**ISSN** 1749-3889 (print), 1749-3897 (online)
**International Journal of Nonlinear Science**
Vol.29(2020) No.1,pp.12-25

# Existence of Multiple Normalized Solutions for the Coupled Nonlocal Elliptic System

Chunhui Wang*
Faculty of Science, Jiangsu University, Zhenjiang, Jiangsu 212013, P.R. China
*(Received 15 February 2020, accepted 20 March 2020)*

**Abstract:** In the present paper we study the existence of multiple positive solutions of the coupled nonlocal elliptic system. This kind of system was considered in the basic quantum chemistry model of small number of electrons interacting with static nucleii which can be approximated by Hartree or Hartree-Fock minimization problems. To accomplish this we use a new minimax argument to dealing with the existence of normalize solutions.

**Keywords:** Positive solutions, Nonlocal elliptic system, Minimax methods, Variational methods.

## 1 Introduction and main results

In the present paper we are concerned with the existence of the multiple positive solutions of the following coupled nonlocal elliptic system

$$\begin{cases} -\Delta u - \lambda_1 u = \mu_1 (\int_{\mathbb{R}^5} \frac{|u(y)|^2}{|x-y|^3} dy) u + \beta (\int_{\mathbb{R}^5} \frac{|v(y)|^2}{|x-y|^3} dy) u, & \text{in } \mathbb{R}^5 \\ -\Delta v - \lambda_2 v = \mu_2 (\int_{\mathbb{R}^5} \frac{|v(y)|^2}{|x-y|^3} dy) v + \beta (\int_{\mathbb{R}^5} \frac{|u(y)|^2}{|x-y|^3} dy) v, & \text{in } \mathbb{R}^5 \\ u, v > 0, \end{cases} \tag{1}$$

and

$$\int_{\mathbb{R}^5} u^2 = A_1^2 \quad \text{and} \quad \int_{\mathbb{R}^5} v^2 = A_2^2, \tag{2}$$

where $A_1, A_2, \mu_1, \mu_2 > 0$ and $\beta < 0$.

The study of the system (1) is main motivated by the recent studying on the existence and nonexistence of solutions of the nonlocal system (1). For example, see [20, 25, 28–30] and the references therein. In fact, this kind of the problem was proposed in the basic quantum chemistry model of small number of electrons interacting with static nucleii which can be approximated by Hartree or Hartree-Fock minimization problems (see [7, 19, 20]). In fact, considering the generalized time-dependent Hartree-Fock model

$$\begin{cases} -i\hbar \frac{\partial \phi_j}{\partial t} = -\hbar^2 \Delta \phi_j + V_j(x)\phi_j + \mu_{jh} \left( \sum_{h=1}^k |\phi_h|^p * J_\alpha \right) |\phi_j|^{p-2} \phi_j \\ \qquad + \beta_{jh} \sum_{h=1}^k \left( |\phi_h|^{\frac{p}{2}-1} \phi_h |\phi_j|^{\frac{p}{2}-1} \phi_j * J_\alpha \right) |\phi_h|^{p-2} \phi_h, \\ \phi_j(x,t) \to 0 \text{ as } |x| \to \infty, \ j = 1, \cdots, k, \end{cases} \tag{3}$$

where $k \geq 2$, $N \geq 1$, $\mu_{jh}, \beta_{jh} \in \mathbb{R}$, $\hbar$ is planck constant, and $V_i(x)$ describes the attractive interaction between the electrons and the nucleii, and $J_\alpha : \mathbb{R}^N \to \mathbb{R}$ is the Riesz potential defined at each point $x \in \mathbb{R}^N \setminus \{0\}$ by

$$J_\alpha(x) = \frac{A_\alpha}{|x|^{N-\alpha}} \quad \text{and} \quad A_\alpha = \frac{\Gamma\left(\frac{N-\alpha}{2}\right)}{\Gamma\left(\frac{\alpha}{2}\right) \pi^{N/2} 2^\alpha}.$$

*Corresponding author. *E-mail address*: wch18682550059@126.com

The system (3) was used to approximate the nonadiabatic case in molecular quantum chemistry at the microscopic scale when $N = 3$ and $p = \alpha = 2$, for example, see [7, 8]. For more general background of this model or similar models, one can refer to [7, 8, 19, 20] and the reference therein. The well-posedness of the Cauchy problem has been proven by Chadam and Glassey in [9]. As in [7, 19, 20] one can study the stationary solution of (3). That is,

$$\begin{cases} -\hbar^2\Delta\phi_j + V_i(x)\phi_j + \mu_{jh}\left(\sum_{h=1}^k |\phi_h|^p * J_\alpha\right)|\phi_j|^{p-2}\phi_j \\ \qquad +\beta_{jh}\sum_{h=1}^k \left(|\phi_h|^{\frac{p}{2}-1}\phi_h|\phi_j|^{\frac{p}{2}-1}\phi_j * J_\alpha\right)|\phi_h|^{p-2}\phi_h = 0, \\ \phi_j(x,t) \to 0 \text{ as } |x| \to \infty,\ 1 \le j \le k. \end{cases} \tag{4}$$

In particular, if $k = 2$ and $N = 3, p = \alpha = 2$, then (4) is reduced to a scalar equation

$$-\Delta\phi_1 + V_1(x)\phi_1 + (\mu_{11} + \beta_{11})\left(\int_{\mathbb{R}^3} \frac{\phi_1^2(y)}{|x-y|}dy\right)\phi_1 = 0,\quad x \in \mathbb{R}^3. \tag{5}$$

The solutions of (5) was considered [11, 20, 21]. If $k = 2$ and $\beta_{jh} = 0 (h \neq j)$, the system (4) equals

$$\begin{cases} -\hbar^2\Delta\phi_1 + V_1\phi_1 = \mu_1(J_\alpha * |\phi_1|^p)|\phi_1|^{p-2}\phi_1 + \beta(J_\alpha * |\phi_2|^p)|\phi_1|^{p-2}\phi_1, \\ -\hbar^2\Delta\phi_2 + V_2\phi_2 = \mu_2(J_\alpha * |\phi_2|^p)|\phi_2|^{p-2}\phi_2 + \beta(J_\alpha * |\phi_1|^p)|\phi_2|^{p-2}\phi_2, \end{cases} \tag{6}$$

where $\mu_1 = -(\beta_{11} + \mu_{11})$, $\mu_2 = -(\beta_{22} + \mu_{22})$ and $\beta = -\mu_{12} = -\mu_{21}$. It is clear that if $N = 5, \alpha = 3, p = 2$, $\hbar = A_\alpha = 1$, and $V_i = -\lambda_i (i = 1, 2)$ is constant, then (6) becomes (1). If $N = 3$ and $\alpha = p = 2$, the paper [33] considered the existence of multiple solutions of (6) for the semiclassical case($\hbar > 0$ small). Recently, the paper [28] proved the existence and nonexistence of positive ground state solutions of (1). Moreover, various qualitative properties of ground state solutions are also obtained. The papers [26, 31] proved the existence of nontrivial solutions of the more general nonlocal interaction case of (1), and the paper [29] proved the existence of nodal solution of (1). More recently, the existence of the normalized solutions for the local system(cubic growth) has been extensively studied by many mathematicians, for instance, see [3–6, 15, 16, 34, 35] and the references therein.

In the paper [27], we study the existence of normalized solution of (1)-(2). Motivated by recent work [6] for the local Schrödinger system, in the present paper we continue the work [27] to study the existence of multiple positive solution of the system (1)-(2) for $\mu_1 = \mu_2$. By using the abstract critical point theory of [27] and Krasnoselskii genus-type theorem, we prove the existence of multiple positive solution of the system (1)-(2). Precisely, the [27] proved existence of one positive radial normalized solution for possibly nonsymmetric systems($\mu_1 \neq \mu_2$). In the present paper we consider the symmetric problem $(1)-(2)(\mu_1 = \mu_2)$. We prove the existence of infinitely many solutions by exploiting the symmetry, which will be found as critical points of the energy functional $\mathcal{E}_\beta : \mathcal{S} \to \mathbb{R}$, defined by

$$\mathcal{E}_\beta(u,v) = \frac{1}{2}\int_{\mathbb{R}^5} |\nabla u|^2 + |\nabla v|^2 - \frac{1}{4}\int_{\mathbb{R}^5} \mu\psi_u|u|^2 + \mu\psi_v|v|^2 + 2\beta\psi_u|v|^2 \tag{7}$$

where $\mathcal{S} := S_A \times S_A (A \in \mathbb{R}^+)$, and

$$S_A = \left\{ w \in H^1_{rad}(\mathbb{R}^5) : \int_{\mathbb{R}^5} w^2 = A^2 \right\} \text{ and } \psi_w(x) = \int_{\mathbb{R}^5} \frac{|w(y)|^2}{|x-y|^3}. \tag{8}$$

Here $H^1_{rad}(\mathbb{R}^5)$ denotes the space of radially symmetric functions in $H^1(\mathbb{R}^5)$. Then we have the following main results.

