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Relative (p, q)-th Order Based on Some Growth Measurement of Composite Entire Functions

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Abstract: The main aim of this paper is to study some comparative growth properties of composite entire functions on the basis of relative (p, q)-th order and relative (p, q)-th lower order of entire function with respect to another entire function where p and q are any two positive integers.

Keywords: Entire function; Index-pair; (p, q)-th order; Relative (p, q)-th order; Composition; Growth

1 Introduction

We assume that the reader is familiar with the fundamental results and the standard notations of the theory of entire functions which are available in [1]. Let f be an entire function defined in the open complex plane. The maximum modulus function $M_f(r)$ is defined as $M_f(r) = \max_{|z|=r} |f(z)|$. Since $M_f(r)$ is strictly increasing and continuous, its inverse function exists. For another entire function g, $M_g(r)$ is defined and the ratio $\frac{M_f(r)}{M_g(r)}$ as $r \to \infty$ is called the growth of f with respect to g in terms of their maximum moduli. The maximum term $\mu_f(r)$ of f can be defined as $\mu_f(r) = \max_{n \geq 0} (|a_n| r^n)$. In fact $\mu_f(r)$ is much weaker than $M_f(r)$ in some sense. So from another angle of view $\frac{\mu_f(r)}{\mu_g(r)}$ as $r \to \infty$ is also called the growth of f with respect to g where $\mu_g(r)$ denotes the maximum term of entire g.

For entire functions, the notions of their growth indicators such as order is classical in complex analysis and during the past decades, several researchers have already been exploring their studies in the area of comparative growth properties of composite entire functions in different directions using the classical growth indicators. But at that time, the concepts of relative orders of entire functions introduced by Bernal [2, 3] and as well as their technical advantages of not comparing with the growths of $\exp z$ are not at all known to the researchers of this area. Therefore the studies of the growths of composite entire functions in the light of their relative orders are the prime concern of this paper. In this paper we establish some newly developed results related to the growth rates of composite entire functions on the basis of relative (p, q)-th order and relative (p, q)-th lower order improving some earlier results where p, q are any two positive integers. In fact some light has already been thrown on such type of works by Datta et al. in [4–9].

2 Definitions

First of all for $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ and $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ where $\mathbb{N}$ is the set of all positive integers. We also denote $\log^{[0]} x = x$ and $\exp^{[0]} x = x$. Further we assume that throughout the present paper a, b, d, m, n, p and q always denote positive integers. Taking this into account, let us recall that Juneja et. al. [10] defined the (p, q)-th order $\rho_f(p, q)$ and (p, q)-th lower order $\lambda_f(p, q)$ of an entire function f respectively as follows:

$$\rho_f(p, q) = \varlimsup_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r} \quad and \quad \lambda_f(p, q) = \varliminf_{r\to\infty} \frac{\log^{[p]} M_f(r)}{\log^{[q]} r},$$

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where $p \geq q$.

In this connection, let us recall that if $0 < \rho_f(p,q) < \infty$, then the following properties hold:

$$\rho_f(p-n,q) = \infty \text{ for } n < p,\ \rho_f(p,q-n) = 0 \text{ for } n < q, \text{ and}$$

$$\rho_f(p+n,q+n) = 1 \text{ for } n = 1,2,\cdots.$$

Similarly for $0 < \lambda_f(p,q) < \infty$, one can easily verify that

$$\lambda_f(p-n,q) = \infty \text{ for } n < p,\ \lambda_f(p,q-n) = 0 \text{ for } n < q,$$

$$\lambda_f(p+n,q+n) = 1 \text{ for } n = 1,2,\cdots.$$

These definitions extend the generalized order and generalized lower order of an entire function considered in [11] for each integer $l \geq 2$.

Since for $0 \leq r < R$,

$$\mu_f(r) \leq M_f(r) \leq \frac{R}{R-r}\mu_f(R) \ \ \{cf. [12]\},$$

it is easy to see that

$$\rho_f(p,q) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]}\mu_f(r)}{\log^{[q]} r} \left(\text{respectively } \lambda_f(p,q) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]}\mu_f(r)}{\log^{[q]} r}\right).$$

Extending the notion of relative order of entire function as introduced Bernal [2, 3], Lahiri and Banerjee [13] introduced the definition of relative (p,q)-th order of entire functions as follows.

Definition 1 *[13] Let p and q be any two positive integers with $p > q$. The relative (p,q)-th order of f with respect to g is defined by*

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p-1]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

Then $\rho_{\exp z}^{(p,q)}(f) = \rho_f(p,q)$ and $\rho_g^{(k+1,1)}(f) = \rho_g^k(f)$ for any $k \geq 1$.

Sánchez Ruiz et al. [14] gave a more natural definition of *r*elative (p,q)-th order and relative (p,q)-th lower order an entire function in the light of index-pair which are as follows:

Definition 2 *[14] Let f and g be any two entire functions with index-pairs (m,q) and (m,p) respectively where $m \geq p$ and $m \geq q$. Then the relative (p,q)-th order and relative (p,q)-th lower order of f with respect to g are defined as*

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} M_g^{-1}(M_f(r))}{\log^{[q]} r}.$$

In fact Definition 2 improves Definition 1 ignoring the restriction $p \geq q$. For details, one may see [14].

