic quasilinear singularly perturbed boundary value problems III. Problems with angular boundary layers. *Differential Equations*, 30(3)(1994): 432-444.

[14] I. P. Boglaev. A numerical methods for a quasi-linear elliptic equations with a small parameter for the highest derivative. *Computational Mathematics and Mathematical Physics*, 28(1988): 118-125.

[15] I. P. Boglaev. A numerical methods for a quasi-linear singular perturbation problem of elliptic type. *USSR Computational Mathematics and Mathematical Physics*, 28(1988): 492-502.

[16] I. Boglaev. On monotone iterative methods for a nonlinear singularly perturbed reaction-diffusion problem. *Journal of Computational and Applied Mathematics*, 162(2004): 445-466.

[17] I. Boglaev. Uniform convergence of monotone iterative methods for semilinear singularly perturbed problems of elliptic and parabolic types. *Electronic Transactions on Numerical Analysis*, 20(2005): 86-103.

[18] I. Boglaev. Robust monotone iterates for nonlinear singularly perturbed boundary value problems. *Boundary Value Problems*, 2009(2009): 1-17.

[19] I. P. Boglaev and V. V. Sirotkin. Domain decomposition technique for singularly perturbed problems and its parallel implementation. In Proc. 13th IMACS Congress on Computation and Applied Mathematics (Edited by J. J. H. Miller and R. Vichnevetsky), Dublin, 1991: 522-523.

[20] I. Boglaev and M. P. Hardy, Monotone finite difference domain decomposition algorithms and applications to nonlinear singularly perturbed reaction-diffusion problems. *Advances in Difference Equations*, 2006(1)(2006): 1-36.

[21] I. P. Boglaev and V. V. Sirotkin, Domain decomposition technique for semiconductor device simulation. In 8th International Conference of Numerical Analysis of Semiconductor Devices and Integrated Circuits (Edited by J. J. H. Miller), Boole Press, Dublin, 1992: 154-155.

[22] K. Böhmer. On finite element methods for fully nonlinear elliptic equations of second order. *SIAM Journal of Numerical Analysis*, 46(3)(2008): 1212-1249.

[23] K. Böhmer. Numerical methods for bifurcation and centre manifolds in nonlinear elliptic and parabolic differential equations. Oxford University Press, Oxford. 2011.

[24] R. F. Boisvert. Families of high order accurate discretization of some elliptic problems. *SIAM Journal of Scientific and Statistical Computing*, 2(1981): 268-285.

[25] F. A. Bornemann. An adaptive multilevel approach to parabolic equations III. 2D error estimation and multilevel preconditioning. *IMPACT of Computing in Science and Engineering*, 4(1)(1992): 1-45.

[26] J. Byeon. On the position of spike layers for mountain pass solutions of singularly perturbed nonlinear Dirichlet problems. *Trends in Mathematics* at: Information Centre for Mathematical Sciences, 8(2)(2005): 33-36.

[27] J. Byeon. Singularly perturbed nonlinear Neumann problems with a general nonlinearity. *Journal of Differential Equations*, 244(2008): 2473-2497.

[28] J. Byeon and J. Perk. Singularly perturbed nonlinear elliptic problems on manifolds. *Calculus of Variations and Partial Differential Equations*, 24(4)(2005): 459-477.

[29] J. Byeon and L. Jeanjean. Standing waves for nonlinear Schr dinger equations with a general nonlinearity. *Archive for Rational Mechanics and Analysis*, 185(2007): 185-200.

[30] D. Cao and E. S. Noussair. Multi-peak solutions for a singularly perturbed semilinear elliptic problem. *Journal of Differential Equations*, 166(2000): 266-289.

[31] D. Cao, N. Dancer, E. Noussair and S. Yan. On the existence and profile of multi-peaked solutions to singularly perturbed semilinear Dirichlet problem. *Discrete and Continuous Dynamical Systems*, 2(2)(1996): 221-236.

[32] M. M. Chawla and P. N. Shivakumar. An efficient finite difference method for two-point boundary value problems. *Neural, Parallel and Scientific Computing*, 4(1996): 387-396.

