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# Carleman Estimation for One-Dimensional Higher Order Viscous Camassa-Holm Equation

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**Abstract:** In this paper, we establish a global Carleman estimate for the higher-order linear Camassa-Holm equation, which can be obtained by combining the inequality for the elliptic operator with the one of parabolic operator. Based on the estimate, we can follow up on its controllability, observability and unique continuation property in the future.

**Keywords:** Higher order; Viscous Camassa-Holm equation; Carleman estimation

## 1 Introduction

The Camassa-Holm(CH) equation describes the unidirectional propagation of waves at the free surface of shallow water under the influence of gravity. In the dispersionless limit, the CH equation is given by

$$y_t - y_{xxt} = -3yy_x + 2y_x y_{xx} + yy_{xxx},$$

where the fluid velocity vector $y$ is a function of position $x \in \mathbb{R}$ and time $t \in \mathbb{R}$ .

The formulation of the higher-order CH equation was recently derived by Coclite, Holden and Karlsen [1]

$$\begin{cases} B_k(y,y) := A_k^{-1} C_k(y) - yy_x, \\ A_k(y) := \sum\limits_{j=0}^{k} (-1)^j \, \partial_x^{2j} y, \\ C_k(y) := -yA_k(\partial_x y) + A_k(y\partial_x y) - 2\partial_x y A_k(y). \end{cases} \tag{1}$$

When $k = 2$, system (1) becomes

$$(y - y_{xx} + y_{xxxx})_t - 3yy_x + 2y_x y_{xx} + yy_{xxx} - 2y_x y_{xxxx} - yy_{xxxxxxx} = 0.$$

In recent years, many researchers extend the studies of the CH equation to the generalized CH equation, higher-order CH equation, and so on. Tian, Shen and Ding gave the optimal control of the viscous CH equation under the boundary condition and proved the existence and uniqueness of optimal solution to the viscous CH equation in a short interval [2]. Using geometrical methods, higher order CH equation has been treated by Constantin and Kolev [3].

Although some researchers have begun to pay attention to the controllability of CH equation [4, 5], it is still in its infancy and remains open. In this paper, we will establish the Carleman estimates for some future researches.

We consider the higher order CH equation with viscous term on a bounded interval

$$\begin{cases} (y - y_{xx} + y_{xxxx})_t - \gamma(y - y_{xx} + y_{xxxx})_{xx} - 3yy_x + 2y_x y_{xx} + yy_{xxx} - 2y_x y_{xxxx} - yy_{xxxxxxx} = 0 \ \ in \ \ Q, \\ y(t,0) = y(t,1) = y_{xx}(t,0) = y_{xx}(t,1) = y_{xxxx}(t,0) = y_{xxxx}(t,1) = 0 \ \ in \ \ (0,T), \\ y(0,x) = y_0(x), \end{cases} \tag{2}$$

where $Q = (0,T) \times (0,1)$ and $\gamma(y - y_{xx} + y_{xxxx})_{xx}$ is the viscous term.

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Let $P$ be higher-order linear viscous CH operator :

$$Py := (y - y_{xx} + y_{xxxx})_t - \gamma(y - y_{xx} + y_{xxxx})_{xx},$$

which is defined on

$$\mathcal{Y} = \left\{ y \in L^2(0,T;H^6(0,1)) \,|y(t,0) = y(t,1) = y_{xx}(t,0) = y_{xx}(t,1) = y_{xxxx}(t,0) = y_{xxxx}(t,1) = 0 \right\}.$$

Let $L_E$ be the fourth-order elliptic operator: $L_E y := y_{xxxx}$, which is defined on

$$\mathcal{Y}_1 = \left\{ y \in L^2(0,T;H^4(0,1)) \,|y(t,0) = y(t,1) = y_{xx}(t,0) = y_{xx}(t,1) = 0 \right\}.$$

Let $L_P$ be the second-order parabolic operator: $L_P y := y_t - \gamma y_{xx}$, which is defined on

$$\mathcal{Y}_2 = \left\{ y \in L^2(0,T;H^2(0,1)) \,|y(t,0) = y(t,1) = y_x(t,0) = y_x(t,1) = 0 \right\}.$$

In this paper, we make the following assumptions:

(i) Let $\omega_0$ be a nonempty open set of the interval $(0,1)$. For $\omega_0$, we choose other two subintervals $\omega_1$ and $\omega$, satisfying $\omega_0 \subset \omega_1 \subset \omega$. $Q^\omega = \omega \times (0,T), Q^{\omega_0} = \omega_0 \times (0,T), Q^{\omega_1} = \omega_1 \times (0,T)$;

(ii) Choose $\psi(x) \in C^\infty[0,1]; \forall x \in [0,1], \psi > 0; \psi'(0) = \psi'(1) = 0; \|\psi\|_{C[0,1]} = 1$ and let $a(x,t) = \frac{e^{\mu(\psi+3)} - e^{5\mu}}{t(T-t)}$, $\phi(x,t) = \frac{e^{\mu(\psi+3)}}{t(T-t)}, \eta(x,t) = \lambda a(x,t)$ for any given parameters $\mu, \lambda > 0$. Here, $\eta(x,t)$ is the weight function, which is very important in Carleman estimate.