In terms of maximum terms of entire functions, Definition 2 can be reformulated as:

Definition 3 *For any positive integer p and q, the growth indicators $\rho_g^{(p,q)}(f)$ and $\lambda_g^{(p,q)}(f)$ of an entire function f with respect to another entire function g are defined as:*

$$\rho_g^{(p,q)}(f) = \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log^{[q]} r} \text{ and } \lambda_g^{(p,q)}(f) = \underline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_g^{-1}\mu_f(r)}{\log^{[q]} r}.$$

In fact, the equivalence of Definitions 2 and 3 has been established in [15].

3 Lemmas

In this section we present some lemmas which will be needed in the sequel.

Lemma 1 *[16] If f and g are any two entire functions with $g(0) = 0$. Let β satisfy $0 < \beta < 1$ and $c(\beta) = \frac{(1-\beta)^2}{4\beta}$. Then for all sufficiently large values of r,*

$$M_f(c(\beta) M_g(\beta r)) \leq M_{f\circ g}(r) \leq M_f(M_g(r)).$$

In addition if $\beta = \frac{1}{2}$, then for all sufficiently large values of r,

$$M_{f\circ g}(r) \geq M_f\left(\frac{1}{8} M_g\left(\frac{r}{2}\right)\right).$$

Lemma 2 *[17] Let f and g be any two entire functions. Then for every $\alpha > 1$ and $0 < r < R$,*

$$\mu_{f\circ g}(r) \leq \frac{\alpha}{\alpha-1}\mu_f\left(\frac{\alpha R}{R-r}\mu_g(R)\right).$$

Lemma 3 *[17] If f and g are any two entire functions with $g(0) = 0$, then for all sufficiently large values of r,*

$$\mu_{f\circ g}(r) \geq \frac{1}{2}\mu_f\left(\frac{1}{8}\mu_g\left(\frac{r}{4}\right) - |g(0)|\right).$$

Lemma 4 *[6] If f be an entire and $\alpha > 1$, $0 < \beta < \alpha$, then for all sufficiently large r,*

$$\mu_f(\alpha r) \geq \beta \mu_f(r).$$

4 Main results

In this section we present the main results of the paper.

Theorem 1 *Let f and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and g be an entire function with non zero (m,n)-th order where $m \geq n$. Then for every positive constant A,*

$$(i) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n+1]} r)}{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} = \infty \quad if \quad q = m,$$

$$(ii) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[q+n+1-m]} r)}{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} = \infty \quad if \quad q > m,$$

$$(iii) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} = \infty \quad if \quad q \leq m-1 \quad and \quad 0 < A < \rho_g(m,n).$$

Proof. From the definition of $\rho_h^{(p,q)}(f)$ in terms of maximum terms, we obtain for arbitrary positive $\varepsilon (> 0)$ and for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}\left(r^A\right)) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) r^A. \tag{1}$$

Let $q > m$. Now from the definition of (m,n)-th order of g in terms of maximum terms, we get for a sequence of values of r tending to infinity that

$$\log^{[m]} \mu_g\left(\frac{\exp^{[q+n+1-m]} r}{4}\right) \geqslant (\rho_g(m,n) - \varepsilon) \log^{[n]}\left(\frac{\exp^{[q+n+1-m]} r}{4}\right),$$

$$i.e.,\ log^{[q-m]}log^{[m]}\mu_g\frac{\exp^{[q+n+1-m]}r}{4}\geq log^{[q-m]}[(\rho_g(m,n)-\varepsilon)\exp^{[q+1-m]}r+O(1)],$$

$$i.e.,\ \log^{[q]}\mu_g\left(\frac{\exp^{[q+n+1-m]}r}{4}\right)\geqslant \exp r+O(1). \tag{2}$$

Further,

$$\log^{[m]}\mu_g\left(\frac{\exp^{[n-1]}r}{4}\right)\geqslant(\rho_g(m,n)-\varepsilon)\log^{[n]}\left(\frac{\exp^{[n-1]}r}{4}\right),$$

$$i.e.,\ \log^{[m-1]}\mu_g\left(\frac{\exp^{[n-1]}r}{4}\right)\geqslant r^{(\rho_g(m,n)-\varepsilon)}+O(1). \tag{3}$$

It follows from Lemma 3 for all sufficiently large values r that

$$\mu_{f\circ g}(r)\geq\frac{1}{2}\mu_f\left(\frac{1}{16}\mu_g\left(\frac{r}{4}\right)\right)\ \{cf.\ [12]\}.$$

Therefore in view of Lemma 4, we obtain from above for all sufficiently large values of r that

$$\mu_{f\circ g}(r)\geq\mu_f\left(\frac{1}{48}\mu_g\left(\frac{r}{4}\right)\right). \tag{4}$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from above for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\left(\frac{1}{48}\mu_g\left(\frac{r}{4}\right)\right),$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(r)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\mu_g\left(\frac{r}{4}\right)+O(1). \tag{5}$$

