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Growth Measurement of Wronskians Generated by Entire or Meromorphic Functions on the Basis of Some Generalized Growth Indicators

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Abstract: In the paper, we establish some new results depending on the comparative growth properties of composite transcendental entire and meromorphic functions using relative ${}_pL^*$-order, relative ${}_pL^*$- type, relative ${}_pL^*$-weak type and that of wronskian generated by one of the factors.

Keywords: Transcendental entire function; Transcendental meromorphic function; Relative ${}_pL^*$-order; Relative ${}_pL^*$-type; Relative ${}_pL^*$-weak type; Growth; Property (A); Slowly changing function; Wronskian

1 Introduction, definitions, and notations

Let us consider that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna theory of meromorphic functions which are available in [1–4]. We also use the standard notations and definitions of the theory of entire functions which are available in [5] and therefore we do not explain those in details. For $x \in [0, \infty)$ and $k \in \mathbb{N}$, we define $\exp^{[k]} x = \exp\left(\exp^{[k-1]} x\right)$ and $\log^{[k]} x = \log\left(\log^{[k-1]} x\right)$ where $\mathbb{N}$ be the set of all positive integers.

Let f be an entire function defined in the open complex plane $\mathbb{C}$. The maximum modulus function $M_f(r)$ corresponding to f is defined on $|z| = r$ as $M_f(r) = \max_{|z|=r} |f(z)|$. In this connection, the following definition is relevant.

Definition 1 *{[6]} A non-constant entire function f is said have the Property (A) if for any $\sigma > 1$ and for all sufficiently large r, $[M_f(r)]^2 \leq M_f(r^\sigma)$ holds. For examples of functions with or without the Property (A), one may see [6].*

When f is meromorphic, one may introduce another function $T_f(r)$ known as Nevanlinna's characteristic function of f, playing the same role as $M_f(r)$.

For a meromorphic function f defined on $\mathbb{C}$, the Wronskian determinant $W(f) = W(a_1, a_2, \cdots, a_k, f)$ is defined as

$$W(f) = \begin{vmatrix} a_1 & a_2 & . & . & . & a_k & f \\ a_1' & a_2' & . & . & . & a_k' & f' \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ . & . & . & . & . & . & . \\ a_1^{(k)} & a_2^{(k)} & . & . & . & a_k^{(k)} & f^{(k)} \end{vmatrix},$$

where $a_1, a_2, \cdots, a_k$ are linearly independent meromorphic functions and small with respect to f (i.e., $T_{a_i}(r) = S(r, f)$ or in other words $\frac{T_{a_i}(r)}{S(r,f)} \to 0$ as $r \to \infty$ for $i = 1, 2, 3, \cdots, k$). From the Nevanlinna's second fundamental theorem, it follows that the set of values of $a \in \mathbb{C} \cup \{\infty\}$ for which $\delta(a; f) > 0$ is countable and $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) \leq 2$ where $\delta(a; f) = 1 - \limsup_{r\to\infty} \frac{N(r,a;f)}{T_f(r)} = \liminf_{r\to\infty} \frac{m(r,a;f)}{T_f(r)}$ [1]. If in particular $\sum_{a \neq \infty} \delta(a; f) + \delta(\infty; f) = 2$, we say that f has the maximum deficiency sum.

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However, in case of any two meromorphic functions f and g, the ratio $\frac{T_f(r)}{T_g(r)}$ as $r \to \infty$ is called as the growth of f with respect to g in terms of their Nevanlinna's Characteristic functions. Further the concept of the growth measuring tools such as order and lower order which are conventional in complex analysis and the growth of entire or meromorphic functions can be studied in terms of their orders and lower orders are normally defined in terms of their growth with respect to the $\exp$ function which are shown in the following definition:

Definition 2 *The order ρ_f (the lower order λ_f) of a meromorphic function f is defined as*

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log(r) + O(1)}$$

$$\left(\lambda_f = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log T_{\exp z}(r)} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left(\frac{r}{\pi}\right)} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log(r) + O(1)}\right).$$

If f is entire, then

$$\rho_f = \overline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \overline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log r}$$

$$\left(\lambda_f = \underline{\lim_{r\to\infty}} \frac{\log\log M_f(r)}{\log\log M_{\exp z}(r)} = \underline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log r}\right).$$