Carleman estimate is an $L^2$-weighted estimate with large parameters for solutions to partial differential equations (PDEs). Carleman estimates were first established by Carleman for a two-dimensional elliptic equation [6]. Local and global Carleman estimates play a central role in the study of partial differential equations regarding questions such as controllability, observation and unique continuation. At present, there are many results about Carleman estimates of partial differential equations, such as transport equation [7], second order parabolic equation [8], second order wave equation [9], second order Schördinger equation [10], Korteweg-de Vries equation [11], and so on.

Before establishing estimates, we need to know the well-posedness results for solutions of the higher-order CH equation. Tian, Zhang and Xia have proved the local well-posedness of the higher-order CH equation and obtained the global existence of solution [12]. Here, we only list their results.

**Proposition 1** $[12, Theorem 2.1]$ *Given $u_0 \in H^s(\mathbb{R}), s > \frac{9}{2}$, and there existing $T > 0$ depending on $\|u_0\|_{H^s}$, then Eq. (2) admits a unique solution $u = u(x,t)$, such that $u \in C([0,T], H^s(\mathbb{R})) \cap C([0,T], H^{s-1}(\mathbb{R}))$.*

*Moreover, the map $u_0 \in H^s(\mathbb{R}) \to u \in C([0,T], H^s(\mathbb{R})) \cap C([0,T], H^{s-1}(\mathbb{R}))$ is continuous.*

**Proposition 2** $[12, Theorem 4.1]$ *Let $u_0 \in H^s(\mathbb{R}), s > \frac{9}{2}$, then the corresponding strong solution $u = u(x,t)$ to Eq. (2) with the initial $u_0$ exists globally in time.*

The rest of this paper is organized as follows. In Section 2.1 and Section 2.2, we establish Carleman estimations of the fourth-order elliptic operator and the second-order parabolic operator respectively. In Section 2.3, we obtain a global Carleman estimate for the higher-order linear CH operator.

# 2 Carleman estimate of the higher-order linear CH operator

## 2.1 Carleman estimate of the fourth-order elliptic operator

**Theorem 1** *There exist constants $\mu_0, C(\psi)$, $C > 0$, such that the following estimation holds:*

$$\begin{aligned}
&\int_Q \left( \frac{1}{\lambda\mu\phi}\theta^2 y_{xxxx}^2 + \lambda^7\mu^7\phi^7\theta^2 y^2 + \lambda^5\mu^5\phi^5\theta^2 y_x^2 + \lambda^3\mu^3\phi^3\theta^2 y_{xx}^2 + \lambda\mu\phi\theta^2 y_{xxx}^2 \right) dxdt \\
&\le C\left[ \int_Q \theta^2 f^2 dxdt + \int_{Q^\omega} (\lambda^7\mu^7\phi^7\theta^2 y^2 + \lambda^5\mu^5\phi^5\theta^2 y_x^2) dxdt \right]
\end{aligned} \tag{3}$$

*for $\mu > \mu_0$, $\lambda > \mu C(\psi)\left(T + T^2\right)$.*

In order to prove Theorem 1, we need the following lemmas.

**Lemma 1** *Let* $L_E y = y_{xxxx} = f, \quad \theta(x,t) = e^{\eta(x,t)}, \quad u = \theta y = e^{\eta} y,$ *then*

$$\theta f = u_t + A_0 u + A_1 u_x + A_2 u_{xx} + A_3 u_{xxx} + u_{xxxx} = I_1(u) + I_2(u) + R(u), \tag{4}$$

*where*

$$A_0 = \eta_x^4 + 4\eta_x\eta_{xx} - 6\eta_x^2\eta_{xx} + 3\eta_{xx}^2 - \eta_{xxxx}, \quad A_1 = -4\eta_x^3 + 12\eta_x\eta_{xx} - 4\eta_{xxx}$$
$$A_2 = 6\eta_x^2 - 6\eta_{xx} A_3 = -4\eta_x$$

*We note that:* $I_1(u) = B_0 u + B_1 u_x + B_3 u_{xxx}$, $I_2(u) = C_0 u + C_1 u_x + B_2 u_{xx} + u_{xxxx}$, *where the coefficients* $B_0, C_0, B_1, C_1, B_2, B_3$ *are undetermined functions such as* $A_i$.

**Lemma 2** *The product of* $I_1(u)$ *and* $I_2(u)$ *in formula (2.2) can be expressed as:*

$$2I_1 \cdot I_2 = u^2\{\cdot\} + u_x^2\{\cdot\} + u_{xx}^2\{\cdot\} + u_{xxx}^2\{\cdot\} + \{\cdot\}_x + \{\cdot\}_{xx} + \{\cdot\}_{xxx} + \{\cdot\}_{xxxx}. \tag{5}$$

The idea comes from [13, 14]. Here we omit the proof process.