Case I. Let $q=m$. Then it follows from (5) for a sequence of values of r tending to infinity that

$$log^{[p]}\mu_h^{-1}\mu_{f\circ g}(exp^{[n+1]}r)\geq\lambda_h^{(p,q)}(f)-\varepsilon)(\rho_g(m,n)-\varepsilon)expr+O(1). \tag{6}$$

Case II. Let $q>m$. Then we obtain from (2) and (5) for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[q+n+1-m]}r\right)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\exp r+O(1). \tag{7}$$

Case III. Again let $q\leq m-1$. Then we get from (3) and (5) for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\log^{[q]}\mu_g\left(\frac{\exp^{[n-1]}r}{4}\right)+O(1),$$

$$i.e.,\ log^{[p]}\mu_h^{-1}\mu_{f\circ g}(exp^{[n-1]}r)\geq(\lambda_h^{(p,q)}(f)-\varepsilon)log^{[m-1]}\mu_g(\frac{exp^{[n-1]}r}{4})+O(1),$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)\geq\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\cdot r^{(\rho_g(m,n)-\varepsilon)}+O(1). \tag{8}$$

Now combining (1) and (6) of Case I, we have for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n+1]}r)}{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))}\geq\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)(\rho_g(m,n)-\varepsilon)\exp r+O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^A}.$$

Since $\frac{\exp r}{r^A}\to\infty$ as $r\to\infty$, then from above the conclusion of the first part of the theorem follows.

Again combining (1) and (7) of Case II we obtain for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)}{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} \geq \frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\exp r+O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^A}$$

$$i.e.\ \overline{\lim_{r\to\infty}}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)}{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} = \infty.$$

This establishes the second part of the theorem.
Once more, it follows from (1) and (8) of Case III for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)}{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} \geq \frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)\cdot r^{(\rho_g(m,n)-\varepsilon)}+O(1)}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^A}. \tag{9}$$

As $A<\rho_g(m,n)$ we can choose $\varepsilon\,(>0)$ in such a way that

$$A<\rho_g(m,n)-\varepsilon. \tag{10}$$

Thus from (9) and (10) we get the third part of the theorem.
Thus the theorem follows. ■
In view of Theorem 1 the following theorem can be carried out.

Theorem 2 *Let f and h be any two entire functions such that $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$ and g be an entire function with g non zero (m,n)-th lower order where $m\geq n$. Then for every positive constant A,*

$$(i)\quad \lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n+1]}r)}{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} = \infty \quad if \quad q=m,$$

$$(ii)\quad \lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[q+n+1-m]}r)}{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} = \infty \quad if \quad q>m,$$

$$(iii)\quad \lim_{r\to\infty}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)}{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} = \infty \quad if \quad q\leq m-1 \quad and \quad 0<A<\lambda_g(m,n).$$

Theorem 3 *Let f and h be any two entire functions such that $0<\lambda_h^{(p,q)}(f)\leq\rho_h^{(p,q)}(f)<\infty$. Suppose g be an entire function with (m,n)-th order $\rho_g(m,n)$ and finite relative (p,n) -th order with respect to h where $m\geq n$. Then for every positive constant A,*

$$(i)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n+1]}r)}{\log^{[p]}\mu_h^{-1}\mu_g(\exp^{[n]}(r^A))} = \infty \quad if \quad q=m,$$

$$(ii)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[q+n+1-m]}r)}{\log^{[p]}\mu_h^{-1}\mu_g(\exp^{[n]}(r^A))} = \infty \quad if \quad q>m,$$

$$(iii)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)}{\log^{[p]}\mu_h^{-1}\mu_g(\exp^{[n]}(r^A))} = \infty \quad if \quad q\leq m-1 \quad and \quad 0<A<\rho_g(m,n).$$

Proof. Suppose $0<A<A_0$.
Case I. Let $q=m$. Then in view of the first part of Theorem 1, we get for a sequence of values of r tending to infinity and $K>1$

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n+1]}r) > \left(\lambda_h^{(p,q)}(f)-\varepsilon\right)r^{A_0}. \tag{11}$$

Case II. Also let $q > m$. Then we obtain from the second part of Theorem 1 for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[q+n+1-m]} r) > \left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}. \tag{12}$$

Case III. Again let $q \leq m-1$. Then we get from the third part of Theorem 1 for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[n-1]} r) > \left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}. \tag{13}$$

Now from the definition of $\rho_h^{(p,n)}(g)$ in terms of maximum terms, we obtain for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1} \mu_g \exp^{[n]}\left(r^A\right)) \leq \left(\rho_h^{(p,n)}(g) + \varepsilon\right) r^A. \tag{14}$$

Now combining (11) of Case I and (14) it follows for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[n+1]} r}{\log^{[p]} \mu_h^{-1} \mu_g \exp^{[n]}\left(r^A\right))} \geq \frac{\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}}{\left(\rho_h^{(p,n)}(g) + \varepsilon\right) r^A}. \tag{15}$$

Since $A_0 > A$, so from (15) the first part of the theorem follows.
Similarly for $A_0 > A$, we obtain from (12) of Case II and (14) for a sequence of values of r tending to infinity that

$$\begin{aligned} \frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[q+n+1-m]} r)}{\log^{[p]} \mu_h^{-1} \mu_g \exp^{[n]}\left(r^A\right))} &\geq \frac{\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}}{\left(\rho_h^{(p,n)}(g) + \varepsilon\right) r^A}, \\ i.e.\ \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[q+n+1-m]} r)}{\log^{[p]} \mu_h^{-1} \mu_g \exp^{[n]}\left(r^A\right))} &= \infty. \end{aligned}$$