**Theorem 1** *Assume that $\mu_1 = \mu_2$ and $A_1 = A_2$ with $\mu, A > 0$. Then for any $k \in \mathbb{N}$ there exists $\beta_k > -R_0$ such that for $\beta < \beta_k$ the problem (1)-(2) has at least k different pairs of radial solutions $(u_{j,\beta}, v_{j,\beta}, \lambda^j_{1,\beta}, \lambda^j_{2,\beta})$, $(v_{j,\beta}, u_{j,\beta}, \lambda^j_{2,\beta}, \lambda^j_{1,\beta})$, $j = 1, 2, \cdots, k$, with increasing energy, where $R_0 < 0$ is a fixed number.*

**Remark 2** *(1) In our previous works [25, 27], we studied the existence of one normalized solution of (6) for $\beta \in \mathbb{R}$. Theorem 1 can be seen as the continuous works of the paper [25, 27].*

*(2) In Theorem 1 we can only study the case $p = \alpha = 2$ and $N = 5$. This is due to the following two reasons. On the one hand, we need use the uniqueness of positive solution of the Choquard equation*

$$-\Delta u + u = \int_{\mathbb{R}^N} \frac{u^2(y)}{|x-y|^{N-2}}dyu, \quad u \in H^1(\mathbb{R}^N), \tag{9}$$

*only holds for $3 \leq N \leq 5$(see [22, 32]). If the uniqueness of positive solution holds true for the general Choquard equation, then we can generalize the results of Theorem 1. On the other hand, we can only study the super-critical growth(that is, $2 > 1 + 4/N$) of* (1)-(2) *which is related to $L^2(\mathbb{R}^N)$-normalized solution. Due to the above two reasons we can only study the case $N = 5$ and $p = \alpha = 2$. We shall continue to pursue to study the existence of normalized solutions of* (1)-(2) *for generalize cases.*

In order to prove Theorem 1, we shall use the Pohozaev identity to make a further constraint. To this purpose we define

$$G(\vec{u}) = \int_{\mathbb{R}^5} |\nabla u_1|^2 + |\nabla u_2|^2 - \frac{3}{4}\sum_{i=1}^{2}\int_{\mathbb{R}^5} \mu_i \psi_{u_i}|u_i|^2 - \frac{3}{2}\int_{\mathbb{R}^5} \beta\psi_{u_1}|u_2|^2 \tag{10}$$

for $\vec{u} := (u_1, u_2)$ and

$$\mathcal{P}_\beta := \{(u_1, u_2) \subset \mathcal{S} : G(u_1, u_2) = 0\}. \tag{11}$$

Recall that the Gagliardo-Nirenberg inequality(see [25, The inequality (2.13)]) is given by

$$\int_{\mathbb{R}^5}\int_{\mathbb{R}^5} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3} dxdy \leq S\left(\int_{\mathbb{R}^5} |u|^2\right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^5} |\nabla u|^2\right)^{\frac{3}{2}}. \tag{12}$$

Also, by using the classical Hardy-Littlewood-Sobolev inequality (see [18]), there exists a constant $C_0 > 0$ such that

$$\int_{\mathbb{R}^5}\int_{\mathbb{R}^5} \frac{|u(x)|^2|u(y)|^2}{|x-y|^3} dxdy \leq C_0\left(\int_{\mathbb{R}^5} |u|^{\frac{20}{7}}\right)^{\frac{7}{5}}. \tag{13}$$

From these two inequalities, one can easily check that the functional $\mathcal{E}_\beta$ is well defined on $H^1(\mathbb{R}^5) \times H^1(\mathbb{R}^5)$.

## 2 Some preliminaries

In the present paper we shall use the following notations.

- $||\cdot||$ is the norm of $H^1(\mathbb{R}^5)$ defined by $||u||^2 = \int_{\mathbb{R}^5}(|\nabla u|^2 + u^2)$.
- $||\cdot||_M$ is also the norm of $H^1(\mathbb{R}^5)$ defined by $||u||_M^2 = \int_{\mathbb{R}^5}(|\nabla u|^2 + Mu^2)$ for some $M > 0$.
- $H^1_{rad}(\mathbb{R}^5)$={$u \in H^1(\mathbb{R}^5)$: $u$ is radial with respect to origin}.
- $S_A = \left\{u \in H^1(\mathbb{R}^5) : \int_{\mathbb{R}^5} u^2 = A^2\right\}$ and $R_A = \left\{u \in H^1_{rad}(\mathbb{R}^5) : \int_{\mathbb{R}^5} u^2 = A^2\right\}$.
- $|\cdot|_p$ is the norm of $L^p(\mathbb{R}^5)$ defined by $|u|_p = \left(\int_{\mathbb{R}^5} |u|^p\right)^{\frac{1}{p}}$ for $0 < p \leq \infty$.
- $H^1_{rad}(\mathbb{R}^5, \mathbb{R}^2) = H^1_{rad}(\mathbb{R}^5, \mathbb{R}) \times H^1_{rad}(\mathbb{R}^5, \mathbb{R})$.
- Let $c > 0$ or $C > 0$ be the arbitrary constants.

In this section, we shall recall some basic conclusions for the nonlocal system (1)-(2). Precisely, the main purpose of this section is to recall the basic abstract critical point theory to the functional $\mathcal{E}_\beta$ on $\mathcal{P}_\beta$. Without loss of generality we assume that $\mu = 1$ in (1). Let $\mathcal{D}^{1,2}(\mathbb{R}^5)$ denote the closure of $\mathcal{C}_c^\infty(\mathbb{R}^5)$ with respect to norm $||w||_{\mathcal{D}^{1,2}} := |\nabla w|_2$. For each $s \in \mathbb{R}$ and $w \in H^1(\mathbb{R}^5)$, we define the function

$$(s \star w)(x) = e^{5s/2} w(e^s x). \tag{14}$$

It's easy to find that $|s \star w|_2 = |w|_2$ for every $s \in \mathbb{R}$. Then given $(u, v) \in \mathcal{S}$, we can easily check that

$$s \star (u, v) := (s \star u, s \star v) \in \mathcal{S} \quad \text{for each } s \in \mathbb{R},$$

where $\mathcal{S} = \mathcal{S}_{A_1} \times \mathcal{S}_{A_1}$.

Next we consider the real valued function

$$\mathfrak{I}^\beta_{(u,v)}(s) := \mathcal{E}_\beta(s \star (u, v)).$$

A direct computation shows that

$$\mathfrak{J}^{\beta}_{(u,v)}(s) = \frac{e^{2s}}{2}\int_{\mathbb{R}^5}\left(|\nabla u|^2 + \nabla v|^2\right) - \frac{e^{3s}}{4}\int_{\mathbb{R}^5}\left(\psi_u|u|^2 + \psi_v|v|^2 + 2\beta\psi_u|v|^2\right). \tag{15}$$

Let

$$\mathcal{W}_\beta := \left\{(u,v)\in\mathcal{S} : \int_{\mathbb{R}^5}\left(\psi_u|u|^2 + \psi_v|v|^2 + 2\beta\psi_u|v|^2\right) > 0\right\}. \tag{16}$$

We infer from Hölder inequality that $\mathcal{W}_\beta = \mathcal{S}$ for $-1 < \beta < +\infty$. On the other hand, for $\beta \le -1$ it results that $\mathcal{W}_\beta \subset \mathcal{S}$ with strict inclusion. By definition and using (15), as in [27, Lemma 2.4] we have the following basic conclusion for the function $\mathfrak{J}^{\beta}_{(u,v)}(s)$.

**Lemma 3** *For each $(u,v)\in\mathcal{S}$, then $s\in\mathbb{R}$ is a critical point of $\mathfrak{J}^{\beta}_{(u,v)}$ if and only if $s\star(u,v)\in\mathcal{P}$. Moreover, we have the following conclusions.*

$(i)$ *If $(u,v)\in\mathcal{W}_\beta$ then there exists a unique critical point $s^{\beta}_{(u,v)}\in\mathbb{R}$ for $\mathfrak{J}^{\beta}_{(u,v)}$, which is a strict maximum point. Furthermore, it is defined by*

$$exp\left\{s^{\beta}_{(u,v)}\right\} = \frac{4\int_{\mathbb{R}^5}\left(|\nabla u|^2 + |\nabla v|^2\right)}{3\int_{\mathbb{R}^5}\left(\psi_u|u|^2 + \psi_v|v|^2 + 2\beta\psi_u|v|^2\right)}. \tag{17}$$

*In particular, if $(u,v)\in\mathcal{P}_\beta$, then $s^{\beta}_{(u,v)} = 0$.*

$(ii)$ *If $(u,v)\notin\mathcal{W}_\beta$, then $\mathfrak{J}^{\beta}_{(u,v)}$ has no critical points in $\mathbb{R}$.*

Next we shall recall the basic abstract critical point theory to the nonlocal system (1). To this purpose we first recall the following definition [13, Definition 3.1].

**Definition 1** *Let $B$ be a closed $\sigma$-invariant subset of a set $X\subset H^1_{rad}(\mathbb{R}^5,\mathbb{R}^2) = H^1_{rad}(\mathbb{R}^5,\mathbb{R})\times H^1_{rad}(\mathbb{R}^5,\mathbb{R})$. We say that a class $\mathcal{F}$ of compact subsets of $X$ is a $\sigma$-homotopy stable family with closed boundary $B$ provided*

$(a)$ *Every set in $\mathcal{F}$ contains $B$.*

$(b)$ *For any $F\in\mathcal{F}$ and any $\sigma$-invariant homotopy $\eta\in C([0,1]\times X, X)$ satisfying $\eta(t,x) = x$ for all $(t,x)\in(\{0\}\times X)\cup([0,1]\times B)$, we have that $\eta(\{1\}\times F)\in\mathcal{F}$.*

From this definition we have the following main results in this section. For the details of the proof of this conclusion we refer to [27, Theorem 2.1].