[33] P. -G. Ciarlet. A justification of the Von Karman equations. *Archive for Rational Mechanics and Analysis*, 73(1980): 349-389.

[34] S. Cingolani and M. Lazzo, Multiple semiclassical standing waves for a class of nonlinear Schrdinger equations. *Topological Methods Nonlinear Analysis*, 10(1997): 1-13.

[35] E. N. Dancer and J. Wei. On the profile of solutions with two sharp layers to a singularly perturbed semilinear Dirichlet problem. *Proceedings of Royal Society of Edinburgh Sect. A*, 127(4)(1997): 691-701.

[36] E. N. Dancer and J. Wei. On the effect of domain topology in a singular perturbation problem. *Topological Methods in Nonlinear Analysis*, 11(1998): 227-248.

[37] E. Dancer and S. Yan. Multipeak solutions for a singularly perturbed Neumann problem. *Pacific Journal of Mathematics*, 189(2)(1999): 241-262.

[38] E. N. Dancer, A. M. Micheletti and A. Pistoia. Multipeak solutions for some singularly perturbed nonlinear elliptic problems on Riemannian manifolds. *Manuscripta Mathematica*, 128(2009): 163-193.

[39] A. J. DeSANTI. Singularly perturbed quadratically nonlinear Dirichlet problems. *Transactions of the American Mathematical Society*, 298(2)(1986): 733-746.

[40] W. Dorfler. Uniform a priori estimates for singularly perturbed elliptic equations in multi-dimensions. *SIAM Journal of Numerical Analysis*, 36(6)(1999): 1878-1900.

[41] G. M. Figueiredo. Multiplicity of solutions for a class of elliptic systems. *Electronic Journal of Differential Equations*, (76)(2006): 1-12.

[42] R. A. FitzHugh. Impulses and physiological states in theoretical models of nerve membrane. *Biophysical Journal*, 1(1961): 445-466.

[43] M. Frontini and E. Sormani. Modified Newton's method with third order convergence. *Applied Mathematics and Computation*, 140(2003): 419-426.

[44] M. Ghimenti and A. M. Micheletti. Positive solutions of singularly perturbed nonlinear elliptic problem on manifolds via Morse theory. *arXiv:1012.5672v1 [math.AP]*: 2010.

[45] M. Ghimenti and A. M. Micheletti. Positive solutions of singularly perturbed nonlinear elliptic problem on Riemannian manifolds with boundary. *arXiv:1012.5668v1 [math.AP]*: 2010.

[46] M. Ghimenti and A. M. Micheletti. On the number of nodal solutions for a nonlinear elliptic problems on symmetric