This establishes the second part of the theorem.
Again it follows from (13) of Case III and (14) for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[n-1]} r)}{\log^{[p]} \mu_h^{-1} \mu_g \exp^{[n]}\left(r^A\right))} \geq \frac{\left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^{A_0}}{\left(\rho_h^{(p,n)}(g) + \varepsilon\right) r^A}. \tag{16}$$

Now suppose A_0 is such that $0 < A < A_0 < \rho_g(m,n)$.
Therefore from (16) we get the conclusion of the third part of the theorem.
Thus the theorem is establish. ■

Theorem 4 *Let f and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$. Suppose g be an entire function with (m,n)-th lower order $\lambda_g(m,n)$ and finite relative (p,n)-th order with respect to h where $m \geq n$. Then for every positive constant A,*

$$\begin{aligned} (i) \quad \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[n+1]} r)}{\log^{[p]} \mu_h^{-1} \mu_g(\exp^{[n]}\left(r^A\right))} &= \infty \quad if \quad q = m, \\ (ii) \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[q+n+1-m]} r)}{\log^{[p]} \mu_h^{-1} \mu_g(\exp^{[n]}\left(r^A\right))} &= \infty \quad if \quad q > m, \\ (iii) \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1} \mu_{f \circ g}(\exp^{[n-1]} r)}{\log^{[p]} \mu_h^{-1} \mu_g(\exp^{[n]}\left(r^A\right))} &= \infty \quad if \quad q \leq m-1 \ \textit{and} \\ 0 &< A < \lambda_g(m,n). \end{aligned}$$

The proof of Theorem 4 is omitted as it can be carried out in the line of Theorem 3.

Theorem 5 *Let f and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and g be an entire function with finite (m,n)-th lower order where $m \geq n$. Then*

$$(i) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1} \mu_f(\exp^{[q]}(r^A))}{\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(\exp^{[n]} r)} = \infty \quad if \quad q \geq m \text{ and } A > 1,$$

$$(ii) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1} \mu_f(\exp^{[q]}(r^A))}{\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(\exp^{[n-1]} r)} = \infty \quad if \quad q \geq m$$

$$\text{or } q = m\,(\neq 1) - 1 \text{ and } \lambda_g(m,n) < A,$$

$$(iii) \quad \overline{\lim_{r\to\infty}} \frac{\log^{[p]} \mu_h^{-1} \mu_f(\exp^{[q]}(r^A))}{\log^{[p+m-q-1]} \mu_h^{-1} \mu_{f\circ g}(\exp^{[n-1]} r)} = \infty \quad if \quad m > q+1 \text{ and } A > \lambda_g(m,n).$$

Proof. From the definition of $\lambda_h^{(p,q)}(f)$ in terms of maximum terms, we obtain for arbitrary positive $\varepsilon\,(>0)$ and for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1} \mu_f(\exp^{[q]}(r^A)) \geq \left(\lambda_h^{(p,q)}(f) - \varepsilon\right) r^A. \tag{17}$$

Also from the definition of (m,n)-th lower order of g in terms of maximum terms, we get for a sequence of values of r tending to infinity that

$$\log^{[m]} \mu_g\left(\beta \exp^{[n-1]} r\right) \leq (\lambda_g(m,n) + \varepsilon) \log^{[n]}(\exp^{[n-1]} \beta r),$$

$$i.e., \ \log^{[m]} \mu_g\left(\beta \exp^{[n-1]} r\right) \leq (\lambda_g(m,n) + \varepsilon) \log r + O(1),$$

$$i.e., \ \log^{[m]} \mu_g\left(\beta \exp^{[n-1]} r\right) \leq \log r^{(\lambda_g(m,n)+\varepsilon)} + O(1), \tag{18}$$

$$i.e., \ \log^{[m-1]} \mu_g\left(\beta \exp^{[n-1]} r\right) \leq r^{(\lambda_g(m,n)+\varepsilon)} + O(1). \tag{19}$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, taking $R = \beta r$ in Lemma 2 and in view of Lemma 4 it follows for all sufficiently large values of r that

$$\log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r) \leq \log^{[p]} \mu_h^{-1} \mu_f\left(\frac{(2\alpha-1)\alpha\beta}{(\alpha-1)(\beta-1)} \mu_g(\beta r)\right),$$

$$i.e., \ \log^{[p]} \mu_h^{-1} \mu_{f\circ g}(r) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} \mu_g(\beta r) + O(1). \tag{20}$$

Case I. Let $q \geq m$. Then it follows from (20) for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1} \mu_{f\circ g}\left(\exp^{[n]} r\right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} \mu_g(\beta \exp^{[n]} r) + O(1),$$

$$i.e., \ \log^{[p]} \mu_h^{-1} \mu_{f\circ g}\left(\exp^{[n]} r\right) \leq \left(\rho_h^{(p,q)}(f) + \varepsilon\right)(\lambda_g(m,n) + \varepsilon) r + O(1). \tag{21}$$