**Theorem 4** *Let $\beta\in\mathbb{R}$, $A_1 = A_2$ and $\mu_1 = \mu_2$. Let $\mathcal{F}$ be a homotopy stable family of compact subsets of $\mathcal{P}_\beta$, with closed boundary $B\subset\mathcal{P}_\beta$. In addition, assume that*

$$B \text{ is contained in a connected component of } \mathcal{P}_\beta,$$

*and $\max\{\sup\mathcal{E}_\beta(B), 0\} < c_{\mathcal{F}} < +\infty$, where $c_{\mathcal{F}} := \inf_{F\in\mathcal{F}}\max_{\vec{u}\in F}\mathcal{E}_\beta(\vec{u})$. Then there exists a sequence $\{(u_n,v_n)\}$ with the following properties:*

$(i)$ *$(u_n,v_n)\in\mathcal{P}_\beta$ for every $n$.*

$(ii)$ *$\mathcal{E}_\beta(u_n,v_n)\to c_{\mathcal{F}}$ as $n\to\infty$.*

$(iii)$ *$||\nabla(\mathcal{E}_\beta|_{(S_{A_1}\times S_{A_2})})(u_n,u_n)||\to 0$ as $n\to\infty$.*

*Moreover, we can find a minimizing sequence $\{F_n\}$ for $c_{\mathcal{F}}$ in such a way that $(u,v)\in F_n$ implies $u,v\ge 0$ a.e., then we can find the sequence $\{(u_n,v_n)\}$ satisfying the additional condition*

$(iv)$ *$u_n,v_n\to 0$ a.e. in $\mathbb{R}^5$ as $n\to\infty$.*

# 3 Properties of the Palais-Smale sequences

In this section we borrow an idea of [6] to study the properties of the Palais-Smale sequences of the nonlocal functional $\mathfrak{J}^{\beta}_{(u,v)}|_{\mathcal{P}}$. To accomplish this we first consider the single equation

$$\begin{cases} -\triangle w + w = \psi_w w, & \text{in } \mathbb{R}^5, \\ w >, \ w \in S_A. \end{cases} \tag{18}$$

Recall that the problem (18) has unique solution $w_0 \in H^1(\mathbb{R}^5)$(see [32]). We define

$$\ell := I(w_0), \quad \text{where} \quad I(w) = \frac{1}{2}\int_{\mathbb{R}^5} |\nabla w|^2 - \frac{1}{4}\int_{\mathbb{R}^5} \psi_u |u|^2. \tag{19}$$

It is well known that $\ell > 0$ is the ground state energy level of (18).

Next we shall investigate the behavior of any Palais-Smale sequence for the constrained functional $\mathcal{E}_\beta|_{\mathcal{P}_\beta}$. We first show that the constrained functional $\mathcal{E}_\beta|_{\mathcal{P}_\beta}$ is coercive.

**Lemma 5** *The constrained functional $\mathcal{E}_\beta|_{\mathcal{P}_\beta}$ is bounded from below and coercive.*

**Proof.** For any $(u, v) \in \mathcal{P}_\beta$, it follows that

$$\mathcal{E}_\beta(u, v) = \frac{1}{6}\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 = \frac{1}{8}\int_{\mathbb{R}^5} \psi_u |u|^2 + \psi_v |v|^2 + 2\beta\psi_u |v|^2. \tag{20}$$

This finishes the proof. ■

From this conclusion, if $\{(u_n, v_n)\} \subset \mathcal{P}_\beta$ is a Palais-Smale sequence for the functional $\mathcal{E}_\beta$, we know that $\{(u_n, v_n)\}$ is bounded. Moreover, we have the following more precisely conclusions for $\beta < 0$.

**Lemma 6** *Let $\beta < 0$ be fixed. Assume that $\{(u_n, v_n)\}$ is a Palais-Smale sequences for $\mathcal{E}_\beta|_{\mathcal{S}}$ at level $c \in (0, +\infty)$ such that*

$$u_n^-, v_n^- \to 0 \ \text{a.e. in } \mathbb{R}^5, \ \text{and } (u_n, v_n) \in \mathcal{P}_\beta.$$

*Then we have the following conclusions.*

- $(a)$ *If $c \neq \ell$, then up to a subsequence $(u_n, v_n) \to (u, v)$ strongly in $H^1(\mathbb{R}^5, \mathbb{R}^2)$, and $(u, v)$ is a solution to* (1)-(2) *for some $\lambda_1, \lambda_2 < 0$.*
- $(b)$ *If $c = \ell$, then one of the following alternatives occurs:*
  - $(i)$ *$(u_n, v_n) \to (u, v)$ strongly in $H^1(\mathbb{R}^5, \mathbb{R}^2)$ up to a subsequence, where $(u, v)$ is a solution to* (1) *for some $\lambda_1, \lambda_2 < 0$ with $\mathcal{E}_\beta(u, v) = \ell$.*
  - $(ii)$ *either $u_n \to w_0$ strongly in $H^1(\mathbb{R}^5)$ and $v_n \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^5)$, or $v_n \to w_0$ strongly in $H^1(\mathbb{R}^5)$ and $u_n \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^5)$, up to a subsequence.*

**Proof.** From Lemma 5 we can know that $(u_n, v_n) \rightharpoonup (\bar{u}, \bar{v}) \in H^1_r(\mathbb{R}^5, \mathbb{R}^2)$ and $(u_n, v_n) \to (\bar{u}, \bar{v}) \in L^q_r(\mathbb{R}^5, \mathbb{R}^2)(\forall q \in (2, 10/3))$(since the embedding $H^1_{rad}(\mathbb{R}^5) \hookrightarrow L^q(\mathbb{R}^5)$ is compact). Moreover, we have $(u_n, v_n) \to (\overline{u}, \overline{v})$ a.e. in $\mathbb{R}^5$. By the definition of $c$ and (20)

$$\lim_{n\to\infty} \frac{1}{6}\int_{\mathbb{R}^5} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) = c.$$

This implies that for every $n$ sufficiently large

$$\int_{\mathbb{R}^5} \left(|\nabla u_n|^2 + |\nabla v_n|^2\right) \geq 5c > 0. \tag{21}$$

Using the fact that the problem is invariant under rotation, we know that $d\mathcal{E}_\beta|_{\mathcal{S}}(u_n, v_n) \to 0$, for every $(\varphi, \phi) \in H^1(\mathbb{R}^5, \mathbb{R}^2)$. Hence by the Lagrange multipliers rule there exist two sequences of real numbers $\{\lambda_{1,n}\}$ and $\{\lambda_{2,n}\}$ such that

$$\begin{aligned}&\int_{\mathbb{R}^5}(\nabla u_n\cdot\nabla\varphi+\nabla v_n\cdot\nabla\phi-\psi_{u_n}u_n\varphi-\psi_{v_n}v_n\phi-\psi_{v_n}u_n\varphi-\psi_{u_n}v_n\phi)\\&\quad-\int_{\mathbb{R}^5}(\lambda_{1,n}u_n\varphi+\lambda_{2,n}v_n\phi)=o(1)||(\varphi,\phi)||_{H^1}\end{aligned}\tag{22}$$

where $o(1) \to 0$ as $n \to \infty$. In light of (21) and $\{(u_n, v_n)\}$ is bounded, we can check as in [25, Lemma 3.7] that $\lambda_{1,n} \to \lambda_1 \in \mathbb{R}$ and $\lambda_{2,n} \to \lambda_2 \in \mathbb{R}$, and in particular $\lambda_1 + \lambda_2 < 0$ , hence at least one of $\lambda_1$ and $\lambda_2$ is strictly negative. Without loss of generality we assume that $\lambda_1 < 0$. Moreover, thanks to [25, Lemma 3.8], we know that if $\lambda_1 < 0$ (resp. $\lambda_2 < 0$), then necessarily $u_n \to \bar{u}$ (resp. $v_n \to \bar{v}$) strongly in $H^1(\mathbb{R}^5)$. Notice that by (22) and weak convergence of $(\bar{u}, \bar{v}) \in H^1(\mathbb{R}^5, \mathbb{R}^2)$ is a solution to

$$\begin{cases}-\Delta\bar{u}-\lambda_1\bar{u}=(\int_{\mathbb{R}^5}\frac{|\bar{u}(y)|^2}{|x-y|^3}dy)\bar{u}+\beta(\int_{\mathbb{R}^5}\frac{|\bar{v}(y)|^2}{|x-y|^3}dy)\bar{u}, & \text{in } \mathbb{R}^5\\-\Delta\bar{v}-\lambda_2\bar{v}=(\int_{\mathbb{R}^5}\frac{|\bar{v}(y)|^2}{|x-y|^3}dy)\bar{v}+\beta(\int_{\mathbb{R}^5}\frac{|\bar{u}(y)|^2}{|x-y|^3}dy)\bar{v}, & \text{in } \mathbb{R}^5\\\bar{u}\ge 0,\ \bar{v}\ge 0, & \text{in } \mathbb{R}^5\end{cases}\tag{23}$$

for some $\lambda_1 < 0$ and $\lambda_2 \in \mathbb{R}$. We have showed that, the Palais-Smale sequence $\{(\bar{u}_n, \bar{v}_n)\}$ tends weakly to a solution of (23). By [25, Lemma 3.8] we know that if $\lambda_2 < 0$, then also $v_n \to \bar{v}$ strongly in $H^1(\mathbb{R}^5)$.