We choose: $B_0 = -4\eta_x^2\eta_{xx}$, $B_1 = -4\eta_x^3$, $B_2 = 6\eta_x^2$, $B_3 = -4\eta_x$, $C_0 = \eta_x^4$, $C_1 = 12\eta_x\eta_{xx}$, then

$$u^2\{\cdot\} = u^2\{2\eta_x^6\eta_{xx} + 96\eta_x^2\eta_{xx}^3 + 48\eta_x^3\eta_{xx}\eta_{xxx} - 8\eta_x^2\eta_{xxxx} + 20\eta_x^4\eta_{xxxx} - 16\eta_{xx}^3$$
$$-480\eta_{xx}\eta_{xxx}^2 - 320\eta_{xx}^2\eta_{xxxx} - 320\eta_x\eta_{xxx}\eta_{xxxx} - 160\eta_x\eta_{xx}\eta_{xxxxx} - 16\eta_x^2\eta_{xxxxxx}\}$$

$$u_x^2\{\cdot\} = u_x^2\left\{12\eta_x^4\eta_{xx} - 576\eta_x\eta_{xx}\eta_{xxx} - 192\eta_{xx}^3 - 96\eta_x^2\eta_{xxxx}\right\}$$

$$u_{xx}^2\{\cdot\} = u_{xx}^2\left\{124\eta_x^2\eta_{xx}\right\}, u_{xxx}^2\{\cdot\} = u_{xxx}^2\{4\eta_{xx}\}$$

$$\{\cdot\}_x + \{\cdot\}_{xx} + \{\cdot\}_{xxx} + \{\cdot\}_{xxxx} = \{G\}_x = \left\{u_x^2\left[-16\eta_x^5 + 96\eta_x\eta_{xx}^2 + 48\eta_x^2\eta_{xx}\right] - 4\eta_x u_{xxx}^2 - 8\eta_x^3 u_x u_{xxx}\right\}_x$$
$$R(u) = R_0 u + R_1 u_x + R_2 u_{xx},$$

where $R_0 = 4\eta_x\eta_{xx} - 2\eta_x^2\eta_{xx} + 3\eta_{xx}^2 - \eta_{xxxx}$, $R_1 = -4\eta_{xxx}$, $R_2 = -6\eta_{xx}$.

Now, Theorem 2.1 can be proved in the following.

**Proposition 3** *There exist constants* $\mu_0$, $C(\psi) > 0$, *such that the following estimate :*

$$\int_Q \left(I_1^2 + I_2^2 + 2\lambda^7\mu^8\phi^7\psi_x^8 u^2 + 12\lambda^5\mu^6\phi^5\psi_x^6 u_x^2 + 124\lambda^3\mu^4\phi^3\psi_x^4 u_{xx}^2 + 4\lambda\mu^2\phi\psi_x^2 u_{xxx}^2\right) dxdt$$
$$+ \int_0^T [G(1,t) - G(0,t)]\, dt \le C\left[\int_Q \theta^2 f^2 dxdt + \int_Q (\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2) dxdt\right] \tag{6}$$

*holds for* $\mu > \mu_0$, $\lambda > \mu C(\psi)(T + T^2)$.

**Proof.** According to the definition and properties of the weight function $\eta(x,t)$, we can estimate the items in above one by one.

$$u^2\{\cdot\} = u^2\left\{2\lambda^7\mu^8\phi^7\psi_x^8 + D_0\right\}, \quad |D_0| \le C(\psi)\lambda^7\mu^7\phi^7;$$

$$u_x^2\{\cdot\} = \left\{12\lambda^5\mu^6\phi^5\psi_x^6 + D_1\right\} u_x^2, \quad |D_1| \le C(\psi)\lambda^5\mu^5\phi^5;$$

$$u_{xx}^2\{\cdot\} = u_{xx}{}^2\left\{124\eta_x^2\eta_{xx}\right\} = u_{xx}^2\left\{124\lambda^3\mu^4\phi^3\psi_x^4 + D_2\right\}, \quad |D_2| \le C(\psi)\lambda^3\mu^3\phi^3;$$

$$u_{xxx}^2\{\cdot\} = u_{xxx}^2\{4\eta_{xx}\} = u_{xxx}^2\left\{4\lambda\mu^2\phi\psi_x^2 + D_3\right\}, \quad |D_3| \le C(\psi)\lambda\mu\phi.$$

Similarly, we have from $R(u) = R_0 u + R_1 u_x + R_2 u_{xx}$ that

$$|R(u)| \le C(\psi)(\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2).$$

We note that: $\int_Q \{G\}_x\, dxdt = \int\limits_0^T \int\limits_0^1 \{G\}_x\, dxdt = \int\limits_0^T [G(1,t) - G(0,t)]\, dt$, $G(1,t) = G_1(1,t) + G_2(1,t)$.
According to the definition and properties of the weight function $\eta(x,t), a(x,t), \phi(x,t)$ and the initial values

$$y(t,0) = y(t,1) = y_{xx}(t,0) = y_{xx}(t,1) = y_{xxxx}(t,0) = y_{xxxx}(t,1) = 0;\ \ \psi_x(1) < 0,$$

we can get that
$G_1(1,t) = \left\{-4\eta_x u_{xxx}^2 - 8\eta_x^3 u_x u_{xxx}\right\}(1,t) \geq \left\{-4\lambda^5\mu^5\phi^5\psi_x^5 u_x^2 - 8\lambda\mu\phi\psi_x u_{xxx}^2\right\}(1,t) \geq 0,$

$$G_2(1,t) = \left\{(-16\eta_x^5 + 96\eta_x\eta_{xx}^2 + 48\eta_x^2\eta_{xx})u_x^2\right\}(1,t) \geq 0.$$