Somasundaram and Thamizharasi [11] introduced the notions of L-order and L-type for entire functions where $L \equiv L(r)$ is a positive continuous function increasing slowly, i.e.,$L(ar) \sim L(r)$ as $r \to \infty$ for every positive constant "a". The more generalized concept of L-order and L-type of meromorphic functions are L^*-order and L^*-type (resp. L^*-lower type) respectively which are as follows:

Definition 3 *[11] The L^*-order $\rho_f^{L^*}$ and the L^*-lower order $\lambda_f^{L^*}$ of a meromorphic function f are defined by*

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log T_f(r)}{\log\left[re^{L(r)}\right]}.$$

If f is entire, then

$$\rho_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]} \text{ and } \lambda_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log^{[2]} M_f(r)}{\log\left[re^{L(r)}\right]}.$$

Definition 4 *[11] The L^*-type $\sigma_f^{L^*}$ and L^*-lower type $\overline{\sigma}_f^{L^*}$ a meromorphic function f such that $0 < \rho_f^{L^*} < \infty$ are defined as*

$$\sigma_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \text{ and } \overline{\sigma}_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}.$$

If f is entire, then

$$\sigma_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}} \text{ and } \overline{\sigma}_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\rho_f^{L^*}}}.$$

Analogously in order to determine the relative growth of two meromorphic functions having same non zero finite L^*-lower order one may introduce the definition of L^*-weak type of meromorphic functions having finite positive L^*-lower order in the following way:

Definition 5 *The L^*-weak type denoted by $\tau_f^{L^*}$ of a meromorphic function f having $0 < \lambda_f^{L^*} < \infty$ is defined as follows:*

$$\tau_f^{L^*} = \underline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}}.$$

Similarly, the growth indicator $\overline{\tau}_f^{L^*}$ is define as

$$\overline{\tau}_f^{L^*} = \overline{\lim_{r\to\infty}} \frac{T_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} \text{ where } 0 < \lambda_f^{L^*} < \infty.$$

If f is entire, then

$$\tau_f^{L^*} = \underline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} \quad \textit{and} \quad \overline{\tau}_f^{L^*} = \overline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[re^{L(r)}\right]^{\lambda_f^{L^*}}} \quad \textit{where } 0 < \lambda_f^{L^*} < \infty .$$

Extending the notion of Somasundaram and Thamizharasi [11], one may introduce concept of ${}_pL^*$-order, ${}_pL^*$-type and ${}_pL^*$-weak type of a meromorphic function f which are as follows:

Definition 6 *For any positive integer p, the ${}_pL^*$-order $\rho_p^{L^*}(f)$ and the ${}_pL^*$-lower order $\lambda_p^{L^*}(f)$ of a meromorphic function f are defined by*

$$\rho_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{\log T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} \quad \textit{and} \quad \lambda_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{\log T_f(r)}{\log\left[r\exp^{[p]} L(r)\right]}.$$

If f is entire, then

$$\rho_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[r\exp^{[p]} L(r)\right]} \quad \textit{and} \quad \lambda_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{\log^{[2]} M_f(r)}{\log\left[r\exp^{[p]} L(r)\right]}.$$

Definition 7 *For any positive integer p, the ${}_pL^*$-type $\sigma_p^{L^*}(f)$ and ${}_pL^*$-lower type $\overline{\sigma}_p^{L^*}(f)$ a meromorphic function f such that $0 < \rho_p^{L^*}(f) < \infty$ are defined as*

$$\sigma_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}} \quad \textit{and} \quad \overline{\sigma}_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}} .$$

If f is entire, then

$$\sigma_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}} \quad \textit{and} \quad \overline{\sigma}_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f)}} .$$

Definition 8 *For any positive integer p, the ${}_pL^*$-weak type denoted by $\tau_p^{L^*}(f)$ of a meromorphic function f having $0 < \lambda_p^{L^*}(f) < \infty$ is defined as follows:*

$$\tau_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}} .$$

Similarly, the growth indicator $\overline{\tau}_p^{L^*}(f)$ is define as

$$\overline{\tau}_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{T_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}}, \quad \textit{where} \quad 0 < \lambda_p^{L^*}(f) < \infty .$$