Riemannian manifolds. *Electronic Journal of Differential Equations*, 18(2010): 15-22.
[47] A. Gierer and H. Meinhardt. A theory of biological pattern formation. *Kybernetika*, 12(1972): 30-39.
[48] D. Gilbarg and N. S. Trudinger. Elliptic partial differential equations of second order. Springer, Berlin. 2001.
[49] C. Gui. Multipeak solutions for a semilinear Neumann problem. *Duke Mathematical Journal*, 84(3)(1996): 739-769.
[50] M. P. Hardy and I Boglaev. Parallel implementation of a monotone decomposition algorithm or nonlinear reaction-diffusion problem. *Australian and New Zealand Industrial and Applied Mathematics Journal*, 46(E)(2005): C290-C303.
[51] N. Hirano. Multiple existence of solutions for a nonlinear elliptic problem on a Riemannian manifold. *Nonlinear Analysis: Theory, Methods and Applications*, 70(2)(2009): 671-692.
[52] K. Ishii and N. Yamada. On the rate of convergence of solutions for the singular perturbations of gradient obstacle problems. *Funkcialaj Ekvacioj*, 33(1990): 551-562.
[53] M. K. Jain, R. K. Jain and R. K. Mohanty. Fourth order difference methods for the system of 2-D non-linear elliptic partial differential equations. *Numerical Methods for Partial Differential Equations*, 7(1991): 227-244.
[54] L. Jeanjean and K. Tanaka. A remark on least energy solutions. *Proceedings of American Mathematical Society*, 131(2003): 2399-2408.
[55] Mo Jiaqi. Singular perturbation for a class of nonlinear reaction diffusion systems. *Science China Mathematics*, 32(11)(1989): 1306-1315.
[56] Mo Jiaqi. A singularly perturbed nonlinear boundary value problem. *Journal of Mathematical Analysis and Applications*, 178(1)(1993): 289-293.
[57] Mo Jiaqi. The singularly perturbed boundary value problems for semilinear elliptic equations of higher order. *Approximation Theory and its Application*, 16(3)(2000): 100-105.
[58] Mo Jiaqi. Singular perturbation for the weakly nonlinear reaction-diffusion equation with boundary perturbation. *Applied Mathematics and Mechanics* (Eng. Ed.), 29(8)(2008): 1105-1110.
[59] Mo Jiaqi and Shao S. The singularly perturbed boundary value problems for higher-order semilinear elliptic equations. *Advances in Mathematics*, 30(2)(2001): 141-148.
[60] Mo Jiaqi and Lin Wantao. A nonlinear singularly perturbed problem for reaction diffusion equations with boundary perturbation. *Acta Mathematicae Applicatae Sinica*, 21(1)(2005): 101-104.
[61] Mo Jiaqi, Zhu Jiang and Wang Hui. Asymptotic behavior of the shock solution for a class of nonlinear equations. *Progresses in Natural Sciences*, 13(9)(2003): 768-770.
[62] Mo Jiaqi, Wang Hui and Lin Wantao et al. Variational iteration method for mechanism of the equatorial eastern Pacific El Nino-Southern Oscillation. *China Physics*, 15(4)(2006): 671-675.
[63] M. K. Kadalbajoo and K. C. Patidar. A survey of numerical techniques for solving singularly perturbed ordinary differential equations. *Applied Mathematics and Computation*, 130(2002): 457-510.
[64] M. K. Kadalbajoo and K. C. Patidar. Singularly perturbed problems in partial differential equations: a survey. *Applied Mathematics and Computation*, 134(2003): 371-429.
[65] W. Kelley and B. Ko. Semilinear elliptic singular perturbation problems with nonuniform behaviour. *Journal of Differential Equations*, 86(1)(1990): 88-101.
[66] B. Ko. Multiple solutions of a class of singular perturbation models with Neumann type. *Journal of Korean Mathematical Society*, 31(1)(1994): 11-28.
[67] M. Kumar and P. Kumar. A finite element approach for finding positive solutions of semilinear elliptic Dirichlet problems. *Numerical Methods for Partial Differential Equations*, 2008(2008): 1-10.
[68] M. Kumar, P. Singh and P. Kumar. A survey on various computational techniques for nonlinear elliptic boundary value problems. *Advances in Engineering Software*, 39(2008): 725-736.
[69] M. Kumar and P. Kumar. Computational method for finding various solutions for a quasilinear elliptic equation of Kirchhoff type. *Advances in Engineering Software*, 40(2009): 1104-1111.
[70] M. Kumar and P. Kumar. Simulation of a nonlinear Steklov eigen value problem using finite element approximation. *Computational Mathematics and Modelling*, 21(1)(2010): 109-116.
[71] M. Kumar and P. Kumar. Finite element approximation for singularly perturbed nonlinear elliptic boundary value problems. *Computational Mathematics and Modelling*, 23(1)(2011): 88-95.
[72] O. A. Ladyženskaja and N. N. Ural'ceva. Linear and quasi-linear elliptic equations. Academic Press, New York. 1968.
[73] J. Li. Quasi-optimal uniformly convergent finite element method for singularly perturbed elliptic boundary value problems. *Computers and Mathematics with Applications*, 33(10)(1997): 11-22.
[74] J. Li and I. M. Navon. Uniformly convergent finite element method for singularly perturbed elliptic boundary value

problems I: Reaction-diffusion type. *Computers and Mathematics with Applications*, 35(3)(1998): 57-70.

[75] J. Li and I. M. Navon. A global uniformly convergent finite element method for a quasi-linear singularly perturbed elliptic problem. *Computers and Mathematics with Applications*, 38(1999): 197-206.