Thus, $G(1,t) \geq 0$. Similarly, $-G(0,t) \geq 0$. Then

$$\int_Q \{G\}_x\, dxdt = \int\limits_0^T [G(1,t) - G(0,t)]\, dt \geq 0. \tag{7}$$

Since $\theta f = I_1(u) + I_2(u) + R(u)$, we have

$$\int_Q \left(I_1^2 + I_2^2 + 2I_1 I_2\right) dxdt = \int_Q |I_1(u) + I_2(u)|^2\, dxdt = \int_Q |\theta f - R(u)|^2\, dxdt. \tag{8}$$

Combining (5), (7) and (8) and all estimates in Proposition 3, there exists a constant $C$ such that (7) holds. This completes the proof of Proposition 3. ■

**Proposition 4** *There exist constants $\mu_0, C(\psi), C > 0$, such that*

$$\begin{aligned}&\int_Q \left(I_1^2 + I_2^2 + \lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2 + \lambda\mu\phi u_{xxx}^2\right) dxdt \\ &\quad \leq C\left[\int_Q \theta^2 f^2 dxdt + \int_{Q^{\omega_0}} (\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2 + \lambda\mu\phi u_{xxx}^2) dxdt\right]\end{aligned} \tag{9}$$

*holds for $\mu > \mu_0, \lambda > \mu C(\psi)(T + T^2)$.*

**Proof.** In fact, by (3), we have

$$\int_Q \left(I_1^2 + I_2^2\right) dxdt \leq C\left[\int_Q \theta^2 f^2 dxdt + \int_Q (\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2) dxdt\right] \tag{10}$$

and

$$\begin{aligned}&\int_Q \left(\lambda^7\mu^8\phi^7\psi_x^8 u^2 + \lambda^5\mu^6\phi^5\psi_x^6 u_x^2 + \lambda^3\mu^4\phi^3\psi_x^4 u_{xx}^2 + \lambda\mu^2\phi\psi_x^2 u_{xxx}^2\right) dxdt \\ &\quad \leq C\left[\int_Q \theta^2 f^2 dxdt + \int_Q (\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2) dxdt\right].\end{aligned} \tag{11}$$

Noticing that $|\psi_x| > 0, \forall x \in (0,1)\backslash\omega_0$, we can choose $\mu_0 = C(\psi) + 1$ such that

$$\begin{aligned}&\int_{Q\backslash Q^{\omega_0}} \left(\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2 + \lambda\mu\phi u_{xxx}^2\right) dxdt \\ &\quad \leq C(\psi)\left[\int_Q \theta^2 f^2 dxdt + \int_{Q^{\omega_0}} (\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2 + \lambda\mu\phi u_{xxx}^2) dxdt\right],\end{aligned}$$

which holds for $\mu > \mu_0$. That is to say that the following estimate holds.

$$\begin{aligned}&\int_Q \left(\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2 + \lambda\mu\phi u_{xxx}^2\right) dxdt \\ &\quad \leq C\left[\int_Q \theta^2 f^2 dxdt + \int_{Q^{\omega_0}} (\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2 + \lambda\mu\phi u_{xxx}^2) dxdt\right].\end{aligned} \tag{12}$$

Then it follows from (10) that

$$\int_Q \left(I_1^2 + I_2^2\right) dxdt \le C\left[\int_Q \theta^2 f^2 dxdt + \int_{Q^{\omega_0}} (\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2 + \lambda\mu\phi u_{xxx}^2) dxdt\right]. \tag{13}$$

In view of (12) and (13), we obtain the desired result of this proposition. ■

**Proposition 5** *Under the same conditions in Proposition 3, we achieve the following estimate:*

$$\begin{aligned}&\int_Q \left(I_1^2 + I_2^2 + \lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2 + \lambda\mu\phi u_{xxx}^2\right) dxdt \\ &\le C\left[\int_Q \theta^2 f^2 dxdt + \int_{Q^{\omega_0}} (\lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2) dxdt\right].\end{aligned} \tag{14}$$

**Proof.** We select a characteristic function $\chi \equiv 1, \forall x \in \omega_0; \chi \equiv 0, \forall x \in \omega_1 \backslash \omega_0$. Using partial integration, we have

$$\begin{aligned}\int_{Q^{\omega_0}} \lambda^3\mu^3\phi^3 u_{xx}^2 dxdt &= \int_{Q^{\omega_0}} \chi\lambda^3\mu^3\phi^3 u_{xx}^2 dxdt \\ &\le \int_{Q^{\omega_1}} \chi\lambda^3\mu^3\phi^3 u_{xx}^2 dxdt \\ &= 3\lambda^3\mu^3 \int_{Q^{\omega_1}} \phi\phi_x^2 u_x^2 dxdt - \int_{Q^{\omega_1}} \chi\lambda^3\mu^3\phi^3 u_x u_{xxx} dxdt\end{aligned}$$