Next, we shall prove that if $c \neq \ell$, then it is necessary that $\lambda_2 < 0$(while if $c = \ell$, then it is possible that $\lambda_2 \geq 0$. However, in such a situation $v_n \to 0$ strongly in $\mathcal{D}^{1,2}(\mathbb{R}^5)$, and $u_n \to w_0$ strongly in $H^1(\mathbb{R}^5)$). In order to prove that if $c \neq \ell$ then $\lambda_2 < 0$. We use the contradiction argument. Assume that $\lambda_2 \geq 0$. Since $\lambda_1 < 0$, by [27, Lemma 3.11], it is obviously to know that the function $\bar{u}$ decays exponentially at infinity. Then if $\lambda_2 \geq 0$ we conclude that $\bar{v}$ satisfies

$$-\Delta\bar{v}+c(x)\bar{v}\ge 0\quad\text{in } \mathbb{R}^5,$$

where $0 \leq c(x) := |\beta|\psi_{\bar{u}} \leq Ce^{-C|x|}$. Since $\bar{v} \geq 0$ in $\mathbb{R}^5$ and $\bar{v} \in H^1(\mathbb{R}^5)$, we can infer that $\bar{v} \equiv 0$ in $\mathbb{R}^5$ by the Liouville-type [27, The inequality (3.20)]. In this case, we can find that $\bar{u}$ is positive and solve (10), and by uniqueness $\bar{u} = w_0$. Recall that any radial positive solution of (18) lies in

$$\mathcal{K}:=\left\{u\in S_A^r:\int_{\mathbb{R}^5}|\nabla u|^2=\frac{3}{4}\int_{\mathbb{R}^5}\psi_u|u|^2\right\}.$$

Hence we obtain

$$\ell=I(w_0)=\frac{1}{8}\int_{\mathbb{R}^5}\psi_{w_0}|w_0|^2.$$

Furthermore, we infer from (20) and $(u_n, v_n) \to (w_0, 0)$ in $L^q(\mathbb{R}^5)(\forall q \in (2, 10/3))$ that

$$c=\lim_n\mathcal{E}_\beta(u_n,v_n)=\lim_{n\to\infty}\frac{1}{8}\int_{\mathbb{R}^5}\left(\psi_{u_n}|u_n|^2+\psi_{v_n}|v_n|^2+2\beta\psi_{u_n}|v_n|^2\right)=\frac{1}{8}\int_{\mathbb{R}^5}\psi_{w_0}|w_0|^2=\ell.$$

Hence, if $c \neq \ell$, then it is necessary that $\lambda_2 < 0$, and as a consequence, $(u_n, v_n) \to (\bar{u}, \bar{v})$ strongly in $H^1$.

Finally, we consider the case $c = \ell$ and $\lambda_2 \geq 0$. This implies that we do not have strong convergence of the whole Palais-Smale sequence. Fortunately, we can prove $u_n \to w_0$ and $v_n \rightharpoonup 0$ in $H^1(\mathbb{R}^5)$. To accomplish this it suffices to show that $v_n \to 0$ strongly in $\mathcal{D}^{1,2}$. We infer from $(u_n, v_n) \in \mathcal{P}_\beta$ that

$$\int_{\mathbb{R}^5}|\nabla v_n|^2=\frac{3}{4}\int_{\mathbb{R}^5}\psi_u|u|^2+\psi_v|v|^2+2\beta\psi_u|v|^2-\int_{\mathbb{R}^5}|\nabla u_n|^2\to\frac{3}{4}\int_{\mathbb{R}^5}\psi_{w_0}|w_0|^2-\int_{\mathbb{R}^5}|\nabla w_0|^2,$$

as $n \to \infty$. Since $w_0 \in \mathcal{K}$, it follows that the last term is equal to 0. That is, $||v_n||_{\mathcal{D}^{1,2}} \to 0$, as desired. ■

# 4 The estimates for the least energy level $\inf_{\mathcal{P}_\beta} \mathcal{E}_\beta$

In this section we study the asymptotical behavior of the least energy level $\inf_{\mathcal{P}_\beta} \mathcal{E}_\beta$ as $\beta$ varies. To this purpose we set

$$h_\beta := \inf_{\mathcal{P}_\beta} \mathcal{E}_\beta,$$

and define the functional $E_\beta : \mathcal{W}_\beta \to \mathbb{R}$ by

$$E_\beta(u,v) := \mathcal{E}_\beta(s_{(u,v)} \star (u,v)).$$

Since (17) and $s^\beta_{(u,v)} \star (u,v) \in \mathcal{P}_\beta$, it is easy to check that

$$\begin{aligned} E_\beta(u,v) &= \frac{1}{6}\int_{\mathbb{R}^5}\left(|\nabla(s^\beta_{(u,v)} \star u)|^2 + |\nabla(s^\beta_{(u,v)} \star v)|^2\right) = \frac{e^{2s^\beta_{(u,v)}}}{6}\int_{\mathbb{R}^5}\left(|\nabla u|^2 + |\nabla v|^2\right) \\ &= \frac{8(\int_{\mathbb{R}^5}|\nabla u|^2 + |\nabla v|^2)^3}{27(\int_{\mathbb{R}^5}\psi_u u^2 + \psi_v v^2 + 2\beta\psi_u v^2)^2} \end{aligned} \tag{24}$$

for every $(u,v) \in \mathcal{W}_\beta$. We first make a different characterize the infimum $h_\beta$.

**Lemma 7** *For each $\beta \le 0$, we have*

$$h_\beta := \inf_{(u,v)\in\mathcal{W}_\beta} E_\beta(u,v).$$

**Proof.** Recall that for each $(u,v) \in \mathcal{P}_\beta$, we have $s^\beta_{(u,v)} = 0$(defined by (17)). Hence we obtain

$$\mathcal{E}_\beta(u,v) = E_\beta(u,v) \ge \inf_{\mathcal{W}_\beta} E_\beta.$$

This implies that

$$h_\beta \ge \inf_{\mathcal{W}_\beta} E_\beta.$$

On the other hand, we note that for any $(u,v) \in \mathcal{W}_\beta$

$$E_\beta(u,v) = \mathcal{E}_\beta(s_{(u,v)} \star (u,v)) \ge h_\beta.$$

Thus, we have

$$\inf_{\mathcal{W}_\beta} E_\beta \ge h_\beta.$$

This completes the proof. ■

The next lemma proves the monotone properties of $h_\beta$ for $\beta$.

**Lemma 8** *The level $h_\beta$ is monotone non-increasing in $\beta$. In particular, $h_\beta \ge h_0$ for every $\beta \le 0$.*

**Proof.** We use the contradiction arguments. Assume that $\beta_1 < \beta_2 \le 0$ but $h_{\beta_1} < h_{\beta_2}$. This leads to

$$\int_{\mathbb{R}^5}\left(\psi_u|u|^2 + \psi_v|v|^2 + 2\beta_1\psi_u|v|^2\right) \le \int_{\mathbb{R}^5}\left(\psi_u|u|^2 + \psi_v|v|^2 + 2\beta_2\psi_u|v|^2\right). \tag{25}$$

Hence one has $\mathcal{W}_{\beta_1} \subset \mathcal{W}_{\beta_2}$. By the definition of the infimum we infer that there exists $(u,v) \in \mathcal{W}_{\beta_1}$ such that $h_{\beta_2} > E_{\beta_1}(u,v) \ge h_{\beta_1}$(since $h_{\beta_2} > h_{\beta_1} = \inf_{(u,v)\in\mathcal{W}_{\beta_1}} E_{\beta_1}$). However, we infer from $(u,v) \in \mathcal{W}_{\beta_2}$ and (25) that

$$h_{\beta_2} > E_{\beta_1}(u,v) \ge E_{\beta_2}(u,v) \ge \inf_{\mathcal{W}_{\beta_2}} E_{\beta_2} = h_{\beta_2}.$$

This is a contradiction. ■

Next we prove the property for $h_0$.