If f is entire, then for $0 < \lambda_p^{L^*}(f) < \infty$,

$$\tau_p^{L^*}(f) = \underline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}} \quad \textit{and} \quad \overline{\tau}_p^{L^*}(f) = \overline{\lim}_{r\to\infty} \frac{\log M_f(r)}{\left[r\exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f)}} .$$

Lahiri and Banerjee [8] introduced the definition of relative order of a meromorphic function with respect to an entire function which is as follows:

Definition 9 *[8] Let f be meromorphic and g be entire. The relative order of f with respect to g denoted by $\rho_g(f)$ is defined as*

$$\begin{aligned} \rho(f,g) &= \inf\{\mu > 0 : T_f(r) < T_g(r^\mu) \textit{ for all sufficiently large } r\} \\ &= \overline{\lim}_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}. \end{aligned}$$

The definition coincides with the classical one [8] if $g(z) = \exp z$.

Similarly, one can define the relative lower order of a meromorphic function f with respect to an entire g denoted by $\lambda_g(f)$ in the following manner :

$$\lambda(f,g) = \underline{\lim}_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log r}.$$

In order to make some progress in the study of relative order, now we introduce relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of a meromorphic function f with respect to an entire g which are as follows:

Definition 10 *The relative ${}_pL^*$-order denoted as $\rho_p^{L^*}(f,g)$ and relative ${}_pL^*$- lower order denoted as $\lambda_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire g are defined as*

$$\rho_p^{L^*}(f,g) = \overline{\lim}_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[r \exp^{[p]} L(r)\right]} \quad \text{and} \quad \lambda_p^{L^*}(f,g) = \underline{\lim}_{r\to\infty} \frac{\log T_g^{-1} T_f(r)}{\log\left[r \exp^{[p]} L(r)\right]},$$

where p is any positive integers.

Further to compare the relative growth of two meromorphic functions having same non zero finite relative ${}_pL^*$-order with respect to another entire function, one may introduce the definitions of relative ${}_pL^*$-type and relative ${}_pL^*$-lower type in the following manner:

Definition 11 *The relative ${}_pL^*$-type and relative ${}_pL^*$-lower type denoted respectively by $\sigma_p^{L^*}(f,g)$ and $\overline{\sigma}_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g such that $0 < \rho_p^{L^*}(f,g) < \infty$ are respectively defined as follows:*

$$\sigma_p^{L^*}(f,g) = \overline{\lim}_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[r \exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,g)}} \quad \text{and} \quad \overline{\sigma}_p^{L^*}(f,g) = \underline{\lim}_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[r \exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,g)}}.$$

Analogously to determine the relative growth of two meromorphic functions having same non zero finite relative ${}_pL^*$-lower order with respect to an entire function one may introduce the definition of relative ${}_pL^*$-weak type in the following way:

Definition 12 *The relative ${}_pL^*$-weak type denoted by $\tau_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g such that $0 < \lambda_p^{L^*}(f,g) < \infty$ is defined as follows:*

$$\tau_p^{L^*}(f,g) = \underline{\lim}_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[r \exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f,g)}}.$$

Similarly, one may define the growth indicator $\overline{\tau}_p^{L^*}(f,g)$ of a meromorphic function f with respect to an entire function g in the following manner :

$$\overline{\tau}_p^{L^*}(f,g) = \overline{\lim}_{r\to\infty} \frac{T_g^{-1} T_f(r)}{\left[r \exp^{[p]} L(r)\right]^{\lambda_p^{L^*}(f,g)}},\ 0 < \lambda_p^{L^*}(f,g) < \infty.$$

Since the natural extension of a derivative is a differential polynomial, we prove our results for a special type of linear differential polynomials viz the Wronskians. In the paper, we establish some new results depending on the comparative growth properties of composite transcendental entire or meromorphic functions using relative ${}_pL^*$-order, relative ${}_pL^*$-type, relative ${}_pL^*$-weak type and that of Wronskian generated by one of the factors.

2 Lemmas

In this section, we present some lemmas which will be needed in the sequel.

Lemma 1 *[9] Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function with regular growth and non zero finite order. Also let $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\lim_{r\to\infty}\frac{\log T_{W(g)}^{-1}T_{W(f)}(r)}{\log T_g^{-1}T_f(r)}=1.$$

Lemma 2 *[9] Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function of regular growth and non zero finite type. Also let $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then*

$$\lim_{r\to\infty}\frac{T_{W(g)}^{-1}T_{W(f)}(r)}{T_g^{-1}T_f(r)}=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}},$$

where $W(f)=W(a_1,a_2,\cdots,a_{k_1},f)$ and $W(g)=W(a_1,a_2,\cdots,a_{k_2},g)$.