Noticing that $\psi \in C_0^\infty(0,1), \omega_0 \subset \omega_1 \subset \omega$, and applying Young's inequality, we can obtain that: for arbitrary $\varepsilon_1, \delta_1 > 0$, there exist constants $C, C(\varepsilon_1)$ and $C(\delta_1)$ such that

$$\begin{aligned}\int_{Q^{\omega_0}} \lambda^3\mu^3\phi^3 u_{xx}^2 dxdt &\le C\int_{Q^{\omega_1}} \lambda^3\mu^5\phi^3 u_x^2 dxdt + \varepsilon_1 \int_{Q^{\omega_1}} \lambda\mu\phi u_{xxx}^2 dxdt + C(\varepsilon_1)\int_{Q^{\omega_1}} \lambda^5\mu^5\phi^5 u_x^2 dxdt \\ &\le C\int_Q \lambda^5\mu^7\phi^5 u^2 dxdt + \varepsilon_1 \int_Q \lambda\mu\phi u_{xxx}^2 dxdt + C(\varepsilon_1)\int_{Q^{\omega_1}} \lambda^5\mu^5\phi^5 u_x^2 dxdt\end{aligned} \tag{15}$$

and

$$\int_{Q^{\omega_1}} \lambda^5\mu^5\phi^5 u_x^2 dxdt \le C\int_Q \lambda^5\mu^7\phi^5 u^2 dxdt + \delta_1 \int_Q \lambda^3\mu^3\phi^3 u_{xx}^2 dxdt + C(\delta_1)\int_{Q^\omega} \lambda^7\mu^7\phi^7 u^2 dxdt. \tag{16}$$

In the same way, for arbitrary $\varepsilon_2, \delta_2 > 0$, there exist constants $C, C(\varepsilon_2)$ and $C(\delta_2)$ such that

$$\int_{Q^{\omega_0}} \lambda\mu\phi u_{xxx}^2 dxdt \le C\int_Q \lambda\mu^3\phi u_{xx}^2 dxdt + \varepsilon_2 \int_Q \frac{1}{\lambda\mu\phi} u_{xxxx}^2 dxdt + C(\varepsilon_2)\int_{Q^{\omega_1}} \lambda^3\mu^3\phi^3 u_{xx}^2 dxdt \tag{17}$$

and

$$\int_{Q^{\omega_1}} \lambda^3\mu^3\phi^3 u_{xx}^2 dxdt \le C\int_Q \lambda^3\mu^5\phi^3 u_x^2 dxdt + \delta_2 \int_Q \lambda\mu\phi u_{xxx}^2 dxdt + C(\delta_2)\int_{Q^\omega} \lambda^5\mu^5\phi^5 u_x^2 dxdt. \tag{18}$$

For $\int_Q \frac{1}{\lambda\mu\phi} {u_{xxxx}}^2 dxdt$, by $I_2(u) = C_0 u + C_1 u_x + B_2 u_{xx} + u_{xxxx}$, we can obtain

$$\begin{aligned}\int_Q \frac{1}{\lambda\mu\phi} u_{xxxx}^2 dxdt &= \int_Q \frac{1}{\lambda\mu\phi}(I_2(u) - C_0 u - C_1 u_x - B_2 u_{xx}^2) dxdt \\ &\le C\int_Q (I_2^2 + \lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2) dxdt.\end{aligned} \tag{19}$$

Substituting (18) and (19) into (17), the following result can be obtained:

$$\begin{aligned}\int_{Q^{\omega_0}} \lambda\mu\phi u_{xxx}^2 dxdt \le\; & C\int_Q \lambda\mu^3\phi u_{xx}^2 dxdt + \varepsilon_2 \cdot C\int_Q (I_2^2 + \lambda^7\mu^7\phi^7 u^2 + \lambda^5\mu^5\phi^5 u_x^2 + \lambda^3\mu^3\phi^3 u_{xx}^2) dxdt \\ &+ C(\varepsilon_2)\cdot C\int_Q \lambda^3\mu^5\phi^3 u_x^2 dxdt + C(\varepsilon_2)\cdot\delta_2 \int_Q \lambda\mu\phi u_{xxx}^2 dxdt \\ &+ C(\varepsilon_2)\cdot C(\delta_2)\int_{Q^\omega} \lambda^5\mu^5\phi^5 u_x^2 dxdt.\end{aligned} \tag{20}$$