**Lemma 9** *We have $h_0 = \inf_{\mathcal{P}_0} \mathcal{E}_0 > 0$.*

**Proof.** We infer from the Gagliardo-Nirenberg inequality (12) that for $(u,v)\in\mathcal{P}_0$

$$\left(\int_{\mathbb{R}^5}\psi_u|u|^2+\psi_v|v|^2\right)^{\frac{2}{3}}\le C_1\left(\int_{\mathbb{R}^5}\psi_u|u|^2\right)^{\frac{2}{3}}+C_1\left(\int_{\mathbb{R}^5}\psi_v|v|^2\right)^{\frac{2}{3}}$$
$$\le C_2\int_{\mathbb{R}^5}\left(|\nabla u|^2+|\nabla v|^2\right)=C_3\int_{\mathbb{R}^5}\left(\psi_u|u|^2+\psi_v|v|^2\right),$$

where the constants $C_1, C_2$ and $C_3$ depend only on $A$. From this, we know that

$$\left(\int_{\mathbb{R}^5}\psi_u|u|^2+\psi_v|v|^2\right)^{\frac{1}{3}}\ge\frac{1}{C_3}. \tag{26}$$

We infer from the Gagliardo-Nirenberg inequality and (24) that

$$h_0=\inf_{(u,v)\in\mathcal{P}_0}\mathcal{E}_0=\inf_{(u,v)\in\mathcal{P}_0}\frac{1}{6}\int_{\mathbb{R}^5}|\nabla u|^2+|\nabla v|^2\ge\inf_{(u,v)\in\mathcal{P}_0}C_4\int_{\mathbb{R}^5}(\psi_u|u|^2+\psi_v|v|^2)^{\frac{2}{3}}\ge\frac{C_4}{C_3^2}>0.$$

This ends the proof. ■

The next lemma proves $h_\beta=\ell$.

**Lemma 10** *For every $\beta\le 0$, we have $h_\beta=\ell$.*

**Proof.** We first prove $h_\beta\ge\ell$. One infers from Lemma 8 that $h_\beta\ge h_0$. Thus we only need to show that $h_0\ge\ell$. Assume that $0<h_0<\ell$. Let $\{(u_n,v_n)\}$ be the minimizing sequence for $h_0$. Without restrictive, we assume that $u_n,v_n\ge 0$ a.e in $\mathbb{R}^5$. Then by Lemma 6, one hase $(u_n,v_n)\to(u_0,v_0)$ in $H^1(\mathbb{R}^5)$. For $\beta=0$, we know that the system (1) is given by two uncoupled equations, and $u_0,v_0$ are positive radial solutions to (18). By uniqueness of the solution of (18), we get $u_0=v_0=w_0$. Then we have

$$\ell>h_0=\mathcal{E}_\beta(u_0,v_0)=I(u_0)+I(v_0)=2\ell>0.$$

This a contradiction. On the other hand, we show that $h_\beta\le\ell$. It is clear that there exists a sequence $\{(u_n,v_n)\}\subset\mathcal{P}_\beta$ such that $\mathcal{E}_\beta(u_n,v_n)\to\ell$, and $u_n\to w_0$ in $H^1(\mathbb{R}^5)$, and $v_n\to 0$ in $\mathcal{D}^{1,2}(\mathbb{R}^5)$ by [27, Lemma 3.10]. Then we have $h_\beta\le\ell$. ■

We define the involution $\sigma: H^1_{rad}(\mathbb{R}^5,\mathbb{R}^2)\to H^1_{rad}(\mathbb{R}^5,\mathbb{R}^2), \sigma(u,v)=(v,u)$. Then by the symmetry of (1), it is easy to find that both $\mathcal{E}_\beta$ and $\mathcal{P}_\beta$ are $\sigma$-invariant. Particularly, for $\beta\le -1$ and $(u,u)\in\mathcal{P}_\beta$, one can check that

$$0<\int_{\mathbb{R}^5}|\nabla u|^2=\frac{3}{4}(1+\beta)\int_{\mathbb{R}^5}\psi_u|u|^2\le 0.$$

This means that $\sigma$ has no fixed points in $\mathcal{P}_\beta$ for $\beta\le -1$.

Finally, we study the fixed point set $\mathcal{P}^\sigma_\beta:=\{(u,v)\in\mathcal{P}_\beta: u=v\}$ as $\beta\to -1$, and the infimum $h^\sigma_\beta:=\inf_{\mathcal{P}^\sigma_\beta}\mathcal{E}_\beta$.

**Lemma 11** *The infimum $h^\sigma_\beta\to\infty$ for $\beta$ decreasing convergence to $-1$.*

**Proof.** We infer from the Gagliardo-Nirenberg inequality (12) that for $(u,u)\in\mathcal{P}^\sigma_\beta$

$$\int_{\mathbb{R}^5}|\nabla u|^2=\frac{3}{4}(1+\beta)\int_{\mathbb{R}^5}\psi_u|u|^2\le\frac{3C_0A}{4}(1+\beta)\left(\int_{\mathbb{R}^5}|\nabla u|^2\right)^{\frac{3}{2}}.$$

Then we have

$$\left(\int_{\mathbb{R}^5}|\nabla u|^2\right)^{\frac{1}{2}}\ge\frac{4}{3C_0A(1+\beta)}.$$

From this we deduce that

$$\mathcal{E}_\beta(u,u)=\frac{1}{6}\int_{\mathbb{R}^5}|\nabla u|^2+|\nabla u|^2=\frac{1}{3}\int_{\mathbb{R}^5}|\nabla u|^2\ge\frac{16}{27C_0^2A^2(1+\beta)^2}.$$

This completes the proof. ■

# 5 Proof of Theorem 1

## 5.1 Minimax theorem

In this subsection we shall follow the idea of [6] to use the Krasnoselskii genus-type arguments to establish a minimax scheme. Let us briefly introduce the genus. To this purpose we define the genus $\gamma(M)$ as the smallest integer $n \in \mathbb{N} \cup \{0\}$ such that there exists a continuous map $g : M \to \mathbb{R}^n\{0\}$ with $g(\sigma(u, v)) = -g(u, v)$ for every $(u, v) \in M$, where $M \subset \mathcal{W}_\beta$ and is closed $\sigma$-invariant set. If no such map exists we set $\gamma(M) = +\infty$. Note that $\gamma(\emptyset) = 0$. Next, we refer [10, Lemma 4.4] for some standard properties of $\gamma$.

**Lemma 12** *Let $M, N \subset \mathcal{W}_\beta$ be closed and $\sigma$-invariant. Then the following conclusions hold.*

$(i)$ *If $M \subset N$, then $\gamma(M) \leq \gamma(N)$.*

$(ii)$ $\gamma(M \cup N) \leq \gamma(M) + \gamma(N)$.

$(iii)$ *If $g : M \to \mathcal{W}_\beta$ is continuous and $\sigma$-equivariant, i.e. $g(\sigma(u, v)) = \sigma(g(u, v))$ for every $(u, v) \in M$, then $\gamma(M) \leq \gamma(\overline{g(M)})$.*

*For a subset $M \subset \mathcal{W}_\beta$ that does not contain fixed points of $\sigma$, there holds:*

$(iv)$ *If $\gamma(M) > 1$, then $M$ is an infinite set.*

$(v)$ *If $M$ is compact, then $\gamma(M) < +\infty$, and there exists a relatively open $\sigma$-invariant neighborhood $N$ of $M$ in $\mathcal{E}_\beta$ such that $\gamma(M) = \gamma(\bar{N})$.*

$(vi)$ *If $S$ is the boundary of a bounded symmetric neighborhood of zero in a $k$-dimensional normed vector space and $\psi : S \to \mathcal{W}_\beta$ is a continuous map satisfying $\psi(-s) = \sigma(\psi(s))$, then $\gamma(\psi(S)) \geq k$.*

Let $\mathcal{M}_\beta := \{M \subset \mathcal{P}_\beta : M \textit{ is closed and } \sigma - \textit{invariant}\}$. Then for any $k \in \mathbb{N}$ we define

$$\mathcal{M}_{k,\beta} := \{M \in \mathcal{M}_\beta : M \textit{ is compact and } \gamma(M) \geq k\}.$$

Next we define the minimax level

$$c_k = c_{k,\beta} = \inf_{M \in \mathcal{M}_{k,\beta}} \max_{(u,v) \in M} \mathcal{E}_\beta(u, v).$$

**Lemma 13** *Any $c_{k,\beta}$ is a real number, that is $\mathcal{M}_{k,\beta} \neq \emptyset$ for every $k$. Furthermore, if $-R_0 \leq \beta < 0$, we have that $\ell \leq c_{k,\beta} < C_k$ for every $k \geq 1$, where $C_k$ is independent of $\beta$.*

**Proof.** First, choose a $k$-dimensional subspace $W$ of $\{w \in H^1_{rad}(\mathbb{R}^5) : \int_{\mathbb{R}^5} w = 0\}$, and set $T := \{w \in W : ||w||_{H^1} = 1\}$. Next we define the mappings $\phi : T \to \mathcal{S}$ and $\psi : T \to \mathcal{P}_\beta$ by

$$\phi(w) := \left( \frac{A}{||w^+||_{L^2}} w^+, \frac{A}{||w^-||_{L^2}} w^- \right) \quad \text{and} \quad \psi(w) := s^\beta_{\phi(w)} \star \phi(w),$$

where $s^\beta_{\phi(w)}$ is given by (17). It is easy to see that for every $\beta$, one sees $\psi(T) \subset \mathcal{P}_\beta$. Moreover, from [27, Lemma 2.6], we know that $\psi$ is continuous. Hence we can use Lemma 12 $(iii)$ to check that

$$\begin{aligned}
\psi(-w) &= s^\beta_{\phi(w)} \star \phi(-w) \\
&= \left( s^\beta_{\phi(w)} \star \left( \frac{A}{||(-w)^+||_{L^2}} (-w)^+ \right), s^\beta_{\phi(w)} \star \left( \frac{A}{||(-w)^-||_{L^2}} (-w)^- \right) \right) \\
&= \left( s^\beta_{\phi(w)} \star \left( \frac{A}{||w^-||_{L^2}} w^- \right), s^\beta_{\phi(w)} \star \left( \frac{A}{||w^+||_{L^2}} w^- \right) \right) \\
&= \sigma \left( s^\beta_{\phi(w)} \star \left( \frac{A}{||w^+||_{L^2}} w^- \right), s^\beta_{\phi(w)} \star \left( \frac{A}{||w^-||_{L^2}} w^- \right) \right) \\
&= \sigma \left( s^\beta_{\phi(w)} \star \phi(w) \right) = \sigma(\psi(w)).
\end{aligned}$$