Case II. Also let $q \geq m$ or $q = m\,(\neq 1) - 1$. Then also we obtain from (19) and (20) for a sequence of values of r tending to infinity that

$$\log^{[p]} \mu_h^{-1} \mu_{f\circ g}\left(\exp^{[n-1]} r\right) \leq$$

$$\left(\rho_h^{(p,q)}(f) + \varepsilon\right) \log^{[q]} \mu_g(\beta \exp^{[n-1]} r) + O(1),$$

$$i.e., \ \log^{[p]} \mu_h^{-1} \mu_{f\circ g}\left(\exp^{[n-1]} r\right) \leq$$

$$\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[m-1]}\mu_g(\beta\exp^{[n-1]}r)+O(1),$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^{(\lambda_g(m,n)+\varepsilon)}+O(1). \quad (22)$$

Case III. Again let $m > q+1$. Then we get from (18) and (20) for a sequence of values of r tending to infinity that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q]}\mu_g(\beta\exp^{[n-1]}r)+O(1),$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[q-m]}\log^{[m]}\mu_g(\beta\exp^{[n-1]}r)+O(1),$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q]}\log^{[m]}\mu_g(\beta\exp^{[n-1]}r)+O(1),$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q]}\log r^{(\lambda_g(m,n)+\varepsilon)}+O(1),$$

$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q-1]}r^{(\lambda_g(m,n)+\varepsilon)}+O(1),$$

$$i.e.,\ \log^{[p+m-q-1]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)\leq r^{(\lambda_g(m,n)+\varepsilon)}+O(1). \quad (23)$$

Now if $q \geq m$ and $\mu > 1$, we get from (17) and (21) of Case I for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))}{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n]}r\right)}\geq\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)r^A}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\left(\lambda_g(m,n)+\varepsilon\right)r+O(1)},$$

from which the first part of the theorem follows.

Again combining (17) and (22) of Case II we obtain for a sequence of values of r tending to infinity when $q \geq m$ or $q = m\,(\neq 1) - 1$

$$\frac{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))}{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)}\geq\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)r^A}{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^{(\lambda_g(m,n)+\varepsilon)}+O(1)}. \quad (24)$$

As $A > \lambda_g(m,n)$ we can choose $\varepsilon\,(> 0)$ in such a way that

$$\lambda_g(m,n)+\varepsilon < A. \quad (25)$$

Thus from (24) and (25) we get the conclusion of second part of the theorem .

When $m > q+1$, it follows from (17) and (23) of Case III for a sequence of values of r tending to infinity that

$$\frac{\log^{[p]}\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))}{\log^{[p+m-q-1]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)}\geq\frac{\left(\lambda_h^{(p,q)}(f)-\varepsilon\right)r^A}{r^{(\lambda_g(m,n)+\varepsilon)}+O(1)}. \quad (26)$$

Now from (25) and (26) we obtain the conclusion of third part of the theorem.

Thus the theorem follows . ■

In the line of Theorem 5 we may state the following theorem without proof.

Theorem 6 *Let f and h be any two entire functions such that $\rho_h^{(p,q)}(f)$ is finite and g be a entire function with (m,n)-th lower order and non zero (p,n)-th relative lower order with respect to h where $m \geq n$. Then*

$$(i)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}\mu_h^{-1}\mu_g(\exp^{[n]}(r^A))}{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n]}r\right)} = \infty \quad if \quad q\geq m \ and\ A>1,$$

$$(ii)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}\mu_h^{-1}\mu_g(\exp^{[n]}(r^A))}{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)} = \infty \quad if \quad q\geq m$$

$$or\ q=m\,(\neq 1)-1\ and\ \lambda_g(m,n)<A,$$

$$(iii)\quad \overline{\lim_{r\to\infty}}\frac{\log^{[p]}\mu_h^{-1}\mu_g(\exp^{[n]}(r^A))}{\log^{[p+m-q-1]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)} = \infty \quad if \quad m>q+1\ and\ A > \lambda_g(m,n).$$

Theorem 7 *Let f and h be any two entire functions such that $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and g be an entire function with finite (m,n)-th order where $m \geq n$. Then*

$$(i) \quad \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A))}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n]} r)} = \infty \quad if \quad q \geq m \text{ and } A > 1,$$

$$(ii) \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A))}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)} = \infty \quad if \quad q \geq m$$
$$\text{or } q = m(\neq 1) - 1 \text{ and } \rho_g(m,n) < A,$$

$$(iii) \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A))}{\log^{[p+m-q-1]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)} = \infty \quad if \quad m > q+1 \text{ and}$$
$$A > \rho_g(m,n).$$

Theorem 8 *Let f and h be any two entire functions such that $\rho_h^{(p,q)}(f)$ is finite and g be a entire function with (m,n)-th order and non zero (p,n)-th relative lower order with respect to h where $m \geq n$. Then*

$$(i) \quad \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_g(\exp^{[n]}(r^A))}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n]} r)} = \infty \quad if \quad q \geq m \text{ and } A > 1,$$

$$(ii) \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_g(\exp^{[n]}(r^A))}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)} = \infty \quad if \quad q \geq m$$
$$\text{or } q = m(\neq 1) - 1 \text{ and } \rho_g(m,n) < A,$$

and

$$(iii) \lim_{r\to\infty} \frac{\log^{[p]} \mu_h^{-1}\mu_g(\exp^{[n]}(r^A))}{\log^{[p+m-q-1]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)} = \infty \quad if \quad m > q+1 \text{ and}$$
$$A > \rho_g(m,n).$$

We omit the proof of Theorems 7 and 8 as those can be carried out in the line of Theorems 5 and 6 respectively.