From (15), (20), (9) and (16), we can derive

$$\begin{aligned}
&\int_Q \left(I_1^2+I_2^2+\lambda^7\mu^7\phi^7u^2+\lambda^5\mu^5\phi^5u_x^2+\lambda^3\mu^3\phi^3u_{xx}^2+\lambda\mu\phi u_{xxx}^2\right)dxdt\\
&\le C\left[\int_Q \theta^2f^2dxdt+\int_{Q^{\omega_0}}(\lambda^7\mu^7\phi^7u^2+\lambda^5\mu^5\phi^5u_x^2)dxdt\right]+(C+C(\varepsilon_1)\cdot C)\int_Q\lambda^5\mu^7\phi^5u^2dxdt\\
&+C(\varepsilon_2)\cdot C\int_Q\lambda^3\mu^5\phi^3u_x^2dxdt+(C(\varepsilon_1)\cdot\delta_1+\varepsilon_2\cdot C)\int_Q\lambda^3\mu^3\phi^3u_{xx}^2dxdt+C\int_Q\lambda\mu^3\phi u_{xx}^2dxdt \qquad (21)\\
&+(C(\varepsilon_2)\cdot\delta_2+\varepsilon_1)\int_Q\lambda\mu\phi u_{xxx}^2dxdt+\varepsilon_2\cdot C\int_Q(I_2^2+\lambda^7\mu^7\phi^7u^2+\lambda^5\mu^5\phi^5u_x^2)dxdt\\
&+C(\varepsilon_1)\cdot C(\delta_1)\int_{Q^\omega}\lambda^7\mu^7\phi^7u^2dxdt+C(\varepsilon_2)\cdot C(\delta_2)\int_{Q^\omega}\lambda^5\mu^5\phi^5u_x^2dxdt.
\end{aligned}$$

In (21), if we select $\varepsilon_1,\varepsilon_2,\delta_1,\delta_2$ sufficiently small and $\lambda$ sufficiently large, we can get (14). The proof of Proposition 5 is completed. ■

Adding the left and the right of (14) and (19) separately, as well as returning $u$ to $\theta y$, we finish the proof of Theorem 1.

## 2.2 Carleman estimate of the second-order parabolic operator

**Theorem 2** *There exist $\mu_0, C(\psi)>0$ such that*

$$\int_Q\left[\frac{1}{\lambda\mu\phi}\theta^2(y_t^2+y_{xx}^2)+\lambda^3\mu^3\phi^3\theta^2y^2+\lambda\mu\phi\theta^2y_x^2\right]dxdt\le C\left[\int_Q\theta^2f^2dxdt+\int_{Q^\omega}\lambda^3\mu^3\phi^3\theta^2y^2dxdt\right] \qquad (22)$$

*holds for $\mu>\mu_0, \lambda>\mu C(\psi)(T+T^2)$.*

**Remark 1** *We will continue to use all the notations and hypothesises in the previous subsection.*

**Lemma 3** *Let* $y_t-\gamma y_{xx}=f$ *and* $u=\theta y=e^\eta y$, *then*

$$\theta f=u_t+\left(-\gamma\eta_x^2-\eta_t+\gamma\eta_{xx}\right)u+2\gamma\eta_xu_x-u_{xx}=I_1(u)+I_2(u)+Ru,$$

*where*

$$I_1(u)=u_t+2\gamma\eta_xu_x+\gamma\eta_{xx}u, I_2(u)=-\gamma\eta_x^2u-u_{xx},\ R(u)=-\eta_tu.$$

Then,

$$I_{1,1}=\gamma\int_0^T\int_0^1\eta_x\eta_{xt}u^2, I_{1,2}=-\int_0^T\int_0^1u_tu_{xx}=0, I_{2,1}=3\gamma^2\int_0^T\int_0^1\eta_x^2u^2\eta_{xx},$$

$$I_{2,2}=\gamma\int_0^T\int_0^1\eta_{xx}u_x^2, I_{3,1}=-\gamma^2\int_0^T\int_0^1\eta_x^2\eta_{xx}u^2, I_{3,2}=\gamma(\int_0^T\int_0^1\eta_{xx}u_x^2-\frac{1}{2}\int_0^T\int_0^1\eta_{xxxx}u^2),$$

$$(I_1u,I_2u)_2=\sum_{i=1}^3\sum_{j=1}^2I_{i,j}=I(u)+I(u_x),$$

where

$$I(u)=\gamma\iint(\eta_x\eta_{xt}+(3-\gamma)\eta_x^2\eta_{xx}-\frac{1}{2}\eta_{xxxx})u^2, I(u_x)=2\gamma\iint^2\eta_{xx}u_x^2.$$

The proof of Theorem 2 is similar to the discussion framework of Theorem 1. For simplicity, we only list the main results in this subsection. We can obtain the following estimates

$$\int_Q\left[I_1^2+I_2^2+(3\gamma-\gamma^2)\lambda^3\mu^4\phi^3\psi_x^4u^2+2\gamma\lambda\mu^2\phi\psi_x^2u_x^2\right]dxdt\le C\left[\int_Q\theta^2f^2dxdt+\int_Q(\lambda^3\mu^3\phi^3u^2+\lambda\mu\phi u_x^2)dxdt\right],$$

$$\int_Q \left( I_1^2 + I_2^2 + \lambda^3\mu^3\phi^3 u^2 + \lambda\mu\phi u_x^2 \right) dxdt \le C \left[ \int_Q \theta^2 f^2 dxdt + \int_{Q^{\omega_0}} (\lambda^3\mu^3\phi^3 u^2 + \lambda\mu\phi u_x^2) dxdt \right],$$

$$\int_Q \left( \mathrm{I}_1^2 + I_2^2 + \lambda^3\mu^3\phi^3 u^2 + \lambda\mu\phi u_x^2 \right) dxdt \le C \left[ \int_Q \theta^2 f^2 dxdt + \int_{Q^{\omega}} \lambda^3\mu^3\phi^3 u^2 dxdt \right]. \tag{23}$$