Then by using Lemma 12 $(vi)$ we have $\gamma(\psi(T)) \geq k$, and $\psi(T) \subset \mathcal{P}_\beta$ is $\sigma$-invariant. Furthermore, $\psi(T)$ is compact as the continuous image of a compact set. As a consequences we know that $\psi(T) \in \mathcal{M}_{k,\beta}$ for each $\beta < 0$. Hence we infer from $-R_0 \leq \beta < 0$ is bounded from below that

$$c_{k,\beta} \leq \max_{\psi(T)} \mathcal{E}_\beta \leq \max_{\psi(T)} \mathcal{E}_{-R_0} := C_k.$$

Finally, by Lemma 10 we know that $\ell = \inf_{\mathcal{P}_\beta} \mathcal{E}_\beta$ for every $\beta \leq 0$. Hence we obtain that

$$\ell = \inf_{\mathcal{P}_\beta} \mathcal{E}_\beta = c_{1,\beta} \leq c_{k,\beta}$$

for every $k \in \mathbb{N}$ and $\beta < 0$. ■

We now define

$$\beta_k := \inf\{\beta \in (-1, 0) : c_{k+1,\beta} \geq h^\sigma_\beta\} \in (-1, 0)$$

for every $k \in \mathbb{N}$. We infer from Lemmas 11 and 13 that if $\beta < \beta_k$ then $c_{k,\beta} < h^\sigma_\beta$.

**Lemma 14** *For any $k \in \mathbb{N}$ and any $\beta < \beta_k$ there exists a Palais-Smale sequence $\{(u^k_n, v^k_n)\}$ for $\mathcal{E}_\beta|_\mathcal{S}$ at level $c_{k,\beta}$, satisfying the additional conditions $(u^k_n)^-, (v^k_n)^- \to 0$ a.e. in $\mathbb{R}^5$ as $n \to \infty$, and $\{(u^k_n, v^k_n)\} \subset \mathcal{P}_\beta$.*

**Proof.** By Definition 1 and Lemma 12 $(iii)$, we can check that the family $\mathcal{M}_{k,\beta}$ is a $\sigma$-homotopy stable family of compact subsets of $\mathcal{P}_\beta$ with boundary $\emptyset$. From Lemma 13 we know that $c_\mathcal{F} < \infty$. Now let $\{M_n\} \subset \mathcal{M}_{k,\beta}$ be a minimizing sequence at level $c_{k,\beta}$ such that $\max_{M_n} \to c_{k,\beta}$ as $n \to \infty$. Next, consider the sequence $|M_n|$, where

$$|M_n| := \{(|u|, |v|) : (u, v) \in M_n\}$$

for all $n$. It is clear that $|M_n|$ is also a minimizing sequence. In fact, $|M_n|$ inherits the equivariancy and the compactness from $M_n$. Moreover, we have $\gamma(|M_n|) \geq \gamma(M_n) \geq k$ for every $k$ by Lemma 12 $(iii)$. Then by Theorem 4 we know the results hold. ■

Next we shall show the Lusternik-Schnirelman type result holds true. To this purpose we consider $\mathcal{B}_\beta := \{M \subset \mathcal{W}_\beta : M\ is\ closed\ and\ \sigma - invariant\}$, and

$$\mathcal{B}_{k,\beta} := \{M \in \mathcal{B}_\beta : M\ is\ compact\ and\ \gamma(M) \geq k\}$$

for any $k \in \mathbb{N}$. Then we can define the mimimax levels

$$e_k = e_{k,\beta} := \inf_{M \in \mathcal{B}_{k,\beta}} \max_{(u,v) \in M} E_\beta(u, v).$$

**Lemma 15** *For every $k \in \mathbb{N}$, we have $e_{k,\beta} = c_{k,\beta}$.*

**Proof.** We define the mapping

$$l(u, v) = s^\beta_{(u,v)} \star (u, v),$$

where $s^\beta_{(u,v)}$ is defined in (17). We infer from (17) and [25, Lemma 2.6] that $l$ is continuous and $\sigma$-equivariant. Let $D \subset \mathcal{B}_{k,\beta}$ such that $\max_D E_\beta < e_{k,\beta} + \varepsilon$, where $E_\beta(u, v) = \mathcal{E}_\beta(s_{(u,v)} \star (u, v))$. By Lemma 12 $(iii)$, the compact $\sigma$-invariant set $M := l(D)$ satisfies $\gamma(M) \geq k$ and hence $M \in \mathcal{M}_{k,\beta}$. From the definition of $E_\beta(u, v)$, we know that for any $(u, v) \in \mathcal{W}_\beta$, $E_\beta(u, v) = \mathcal{E}_\beta(u, v)$. Particularly, we have

$$c_{k,\beta} \leq \max_M \mathcal{E}_\beta = \max_D E_\beta \leq e_{k,\beta} + \varepsilon$$

for each $\varepsilon > 0$. Hence we get $c_{k,\beta} \leq e_{k,\beta}$. On the other hand, we infer from $\mathcal{M}_{k,\beta} \subset \mathcal{B}_{k,\beta}$ and $E_\beta = \mathcal{J}_\beta$ on $\mathcal{P}_\beta$ that for any $M \in \mathcal{M}_{k,\beta}$

$$c_{k,\beta} + \varepsilon \geq \max_M \mathcal{E}_\beta = \max_M E_\beta \geq e_{k,\beta}.$$

This implies that $c_{k,\beta} \geq e_{k,\beta}$. ■

We define now the critical set

$$\mathcal{K}^+_c := \{(u, v) \in \mathcal{S} : u, v \geq 0\ a.e.\ \text{in}\ \mathbb{R}^5,\ \mathcal{E}_\beta(u, v) = c,\ d\mathcal{E}_\beta|_\mathcal{S}(u, v) = 0\}.$$

Note that $\mathcal{K}^+_c$ is $\sigma$-invariant, and $\mathcal{K}^+_c \subset \mathcal{P}_\beta$ by the Pohozaev identity. The next lemma states the Palais-Smale sequence relationship between the functionals $E_\beta$ and $\mathcal{E}_\beta$. For the proof one can refer to [27, lemma 2-10].

**Proposition 16** *Let $\{(\tilde{u}_n, \tilde{v}_n)\}$ be a Palais-Smale sequence for $E_\beta$ at level $e_\beta \in (0, +\infty)$. Suppose that for every $n$ there exists $(w_n, z_n) \in \mathcal{P}_\beta$ such that*

(a) $||(\tilde{u}_n, \tilde{v}_n) - (w_n, z_n)||_{H^1} \to 0$ *as* $n \to \infty$;

(b) $||(w_n, z_n)||_{\mathcal{D}^{1,2}} \leq C$.

*Then* $s_n^\beta := s^\beta_{(\tilde{u}_n, \tilde{v}_n)}$ *tends to* $0$ *as* $n \to \infty$, *and* $(u_n, v_n) := s_n^\beta \star (\tilde{u}_n, \tilde{v}_n)$ *satisfies*

(i) $(u_n, v_n) \in \mathcal{P}_\beta$ *for every* $n$;

(ii) $\mathcal{E}_\beta(u_n, v_n) \to e_\beta$ *as* $n \to \infty$;

(iii) $||\nabla(\mathcal{E}_\beta|_\mathcal{S})(u_n, v_n)|| \to 0$ *as* $n \to \infty$.

*If moreover* $w_n, z_n \geq 0$ *a.e. in* $\mathbb{R}^5$, *then we have*

(iv) $u_n^-, v_n^- \to 0$ *a.e. in* $\mathbb{R}^5$ *as* $n \to \infty$.

Finally, we prove the Lusternik-Schnirelman type result.

**Lemma 17** *Fix* $\beta < \beta_{k+p}$ *and suppose that* $c = c_{j,\beta} = c_{j+1,\beta} = \cdots = c_{j+p,\beta}$ *for some* $j \geq 1$, $p \geq 0$. *If* $c \neq \ell$, *then* $\gamma(\mathcal{K}_c^+) > p$.

**Proof.** We use the contradiction arguments. Assume that $\gamma(\mathcal{K}_c^+) \leq p$. We have that $\gamma(\mathcal{K}_c^+)$ is compact from Lemma 6. Then by Lemma 12 $(v)$, there exists an open $\sigma$-invariant neighborhood $N$ of $\mathcal{K}_c^+$ in $\mathcal{W}_\beta$ such that $\gamma(\bar{N}) \leq p$. For each fixed $D \in \mathcal{B}_{j+p,\beta}$, we have $\gamma(D) \geq j + p$. Note that $D \subset (D \setminus N) \cup \bar{N}$, then we can check that $\gamma(D \setminus N) \geq j$ by Lemma 12 $(ii)$. This means that $D \setminus N \in \mathcal{B}_{j,\beta}$. On the other hand, by the definition of $e_j = c_j$, we infer that $(D \setminus N) \cap E_{\beta,e_j} \neq \emptyset$, where $E_{\beta,e_j}$ is the superlevel set $\{E_\beta \geq e_j\}$. Let $F := E_{\beta,c_j} \setminus N$. Then we deduce that $F$ is closed $\sigma$-invariant. In particular, $F \cap D \neq \emptyset$ for every $D \in \mathcal{B}_{j+p,\beta}$.