As an application of Theorems 1 and 5, we may state the following theorem:

Theorem 9 *Let f, g and h be any three entire functions such that $g(0) = 0$, $0 < \lambda_h^{(p,q)}(f) \leq \rho_h^{(p,q)}(f) < \infty$ and $\lambda_g(m,n) < A < \rho_g(m,n)$ where $m \geq n$. Then for $q = m(\neq 1) - 1$,*

$$\underline{\lim}_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} \leq 1 \leq \overline{\lim}_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))}.$$

Proof. In view of Theorem 1 we get for a sequence of values of r tending to infinity and for $K > 1$

$$\log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r) > \log\left\{\log^{[p-1]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A))\right\}^K,$$
$$i.e., \ \log^{[p]} \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r) > \log\left\{\log^{[p-1]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A))\right\},$$

$$i.e., \ \mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r) > \mu_h^{-1}\mu_f(\exp^{[q]}(r^A)),$$
$$i.e., \ \overline{\lim}_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} > 1. \tag{27}$$

Again from Theorem 5 we obtain for a sequence of values of r tending to infinity and for $P > 1$

$$\log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A)) > \log\left\{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]} r\right)\right\}^P,$$
$$i.e., \ \log^{[p]} \mu_h^{-1}\mu_f(\exp^{[q]}(r^A)) > \log\left\{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]} r\right)\right\},$$

$$\begin{aligned} i.e., \ \mu_h^{-1}\mu_f(\exp^{[q]}\left(r^A\right)) &> \mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]} r\right), \\ i.e., \ \varliminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\mu_h^{-1}\mu_f(\exp^{[q]}(r^A))} &< 1. \end{aligned} \tag{28}$$

Thus the theorem follows from (27) and (28). ■

In view of Theorems 3 and 6, the following theorem can be carried out:

Theorem 10 *Let f, g and h be any three entire functions such that $g(0) = 0$, $0 < \lambda_h^{(p,q)}(f) \le \rho_h^{(p,q)}(f) < \infty$, $0 < \lambda_h^{(p,n)}(g) \le \rho_h^{(p,n)}(g) < \infty$ and $\lambda_g(m,n) < A < \rho_g(m,n)$ where $m \ge n$. Then for $q = m(\ne 1) - 1$,*

$$\varliminf_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\mu_h^{-1}\mu_g(\exp^{[n]}(r^A))} \le 1 \le \varlimsup_{r\to\infty} \frac{\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]} r)}{\mu_h^{-1}\mu_g(\exp^{[n]}(r^A))}.$$

The proof is omitted.

Theorem 11 *Let l, f and h be any three entire functions such that $\lambda_h^{(p,d)}(l) > 0$ and $\rho_h^{(p,q)}(f) < \infty$ .Also let g and k are two entire function with $\rho_g(m,n) < \lambda_k(a,b)$. Then*

$$(i)\ \lim_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{l\circ k}\left(\exp^{[b-1]} r\right)}{\log^{[p-1]} \mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]} r\right) \cdot \log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r\right)} = \infty$$

$$if \quad d \le a-1 \text{ and } q \ge m-1,$$

$$(ii)\ \lim_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{l\circ k}\left(\exp^{[b-1]} r\right)}{\log^{[p]} \mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]} r\right) \cdot \log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r\right)} = \infty$$

$$if \quad d \le a-1 \text{ and } m-q = 2,$$

$$(iii)\ \lim_{r\to\infty} \frac{\log^{[p-1]} \mu_h^{-1}\mu_{l\circ k}\left(\exp^{[b-1]} r\right)}{\log^{[p+m-q-2]} \mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]} r\right) \cdot \log^{[p-1]} \mu_h^{-1}\mu_f\left(\exp^{[q-1]} r\right)} = \infty$$

$$if \quad d \le a-1 \text{ and } m-q > 2,$$

where $m \ge n$ and $a \ge b$.