[76] Y. Li and L. Nirenberg, The Dirichlet problem for singularly perturbed elliptic equations. *Communication in Pure and Applied Mathematics*, 51(11-12)(1998): 1445-1490.

[77] C. S. Lin, W. M. Ni and I. Takagi. Large amplitude stationary solutions to a chemotaxis system. *Journal of Differential Equations*, 72(1988): 1-27.

[78] G. Q. Liu and Y. C. Su. A uniformly convergent difference scheme for the singular perturbation problem of a higher order elliptic differential equation. *Applied Mathematics and Mechanics*, 17(1996): 413-421.

[79] A. M. Micheletti and A. Pistoia. Nodal solutions for a singularly perturbed nonlinear elliptic problems on Riemannian manifolds. *Advanced Nonlinear Studies*, 9(3)(2009): 565-577.

[80] A. M. Micheletti and A. Pistoia. On the existence of nodal solutions for a nonlinear elliptic problem on symmetric Riemannian manifolds. *International Journal of Differential Equations*, 2010(2010): 1-11.

[81] J. J. H. Miller, E. O'Riordan and G. I. Shiskin. Fitted numerical methods for singular perturbation problems. World Scientific, Singapore. 1996.

[82] J. J. H. Miller, E. O'Riordan and G. I. Shishkin. Fitted numerical methods for singular perturbation problems: Error estimates in the maximum norm for linear problems in one and two dimensions. World Scientific, Singapore. 1996.

[83] R. K. Mohanty. Order difference methods for a class of singular two-space dimensional elliptic boundary value problems. *Journal of Computational and Applied Mathematics*, 81(1997): 229-247.

[84] R. K. Mohanty and S. Dey. A new finite difference discretization of order four for for two-dimensional quasi-linear elliptic boundary value problems. *International Journal of Computer Mathematics*, 76(2001): 505-576.

[85] R. K. Mohanty and S. Singh. A new fourth order discretization for singularly perturbed two dimensional non-linear elliptic boundary value problems. *Applied Mathematics and Computation*, 175(2006): 1400-1414.

[86] R. K. Mohanty and S. Singh. A new highly accurate discretization for three dimensional singularly perturbed nonlinear elliptic partial differential equations. *Numerical Methods for Partial Differential Equations*, 22(6)(2006): 1379-1395.

[87] R. Molle. Semilinear elliptic problems in unbounded domains with unbounded boundary. *Asymptotic Analysis*, 38(3-4)(2004): 293-307.

[88] J. Nagumo, S. Arimoto and Y. Yoshikawa. An active pulse transmission line simulating nerve axon. *Proceedings of the Institute of Radio Engineers*, 50(10)(1962): 2061-2070.

[89] W. M. Ni and I. Takagi. On the shape of least energy solutions to a semilinear Neumann problem. *Communications in Pure and Applied Mathematics*, 44(1991): 819-851.

[90] W.-M. Ni and J. Wei. On the location and profile of spike-layer solutions to a singularly perturbed semilinear Dirichlet problems. *Communication in Pure and Applied Mathematics*, 48(7)(1995): 731-768.

[91] I. M. Ortega and W. C. Rheinboldt. Iterative solutions of nonlinear equations in several variables. Academic Press, New York, NY, U.S.A. 1970.

[92] A. Y. Ozban. Some new variants of Newton's method. *Applied Mathematics Letters*, 17(2004): 677-682.

[93] C. Pao. Monotone iterative methods for finite difference system of reaction-diffusion equations. *Numeriche Mathematik*, 46(4)(1985): 571-586.

[94] M. del Pino and P. L. Felmer. Spike- layered solutions of singularly perturbed elliptic problems in a degenerate setting. *Indiana University Mathematics Journal*, 48(1999): 63-79.

[95] M. Del Pino and P. Felmer, Semi-classical states of nonlinear Schrdinger equations: A variational reduction method. *Mathematische Annalen*, 324(2002): 1-32.

[96] M Del Pino, P. Felmer and J. Wei. Multiple peak solutions for singular perturbation problems. *Calculus of variations and Partial Differential Equations*, 10(2000): 119-134.