Let $D_n$ be a minimizing sequence for $e_{j+p}$. By using the same argument as in [27, lemma 2-10], we can suppose that $D_n \subset \mathcal{P}_\beta$ for every $n$. Also, as in the proof of Lemma 14, we can assume that any $(u, v) \in D_n$ such that $u, v \geq 0$. Then by applying [13, Theorem 7-2], we deduce that there exists a Palais-Smale sequence $\{(\tilde{u}_n, \tilde{v}_n)\}$ for $E_\beta$ on $\mathcal{S}$ at level $e_{j+p}$ such that

$$dist_{H^1(\mathbb{R}^5,\mathbb{R}^2)}((\tilde{u}_n, \tilde{v}_n), D_n) \to 0, \tag{27}$$

and

$$dist_{H^1(\mathbb{R}^5,\mathbb{R}^2)}((\tilde{u}_n, \tilde{v}_n), E_{\beta,e_j} \setminus N) \to 0. \tag{28}$$

Since $D_n$ is compact, the first condition implies the existence of $(w_n, z_n) \in \mathcal{P}_\beta$ with the properties $(a)$ and $(b)$ of Proposition 16, and $w_n, z_n \geq 0$ a.e. in $\mathbb{R}^5$ for every $n$. Therefore, we infer from Proposition 16 that $s_n^\beta = s_(\tilde{u}_n, \tilde{v}_n)^\beta$ tends to 0 as $n \to \infty$, and $(u_n, v_n) = s_n^\beta \star (\tilde{u}_n, \tilde{v}_n)$ is a Palais-Smale sequence for $\mathcal{E}_\beta$ at level $e_j$, with $(u_n, v_n) \in \mathcal{P}_\beta$ for every $n$ and $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^5$. Furthermore, we infer from Proposition 6 that $(u_n, v_n) \to (u, v) \in \mathcal{K}_c^+$ strongly in $H^1$ up to a subsequence since $e_j = c_j \neq \ell$.

We infer from [27, lemma 2.6] that $(\tilde{u}_n, \tilde{v}_n) = (-s_n^\beta) \star (u_n, v_n) \to (u, v) \in \mathcal{K}_c^+$, as $n \to \infty$. Moreover, we have

$$dist_{H^1(\mathbb{R}^5,\mathbb{R}^2)}((\tilde{u}_n, \tilde{v}_n), \mathcal{K}_c^+) \to 0.$$

On the other hand, by (27) and by (28), we infer that there exists a constant $C > 0$ such that for every $n$ large

$$dist_{H^1(\mathbb{R}^5,\mathbb{R}^2)}((\tilde{u}_n, \tilde{v}_n), \mathcal{K}_c^+) \geq \inf_{(w,z) \in E_{\beta,e_j} \setminus N} dist_{H^1(\mathbb{R}^5,\mathbb{R}^2)}((w, z), \mathcal{K}_c^+) - o(1) \geq C$$

by definition of $N$. This is a contradiction. ■

## 5.2 Proof of the main results

In this subsection we are devoted to giving the proof Theorem 1. Combining Proposition 6 and Lemmas 14 and 17, it suffices to show that

$$\ell < c_{2,\beta} \leq \cdots \leq c_{k+1,\beta} < h_\beta^\sigma$$

for $\beta < \beta_k$. We first prove the first inequality $\ell < c_{2,\beta}$. This is a consequences of the following statement.

**Lemma 18** *There exists $\delta > 0$ such that the nonnegative closed sublevel set*

$$\mathcal{E}_{\mathcal{P}_\beta^+}^{\ell+\delta} := \left\{(u,v) \in \mathcal{P}_\beta : u, v \geq 0 \; a.e. \; in \; \mathbb{R}^5, \mathcal{E}_\beta(u,v) \leq \ell + \delta\right\}$$

*has genus* 1.

**Proof.** We first claim that for every $\varepsilon > 0$, there exists $\delta > 0$ such that $(u,v) \in \mathcal{P}_\beta$, $u, v \geq 0$ a.e. in $\mathbb{R}^5$ and $\mathcal{E}_\beta(u,v) \leq \ell+\delta$ implies

$$\text{either } ||u - w_0||_{H^1} + ||v||_{\mathcal{D}^{1,2}} < \varepsilon, \text{ or } ||v - w_0||_{H^1} + ||u||_{\mathcal{D}^{1,2}} < \varepsilon. \tag{29}$$

If this is not true, it means that we could find $\varepsilon > 0$ and a sequence $\{(w_n, z_n)\} \subset \mathcal{P}_\beta$, $w_n, z_n \geq 0$ a.e. in $\mathbb{R}^5$, such that $\mathcal{E}_\beta(w_n, z_n) \to \ell$ and

$$\text{both } ||w_n - w_0||_{H^1} + ||z_n||_{\mathcal{D}^{1,2}} \geq \varepsilon, \text{ and } ||z_n - w_0||_{H^1} + ||w_n||_{\mathcal{D}^{1,2}} \geq \varepsilon. \tag{30}$$

Since $\{(w_n, z_n)\} \subset \mathcal{P}_\beta$, we have

$$E_\beta(w_n, z_n) = \mathcal{E}_\beta(w_n, z_n) = \frac{1}{6}||(w_n, z_n)||^2_{\mathcal{D}^{1,2}} \to \ell.$$

We infer from Lemmas 7 and 9 that $\{(w_n, z_n)\}$ is a bounded minimizing sequence for $E_\beta$ on $\mathcal{W}_\beta$. By Ekeland's variational principle, we know that there exists a Palais-Smale sequence $\{(\tilde{u}_n, \tilde{v}_n)\}$ for $E_\beta$ on $\mathcal{W}_\beta$, where $||(\tilde{u}_n, \tilde{v}_n) - (w_n, z_n)||_{H^1} \to 0$ as $n \to \infty$. Next, let $s_n^\beta := s^\beta_{(\tilde{u}_n, \tilde{v}_n)}$ and $(u_n, v_n) = s_n^\beta \star (\tilde{u}_n, \tilde{v}_n)$. Then by Proposition 16 we can find that $\{(u_n, v_n)\}$ is a Palais-Smale sequence for $\mathcal{E}_\beta$ on $\mathcal{S}$ at level $\ell$, with $(u_n, v_n) \in \mathcal{P}_\beta$ for every $n$, and $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^5$, and $s_n^\beta \to 0$ as $n \to \infty$.

From Proposition 6, we infer that one of the alternatives $(i)$ and $(ii)$ holds. First, we show that $(ii)$ cannot occur. Using the fact $||(\tilde{u}_n, \tilde{v}_n) - (w_n, z_n)||_{H^1} \to 0$ and (30) we can conclude that

$$\text{both } ||\tilde{u}_n - w_0||_{H^1} + ||\tilde{v}_n||_{\mathcal{D}^{1,2}} \geq \varepsilon, \text{ and } ||\tilde{v}_n - w_0||_{H^1} + ||\tilde{u}_n||_{\mathcal{D}^{1,2}} \geq \varepsilon. \tag{31}$$

Assume that $(ii)$ holds. Then we have $u_n \to w_0$ strongly in $H^1$ and $v_n \to 0$ strongly in $\mathcal{D}^{1,2}$. Moreover, by using the fact $s_n^\beta \to 0$, we deduce that $\tilde{u}_n = -s_n^\beta \star u_n \to w_0$ strongly in $H^1$ and $||\tilde{v}_n||_{\mathcal{D}^{1,2}} = s^{-s_n^\beta}||v_n||_{\mathcal{D}^{1,2}} \to 0$. This contradicts with (31). Thus, we know that the alternative $(i)$ holds. That is, $(\tilde{u}_n, \tilde{v}_n) \to (u_\beta, v_\beta)$ in $H^1$, where $(u_\beta, v_\beta)$ is a positive solution to (1). Moreover, $(u_\beta, v_\beta)$ achieves the minimum of $E_\beta$ on $\mathcal{W}_\beta$. By the strong maximum principle we have $u_\beta, v_\beta > 0$ in $\mathbb{R}^5$, then $\int_{\mathbb{R}^5} \psi_{u_\beta}|v_\beta|^2 = \int_{\mathbb{R}^5} \psi_{v_\beta}|u_\beta|^2 > 0$. But by (29) we have

$$\ell = E_\beta(u_\beta, v_\beta) > E_0(u_\beta, v_\beta) \geq \inf_{(u,v)\in\mathcal{W}_0} \mathcal{E}_0 = h_0 = \ell.$$

This is a contradiction. Then the claim (29) holds.