Proof. From the definition of (m,n)-th order of g in terms of maximum terms, we get for arbitrary positive ε and for all sufficiently large values of r that

$$\begin{aligned} \log^{[m]} \mu_g\left(\beta \exp^{[n-1]} r\right) &\le (\rho_g(m,n) + \varepsilon) \log^{[n]}(\beta \exp^{[n-1]} r), \\ i.e., \ \log^{[m]} \mu_g\left(\beta \exp^{[n-1]} r\right) &\le (\rho_g(m,n) + \varepsilon) \log r + O(1), \\ i.e., \ \log^{[m]} \mu_g\left(\beta \exp^{[n-1]} r\right) &\le \log r^{(\rho_g(m,n)+\varepsilon)} + O(1), \qquad (29) \\ i.e., \ \log^{[m-1]} \mu_g\left(\beta \exp^{[n-1]} r\right) &\le r^{(\rho_g(m,n)+\varepsilon)} + O(1). \qquad (30) \end{aligned}$$

Also from the definition of (a,b)-th lower order of k in terms of maximum terms, we get for all sufficiently large values of r that

$$\begin{aligned} \log^{[a]} \mu_k\left(\frac{\exp^{[b-1]} r}{4}\right) &\geqslant (\lambda_k(a,b) - \varepsilon) \log^{[b]}\left(\frac{\exp^{[b-1]} r}{4}\right), \\ i.e., \ \log^{[a]} \mu_k\left(\frac{\exp^{[b-1]} r}{4}\right) &\geqslant \log r^{(\lambda_k(a,b)-\varepsilon)} + O(1), \qquad (31) \\ i.e., \ \log^{[a-1]} \mu_k\left(\frac{\exp^{[b-1]} r}{4}\right) &\geqslant r^{(\lambda_k(a,b)-\varepsilon)} + O(1). \qquad (32) \end{aligned}$$

Again from the definition of (p,q)-th relative order of f with respect to h in terms of maximum terms, we have for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log r,$$
$$i.e.,\ \log^{[p-1]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r\right)\leq r^{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)}. \tag{33}$$

Since $\mu_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 3 and in view of (4), for all sufficiently large values r that

$$\log^{[p]}\mu_h^{-1}\mu_{l\circ k}(r)\geqslant\left(\lambda_h^{(p,d)}(l)-\varepsilon\right)\log^{[d]}\mu_k\left(\frac{r}{4}\right)+O(1). \tag{34}$$

Case I. Let $d\leq a-1$. Then we get from (32) and (34) for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_{l\circ k}\left(\exp^{[b-1]}r\right)\geqslant\left(\lambda_h^{(p,d)}(l)-\varepsilon\right)\log^{[a-1]}\mu_k\left(\frac{\exp^{[b-1]}r}{4}\right)+O(1),$$
$$i.e.,\ \log^{[p-1]}\mu_h^{-1}\mu_{l\circ k}\left(\exp^{[b-1]}r\right)\geqslant\exp\left\{\left(\lambda_h^{(p,d)}(l)-\varepsilon\right)r^{(\lambda_k(a,b)-\varepsilon)}+O(1)\right\}. \tag{35}$$

Case II. Again let $q\geq m-1$. Then we obtain from (20) and (30) for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\log^{[m-1]}\mu_g\left(\beta\exp^{[n-1]}r\right)+O(1),$$
$$i.e.,\ \log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\leq\exp\left\{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^{(\rho_g(m,n)+\varepsilon)}+O(1)\right\}. \tag{36}$$

Case III. Also let $q<m$. Then it follows from (20) and (29) for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q]}\log^{[m]}\mu_g(\beta\exp^{[n-1]}r),$$
$$i.e.,\ \log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q-1]}r^{(\rho_g(m,n)+\varepsilon)}+O(1). \tag{37}$$

Now if $m-q=2$, then we get from (37) for all sufficiently large values of r that

$$\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\leq\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp r^{(\rho_g(m,n)+\varepsilon)}+O(1). \tag{38}$$

Also if $m-q>2$, then we get from (37) for all sufficiently large values of r that

$$\begin{aligned}&\log^{[m-q-2]}\left[\log^{[p]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\right]\\ \leq\ &\log^{[m-q-2]}\left[\left(\rho_h^{(p,q)}(f)+\varepsilon\right)\exp^{[m-q-1]}r^{(\rho_g(m,n)+\varepsilon)}+O(1)\right],\end{aligned}$$
$$i.e.,\ \log^{[p+m-q-2]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\leq\exp r^{(\rho_g(m,n)+\varepsilon)}+O(1). \tag{39}$$

Now as $\rho_g(m,n)<\lambda_k(a,b)$, we can choose $\varepsilon\,(>0)$ in such a manner that

$$\rho_g(m,n)+\varepsilon<\lambda_k(a,b)-\varepsilon. \tag{40}$$

Therefore combining (33) , (35) of Case I and (36) of Case II it follows for all sufficiently large values of r that

$$\frac{\log^{[p-1]}\mu_h^{-1}\mu_{l\circ k}\left(\exp^{[b-1]}r\right)}{\log^{[p-1]}\mu_h^{-1}\mu_{f\circ g}\left(\exp^{[n-1]}r\right)\cdot\log^{[p-1]}\mu_h^{-1}\mu_f\left(\exp^{[q-1]}r\right)}$$
$$\geqslant\frac{\exp\left\{\left(\lambda_h^{(p,d)}(l)-\varepsilon\right)r^{(\lambda_k(a,b)-\varepsilon)}+O(1)\right\}}{r^{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)}\cdot\exp\left\{\left(\rho_h^{(p,q)}(f)+\varepsilon\right)r^{(\rho_g(m,n)+\varepsilon)}+O(1)\right\}}.$$

Thus in view of (40) first part of the theorem follows from above.