By the similar argument as (19), we can deduce from the definition of $I_1$ and $I_2$ that

$$\int_Q \frac{1}{\lambda\mu\phi} u_t^2 dxdt = \int_Q \frac{1}{\lambda\mu\phi} (I_1 - 2\gamma\eta_x u_x - \gamma\eta_{xx} u)^2 dxdt \le C \int_Q (I_1^2 + \lambda\mu^3\phi u^2 + \lambda\mu\phi u_x^2) dxdt, \tag{24}$$

$$\int_Q \frac{1}{\lambda\mu\phi} u_{xx}^2 dxdt = \int_Q \frac{1}{\lambda\mu\phi} (I_2 + \gamma\eta_x^2 u)^2 dxdt \le C \int_Q (I_2^2 + \lambda^3\mu^3\phi^3 u^2) dxdt. \tag{25}$$

Combining (23)-(25), we derive

$$\int_Q \left[ \frac{1}{\lambda\mu\phi} (u_t^2 + u_{xx}^2) + \lambda^3\mu^3\phi^3 u^2 + \lambda\mu\phi u_x^2 \right] dxdt \le C [\int_Q \theta^2 f^2 dxdt + \int_{Q^\omega} \lambda^3\mu^3\phi^3 u^2 dxdt].$$

By substituting $\theta y$ for $u$, we can also get (22).

## 2.3 Carleman estimate of higher-order linear CH operator

In this section, based on the above estimates, we shall establish the Carleman estimate for higher-order linear CH operator:

$$Py := (y - y_{xx} + y_{xxxx})_t - \gamma(y - y_{xx} + y_{xxxx})_{xx}.$$

**Theorem 3** *There exist constants $\mu_0, C(\psi),\ C > 0$ such that*

$$\begin{aligned} &\int_Q \left( \frac{1}{\lambda\mu\phi} \theta^2 y_{xxxxxx}^2 + \lambda\mu\phi\theta^2 y_{xxxxx}^2 + \lambda^3\mu^3\phi^3\theta^2 y_{xxxx}^2 \right. \\ &\qquad \left. + \lambda^7\mu^7\phi^7\theta^2 y^2 + \lambda^5\mu^5\phi^5\theta^2 y_x^2 + \lambda^3\mu^3\phi^3\theta^2 y_{xx}^2 + \lambda\mu\phi\theta^2 y_{xxx}^2 \right) dxdt \\ &\le C \left[ \int_Q \theta^2 f^2 dxdt + \int_{Q^\omega} (\lambda^7\mu^7\phi^7\theta^2 y^2 + \lambda^5\mu^5\phi^5\theta^2 y_x^2) dxdt \right] \end{aligned} \tag{26}$$

*for $\mu > \mu_0$ and $\lambda > \mu C(\psi)(T + T^2)$.*

**Proof.** Let $y - y_{xx} + y_{xxxx} = u$, then $Py = f$ can be written in the following forms:

$$\begin{cases} u_t - \gamma u_{xx} = f, \\ u(t, 0) = u(t, 1) = 0 \end{cases} \tag{27}$$

and

$$\begin{cases} y_{xxxx} = u - y + y_{xx}, \\ y(t, 0) = y(t, 1) = y_{xx}(t, 0) = y_{xx}(t, 1) = 0. \end{cases} \tag{28}$$

We shall establish two Carleman estimates for the elliptic equation and the parabolic equation with the same weights. Then we combine the two Carleman estimates into a single one for higher-order linear viscous CH operator. A similar analysis for a coupled system of elliptic-hyperbolic equations can be found in [15].

By applying the results of Theorem 1 and Theorem 2 to $y$ in (28) and $u$ in (27) respectively, we can get:

$$\begin{aligned} &\int_Q \left( \frac{1}{\lambda\mu\phi} \theta^2 y_{xxxx}^2 + \lambda^7\mu^7\phi^7\theta^2 y^2 + \lambda^5\mu^5\phi^5\theta^2 y_x^2 + \lambda^3\mu^3\phi^3\theta^2 y_{xx}^2 + \lambda\mu\phi\theta^2 y_{xxx}^2 \right) dxdt \\ &\le C \left[ \int_Q \theta^2 |u - y + y_{xx}|^2 dxdt + \int_{Q^\omega} (\lambda^7\mu^7\phi^7\theta^2 y^2 + \lambda^5\mu^5\phi^5\theta^2 y_x^2) dxdt \right] \end{aligned} \tag{29}$$

and

$$\int_Q \left[\frac{1}{\lambda\mu\phi}\theta^2(u_t^2+u_{xx}^2)+\lambda^3\mu^3\phi^3\theta^2u^2+\lambda\mu\phi\theta^2u_x^2\right]dxdt \le C\left[\int_Q \theta^2 f^2dxdt+\int_{Q^\omega}\lambda^3\mu^3\phi^3\theta^2u^2dxdt\right]. \tag{30}$$