We take $\varepsilon > 0$ sufficiently small such that $||u - w_0||_{H^1}$. This implies $||u||_{\mathcal{D}^{1,2}} > \varepsilon$. For any such $\varepsilon > 0$, there exists positive $\delta > 0$ such that $\mathcal{E}_{\mathcal{P}_\beta}^{\ell+\delta}$, where

$$D := \left\{(u,v) \in \mathcal{P}_\beta \middle| \begin{array}{l} \text{either } ||u - w_0||_{H^1} + ||v||_{\mathcal{D}^{1,2}} < \varepsilon, \\ \text{or } ||v - w_0||_{H^1} + ||u||_{\mathcal{D}^{1,2}} < \varepsilon \end{array} \right\}.$$

We define $D = D_1 \cup D_2$, where

$$D_1 = \{(u,v) \in \mathcal{P}_\beta : ||u - w_0||_{H^1} + ||v||_{\mathcal{D}^{1,2}} < \varepsilon\} \quad \text{and} \quad D_2 = \sigma(D_1).$$

By the choice of $\varepsilon$, we know that $D_1 \cap D_2 = \emptyset$. From this we find that $D$ is the disjoint of two closed sets with $D_2 = \sigma(D_1)$. Then by the definition of genus, we have $\gamma(D) = 1$. Finally, by the monotonicity property of the genus i.e. 12 $(i)$, we have $\gamma(\mathcal{E}^{\ell+\delta}_{\mathcal{P}_\beta}) \leq \gamma(D) = 1$, and $\mathcal{E}^{\ell+\delta}_{\mathcal{P}_\beta}$ is not empty. Hence $\gamma(\mathcal{E}^{\ell+\delta}_{\mathcal{P}_\beta}) = 1$. This completes the proof. ■

Now we are ready to give the proof of Theorem 1.

**Proof of Theorem 1.** Since $\beta$ is fixed, we omit such dependence. Since $c_k \leq c_{k+1}$ for every $k \geq 1$. Thus, we need to prove that $c_2 > c_1 = \inf_{\mathcal{P}} \mathcal{E} = \ell$. To accomplish this we use the contradiction arguments. Suppose that $c_2 = c_1$. Then there exists a sequence $\{M_n\} \subset \mathcal{M}_2$ with $\sup_{M_n} \mathcal{E} \to \ell$, and for every $n$ large enough. Moreover, we have $\sup_{M_n} \mathcal{E} \leq \ell + \varepsilon$. Then by Lemma 12 $(iii)$ we have $\gamma(|M_n|) \geq \gamma(M_n) \geq 2$ for every $n$, where $|M_n| := \{(|u|, |v|) : (u, v) \in M_n\}$. On the other hand, for every $(u, v) \in \mathcal{P}$ we have $\mathcal{E}(u, v) = \mathcal{E}(|u|, |v|)$. Hence we deduce that $|M_n| \subset \mathcal{E}^{\ell+\delta}_{\mathcal{P}^+}$. We infer from Lemma 12 $(i)$ that $\gamma(|M_n|) \leq \gamma(\mathcal{E}^{\ell+\delta}_{\mathcal{P}^+})$. Finally, From Lemma 18 we infer that $\gamma(|M_n|) \leq \gamma(\mathcal{E}^{\ell+\delta}_{\mathcal{P}^+}) = 1$. This is a contradiction. ■

# Acknowledgments

This work was supported by the National Natural Science Foundation of China (Nos: 11971202, 11571140, 11671077) and the Natural Science Foundation of Jiangsu Province (Nos: BK20150478, BK20170542).

# References

[1] A. Ambrosetti, E. Colorado. Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc.*, 75(2007):67–82.

[2] M. Badiale, E. Serra, Semilinear Elliptic Equations for Beginners. Existence Results via the Variational Approach. Universitext, Springer, London. 2011.

[3] T. Bartsch, S. de Valeriola. Normalized solutions of nonlinear Schrödinger equations. *Arch. Math.(Basel)*, 100(2013):75–83.

[4] T. Bartsch, L. Jeanjean, N. Soave. Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$. *J. Math. Pures Appl.*, 106(2016): 583–614.

[5] T. Bartsch, N. Soave. A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. *J. Funct. Anal.*, 272(2017). 4998–5037.

[6] T. Bartsch, N. Soave. Multiple normalized solutions for a competing system of Schrödinger equations. *Calc. Var. Partial Differential Equations*, 22(2019):1–32.

[7] C. Le Bris, P.-L. Lions. From atoms to crystals: a mathematical journey. *Bull. Amer. Math. Soc.(N.S.)*, 42(2005):291–363.

[8] E. Cancés, C. Le Bris. On the time-dependent Hartree-Fock equations coupled with a classical nuclear dynamics. *Math. Models Methods Appl. Sci.*, 9(1999):963–990.

[9] J.-M. Chadam, R.-T. Glassey, Global existence of solutions to the Cauchy problem for time-dependent Hartree equations, *J. Mathematical Phys.*, 16(1975): 1122–1130.

[10] E. Dancer, J. Wei, T. Weth. A priori bounds versus multiple existence of positive solutions for a nonlinear Schrödinger system. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 27(2010):953–969.

[11] V. Georgiev, G. Venkov. Symmetry and uniqueness of minimizers of Hartree type equations with external Coulomb potential. *J. Differential Equations*, 251(2011):420–438.

[12] M. Ghimenti, J. Van Schaftingen. Nodal solutions for the Choquard equation. *J. Funct. Anal.*, 271(2016):107–135.

[13] N. Ghoussoub. Duality and Perturbation Methods in Critical Point Theory, Cambridge Tracts in Mathematics. Cambridge Univerity Press, Cambridge. 1993.

[14] L. Jeanjean. Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.*, 28(1997): 1633–1659.

[15] L. Jeanjean, T. Bartsch. Normalized solutions for nonlinear schrödinger systems. *Proceedings of the Royal Society of Edinburgh A.*, 148(2019):225–242.

[16] L. Jeanjean, T.-J. Luo, Z.-Q. Wang. Multiple normalized solutions for quasi-linear Schrödinger equations. *J. Differential Equations*, 259(2015):3894–3928.

[17] E.-H. Lieb. Existence and uniqueness of theminimizing solution of Choquard's nonlinear equation. *Stud. Appl. Math.*, 57(1976):93–105.

[18] E.-H. Lieb, M. Loss. Analysis, Graduate Studies in mathematics. American Mathematical Society, Providence, RI, second edition(2001).

[19] E.-H. Lieb, B. Simon. The Hartree-Fock theory for Coulomb systems. *Comm. Math. Phys.*, 53(1977):185–194.

[20] P.-L. Lions. Solutions of Hartree-Fock equations for Coulomb systems. *Comm. Math. Phys.*, 109(1987):33–97.

[21] P.-L. Lions. Some remarks on Hartree equation. *Nonlinear Anal.*, 5(1981):1245–1256.

[22] L. Ma, L. Zhao. Classification of positive solitary solutions of the nonlinear Choquard equation. *Arch. Ration. Mech. Anal.*, 195(2010):455–467.

[23] V. Moroz, J. Van Schaftingen. Groundstates of nonlinear Choquard equations:existence, qualitative properties and decay asymptotics. *J.Funct.Anal.*, 265(2013):153–184.

[24] R.-S. Palais. The principle of symmetic criticality, *Comm. Math. Phys.*, 69(1979):19–30.

[25] J. Wang. Standing waves solutions for the coupled Hartree-Fock type nonlocal elliptic system, *Submitted*, 2018.

[26] J. Wang, Y.-Y. Dong. Bonded states solutions for a coupled nonlinear hartree equations with nonlocal interaction, *Submitted*, 2018.

[27] J. Wang, C. Wang, L. Xiao. Existence of normalized positive ground state solution of the coupled nonlocal elliptic system, *Submitted*, 2019.

[28] J. Wang, J.-P. Shi. Standing waves for a coupled nonlinear Hartree equations with nonlocal interaction, *Calc. Var. Partial Differential Equations*, 56(2017):1-38.

[29] J. Wang, Z.-Q. Wang. Existence of odd solutions for the weakly coupled hartree type elliptic system with nonlocal interaction, *Preprint*, 2018.

[30] J. Wang, W. Yang. Normalized solutions and asymptotical behavior of minimizer for the coupled Hartree equations. *J. Differential Equations*, 265(2018):501–544.

[31] J. Wang, W. Yang. Bound and ground state solutions of coupled nonlinear hartree system with nonlocal interaction. *Preprint, Submitted*, 2018.

[32] T. Wang, T.-S. Yi. Uniqueness of positive solutions of the Choquard type equations. *Appl. Anal.*, 96(2017):409–417.

[33] M.-B. Yang, Y.-H. Wei, Y.-H. Ding. Existence of semiclassical states for a coupled Schrödinger system with potentials and nonlocal nonlinearities. *Z. Angew. Math. Phys.*, 65(2014):41–68.

[34] X.-Y. Zeng, Y.-J. Guo, H.-S. Zhou. Blow-up solutions for a nonlinear Schrödinger system with Intraspecies and Interspecies Interactions. *J. Differential Equations*, 264(2018):1411–1441.

[35] X.-Y. Zeng, Y.-J. Guo, Z.-Q. Wang, H.-S. Zhou. Properties of ground states of attractive Gross-Pitaevskii equations with multi-well potentials. *Nonlinearity*, 31(2018):957–979.