Again combining (33) , (35) of Case I, (38) of Case III and (40) we obtain for all sufficiently large values of r that

$$\frac{log^{[p-1]}\mu_h^{-1}\mu_{l\circ k}(exp^{[b-1]}r)}{\log^{[p]}\mu_h^{-1}\mu_{f\circ g}(exp^{[n-1]}r)\cdot\log^{[p-1]}\mu_h^{-1}\mu_f(\exp^{[q-1]}r)} \geq \frac{exp(\lambda_h^{(p,d)}(l)-\varepsilon)r^{(\lambda_k(a,b)-\varepsilon)}+O(1)}{r^{(\rho_h^{(p,q)}(f)+\varepsilon)}\cdot[(\rho_h^{(p,q)}(f)+\varepsilon)\exp r^{(\rho_g(m,n)+\varepsilon)}+O(1)]}.$$

Hence from above the second part of the theorem follows.

Similarly combining (33) , (35) of Case I, (39) of Case III and (40) we get for all sufficiently large values of r that

$$\frac{\log^{[p-1]}\mu_h^{-1}\mu_{l\circ k}(\exp^{[b-1]}r)}{\log^{[p+m-q-2]}\mu_h^{-1}\mu_{f\circ g}(\exp^{[n-1]}r)\cdot\log^{[p-1]}\mu_h^{-1}\mu_f(\exp^{[q-1]}r)} \geq \frac{exp(\lambda_h^{(p,d)}(l)-\varepsilon)r^{(\lambda_k(a,b)-\varepsilon)}+O(1)}{r^{(\rho_h^{(p,q)}(f)+\varepsilon)}\cdot[expr^{(\rho_g(m,n)+\varepsilon)}+O(1)]}.$$

Thus the conclusion of the third part of the theorem follows from above.

Hence the theorem follows. ■

Remark 12 *If we consider $\rho_g(m,n) < \rho_k(a,b)$ instead of $\rho_g(m,n) < \lambda_k(a,b)$ and the other conditions remain the same, the conclusion of Theorem 11 remains valid with "limit superior " replaced by " limit ".*

Remark 13 *The same results of above theorems and remarks in terms of maximum modulus of entire functions can also be deduced with the help of Lemma 1.*

References

[1] G. Valiron. Lectures on the General Theory of Integral Functions. *Chelsea Publishing Company*, 1949.

[2] L.B. Gonzaléz. Crecimiento relativo de funciones enteras. Aportaciones al estudio de las funciones enteras con índice exponencial finito. *Doctoral Thesis, Universidad de Sevilla, Spain,* 1984.

[3] L. Bernal. Orden relative de crecimiento de funciones enteras. *Collect. Math.*, 39(1988): 209 – 229.

[4] S.K. Datta and T. Biswas. Growth of entire functions based on relative order. *Int. J. Pure Appl. Math.*, 51(1)(2009): 49 – 58.

[5] S.K. Datta and T. Biswas. Relative order of composite entire functions and some related growth properties. *Bull. Cal. Math. Soc.*, 102(3)(2010): 259 – 266.

[6] S.K. Datta and A.R. Maji. Relative order of entire functions in terms of their maximum terms. *Int. Journal of Math. Analysis*, 5(43)(2011): 2119 – 2126.

[7] S.K. Datta, T. Biswas and D.C. Pramanik. On relative order and maximum term-related comparative growth rates of entire functions. *Journal of Tripura Mathematical Society*, 14(2012): 60 – 68.

[8] S.K. Datta, T. Biswas and R. Biswas. On relative order based growth estimates of entire functions. *Int. J. Math. Sci. Eng. Appl.*, 7(II)(2013): 59 – 67.

[9] S.K. Datta, T. Biswas and R. Biswas. Comparative growth properties of composite entire functions in the light of their relative order. *Math. Student*, 82(1-4)(2013): 209 – 216.

[10] O.P. Juneja, G.P. Kapoor and S.K. Bajpai. On the (p,q)-order and lower (p,q)-order of an entire function. *J. Reine Angew. Math.*, 282(1976): 53 – 67.

[11] D. Sato. On the rate of growth of entire functions of fast growth. *Bull. Amer. Math. Soc.*, 69(1963): 411 – 414.

[12] A.P. Singh and M.S. Baloria. On the maximum modulus and maximum term of composition of entire functions. *Indian J. Pure Appl. Math.*, 22(12)(1991): 1019 – 1026.

[13] B.K. Lahiri and D. Banerjee. Entire functions of relative order (p, q). *Soochow J. Math.*, 31(4)(2005): 497 – 513.

[14] L.M.S. Ruiz, S.K. Datta, T. Biswasm and G.K. Mondal. On the (p,q)-th relative order oriented growth properties of entire functions. *Abstr. Appl. Anal.*, Hindawi Publishing Corporation, 2014.

[15] T. Biswas. Growth analysis of composite entire functions from the view point of relative (p,q)-th order. *Korean J. Math.*, 26(3)(2018): 405 – 425.

[16] J. Clunie. The composition of entire and meromorphic functions, Mathematical Essays dedicated to A. J. Macintyre. *Ohio University Press*, 1970: 75 – 92.

[17] A.P. Singh. On maximum term of composition of entire functions. *Proc. Nat. Acad. Sci. India*, 59(A)(1989): 103 – 115.