[97] L. Prandtl. Uber Flussigkeits-bewegung bei kleiner Reibung in: Verhandlungen III. Int. Math. Kongresses. Tuebner, Leipzig. (1905): 484-491.

[98] M. Ramos. On singular perturbation of superlinear elliptic systems. *Journal of Mathematical Analysis and Applications*, 352(2009): 246-258.

[99] M. Ramos and S. Soares. On the concentration of solutions of singularly perturbed Hamiltonian systems. *Portugaliae Mathematica (N.S.)*, 63(2006): 157-171.

[100] M. Ramos and H. Tavares. Solutions with multiple spike patterns for an elliptic system. *Calculus of Variations and Partial Differential Equations*, 31(2008): 1-25.

[101] H. G. Roos, M. Stynes and L. Tobiska. Numerical Methods for Singularly Perturbed Differential Equations.

Springer, Berlin. 1996.

[102] A. A. Samarskil. Theory of difference schemes. Marcel Dekker, New York, 2001.

[103] H. A. Schwarz. Uber einige abbildungsaufgaben. *Journal fur die Reine and Angewandte Mathematik*, 70(1869): 105-120.

[104] S. Selberherr. Analysis and simulation of semiconductor devices. Springer-Verlag, Wien. 1984.

[105] G. I. Shiskin. Grid approximation of boundary value problems for singularly perturbed quasilinear elliptic equations in the presence of transitional layer. *Mathematical Modelling*, 7(2)(1995): 72-88.

[106] G. I. Shiskin and I. V. Tselishcheva. Parellel methods for solving singularly perturbed boundary value problems for elliptic equations. *Matematicheskoe Modelirovanic*, 8(3)(1996): 111-127.

[107] V. V. Sirotkin. The solution of singularly perturbed semilinear elliptic problems by iterative domain decomposition algorithms. *Computers and Mathematics with Applications*, 33(11)(1997): 99-116.

[108] V. Sirotkin. Two level parallel Schwarz methods for singularly perturbed semilinear elliptic problems. *Computers and Mathematics with Applications*, 40(2000): 645-647.

[109] Y. C. Su and G. Q. Liu. Asymptotic solution of singular perturbation problems for the fourth order elliptic differential equations. *Applied Mathematics and Mechanics*, 11(1990): 597-6109. (In Chinese)

[110] Y. Sun and S. Wu. Iterative solutions for a singular nonlinear elliptic problem. *Applied Mathematics and computation*, 118(2001): 53-62.

[111] S. Sze. Physics of semiconductor devices. Wiley, New York. 1969.

[112] I. V. Tselishcheva and G. I. Shishkin. Monotone domain decomposition schemes for a singularly perturbed semilinear elliptic reaction-diffusion equation with Robin boundary condition. Mathematical Modelling and Analysis at: Proceedings of the 10th International Conference, MMA2005, CMAM2, Trakai, (2005): 271-276.

[113] D. Visetti. Multiplicity of solutions of a zero mass nonlinear equation on a Riemannian manifold. *Journal of Differential Equations*, 245(9)(2008): 2397-2439.

[114] J. Wei. On the construction of single-peaked solutions to a singularly perturbed semilinear Dirichlet problems. *Journal of Differential Equations*, 129(2)(1996): 315-333.

[115] J. Wei. On the effect of domain geometry and boundary geometry in some singular perturbation problems. in: Differential Equations and Applications, Hangzhou, (1996), Internat. Press, Cambridge, MA, (1997): 326-339.

[116] J. Wei. On the boundary spike layer solutions to a singularly perturbed Neumann problem. *Journal of Differential Equations*, 134(1)(1997); 104-133.

[117] J. Wei and M. Winter. Multi-peak solutions for a wide class of singular perturbation problems. *Journal of London Mathematical Society*, 59(2)(1999): 585-606.

[118] S. Weerakoon and T. G. I. Fernando. A variant of Newton's method with accelerated third order convergence. *Applied Mathematics Letters*, 13(2000): 87-93.

[119] C. Yusen. The singular perturbation of nonlocal boundary value problems for nonlinear elliptic equation of fourth order. *Journal of Systems Sciences and Mathematical Sciences*, 13(1)(2000): 92-96.