Adding the left and the right sides in (29) and (30) respectively, we obtain

$$\begin{aligned}&\int_Q \left[\frac{1}{\lambda\mu\phi}\theta^2(u_t^2+u_{xx}^2)+\lambda^3\mu^3\phi^3\theta^2u^2+\lambda\mu\phi\theta^2u_x^2+\frac{1}{\lambda\mu\phi}\theta^2y_{xxxx}^2\right.\\&\qquad\left.+\lambda^7\mu^7\phi^7\theta^2y^2+\lambda^5\mu^5\phi^5\theta^2y_x^2+\lambda^3\mu^3\phi^3\theta^2y_{xx}^2+\lambda\mu\phi\theta^2y_{xxx}^2\right]dxdt\\&\le C\left[\int_Q\theta^2f^2dxdt+\int_Q\theta^2\left|u-y+y_{xx}\right|^2dxdt\right.\\&\qquad\left.+\int_{Q^\omega}(\lambda^7\mu^7\phi^7\theta^2y^2+\lambda^5\mu^5\phi^5\theta^2y_x^2+\lambda^3\mu^3\phi^3\theta^2u^2)dxdt\right].\end{aligned} \tag{31}$$

The term $\int_Q\theta^2\left|u-y+y_{xx}\right|^2dxdt$ at the right side of (31) can be absorbed by the left side

$$\begin{aligned}&\int_Q \left[\frac{1}{\lambda\mu\phi}\theta^2(u_t^2+u_{xx}^2)+\lambda^3\mu^3\phi^3\theta^2u^2+\lambda\mu\phi\theta^2u_x^2+\frac{1}{\lambda\mu\phi}\theta^2y_{xxxx}^2\right.\\&\qquad\left.+\lambda^7\mu^7\phi^7\theta^2y^2+\lambda^5\mu^5\phi^5\theta^2y_x^2+\lambda^3\mu^3\phi^3\theta^2y_{xx}^2+\lambda\mu\phi\theta^2y_{xxx}^2\right]dxdt\\&\le C\left[\int_Q\theta^2f^2dxdt+\int_{Q^\omega}(\lambda^7\mu^7\phi^7\theta^2y^2+\lambda^5\mu^5\phi^5\theta^2y_x^2+\lambda^3\mu^3\phi^3\theta^2u^2)dxdt\right].\end{aligned} \tag{32}$$

Since $y_{xxxx}=u-y+y_{xx}$, we can deduce

$$\int_Q\frac{1}{\lambda\mu\phi}\theta^2y_{xxxxxx}^2dxdt=\int_Q\frac{1}{\lambda\mu\phi}\theta^2(u-y+y_{xx})_{xx}^2dxdt\le\int_Q\frac{1}{\lambda\mu\phi}\theta^2(u_{xx}^2+y_{xx}^2+y_{xxxx}^2)dxdt,$$

$$\int_Q\lambda\mu\phi\theta^2y_{xxxxx}^2dxdt=\int_Q\lambda\mu\phi\theta^2(u-y+y_{xx})_x^2dxdt\le\int_Q\lambda\mu\phi\theta^2(u_x^2+y_x^2+y_{xxx}^2)dxdt,$$

$$\int_Q\lambda^3\mu^3\phi^3\theta^2y_{xxxx}^2dxdt=\int_Q\lambda^3\mu^3\phi^3\theta^2(u-y+y_{xx})dxdt\le\int_Q\lambda^3\mu^3\phi^3\theta^2(u^2+y^2+y_{xx}^2)dxdt,$$

$$\int_{Q^\omega}\lambda^3\mu^3\phi^3\theta^2u^2dxdt\le\int_{Q^\omega}\lambda^3\mu^3\phi^3\theta^2(y^2+y_{xx}^2+y_{xxxx}^2)dxdt$$

and

$$\int_{Q^\omega}\lambda^3\mu^3\phi^3\theta^2y_{xxxx}^2dxdt\le\int_{Q^\omega}\lambda^3\mu^3\phi^3\theta^2(u^2+y^2+y_{xx}^2)dxdt.$$

Combining the above five estimates, we can obtain from (32) that (26) holds. The proof of Theorem 3 is completed. ■

# 3 Conclusion

In this paper, based on the Carleman estimates of the elliptic and parabolic operators, we develop a global Carleman estimate for the higher order viscous CH operator. This global estimate has a complete structure and we can find that all the high-order derivatives can be constrained by the control term and the low-order derivatives in a local interval. With Carleman estimates, we can consider the controllability, observation and unique continuation property of the higher-order viscous CH equation in the future.

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