Lemma 3 *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function with regular growth and non zero finite order and $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then for any positive integer p, the relative ${}_pL^*$-order and relative ${}_pL^*$-lower order of $W(f)$ with respect to $W(g)$ are same as those of f with respect to g.*

Proof. By Lemma 1, we obtain that

$$\begin{aligned}\rho_p^{L^*}(W[f],W[g]) &= \overline{\lim_{r\to\infty}}\frac{\log T_{W[g]}^{-1}T_{W[f]}(r)}{\log\left[r\exp^{[p]}L(r)\right]}\\ &= \overline{\lim_{r\to\infty}}\left\{\frac{\log T_g^{-1}T_f(r)}{\log\left[r\exp^{[p]}L(r)\right]}\cdot\frac{\log T_{W[g]}^{-1}T_{W[f]}(r)}{\log T_g^{-1}T_f(r)}\right\}\\ &= \overline{\lim_{r\to\infty}}\frac{\log T_g^{-1}T_f(r)}{\log\left[r\exp^{[p]}L(r)\right]}\cdot\lim_{r\to\infty}\frac{\log T_{W[g]}^{-1}T_{W[f]}(r)}{\log T_g^{-1}T_f(r)}\\ &= \rho_p^{L^*}(f,g)\cdot 1\\ &= \rho_p^{L^*}(f,g)\ .\end{aligned}$$

In a similar manner, $\lambda_p^{L^*}(W[f],W[g])=\lambda_p^{L^*}(f,g)$.

This proves the lemma. ■

Lemma 4 *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function with regular growth and non zero finite type with $\sum\limits_{a\neq\infty}\delta(a;g)+\delta(\infty;g)=2$. Then for any positive integer p, the relative ${}_pL^*$-type and relative ${}_pL^*$-lower type of $W[f]$ with respect to $W[g]$ are $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ times that of f with respect to g if $\rho_p^{L^*}(f,g)$ is positive finite where $W(f)=W(a_1,a_2,\cdots,a_{k_1},f)$ and $W(g)=W(a_1,a_2,\cdots,a_{k_2},g)$.*

Proof. By Lemma 3 and Lemma 2 and above we get that

$$\begin{aligned}\sigma_p^{L^*}(W[f],W[g]) &= \overline{\lim_{r\to\infty}}\frac{T_{W[g]}^{-1}T_{W[f]}(r)}{\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(W[f],W[g])}}\\ &= \lim_{r\to\infty}\frac{T_{W[g]}^{-1}T_{W[f]}(r)}{T_g^{-1}T_f(r)}\cdot\overline{\lim_{r\to\infty}}\frac{T_g^{-1}T_f(r)}{\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,g)}}\\ &= \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\sigma_p^{L^*}(f,g)\ .\end{aligned}$$

Similarly, $\overline{\sigma}_p^{L^*}\left(W\left[f\right],W\left[g\right]\right)=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}\cdot\overline{\sigma}_p^{L^*}\left(f,g\right).$

This proves the lemma. ■

In the line of Lemma 4, we may state the following lemma without its proof :

Lemma 5 *Let f be a transcendental meromorphic function having the maximum deficiency sum and g be a transcendental entire function with regular growth and non zero finite type with* $\sum\limits_{a\neq\infty}\delta\left(a;g\right)+\delta\left(\infty;g\right)=2$. *Then for* $0<\lambda_p^{L^*}\left(f,g\right)<\infty$, $\tau_p^{L^*}\left(W\left[f\right],W\left[g\right]\right)$ *and* $\overline{\tau}_p^{L^*}\left(W\left[f\right],W\left[g\right]\right)$ *are* $\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;g)}\right)^{\frac{1}{\rho_g}}$ *times that of f with respect to g i.e.,*

$$\tau_p^{L^*}\left(W\left[f\right],W\left[g\right]\right)=\left(\frac{1+k_1-k_1\delta\left(\infty;f\right)}{1+k_2-k_2\delta\left(\infty;g\right)}\right)^{\frac{1}{\rho_g}}\cdot\tau_p^{L^*}\left(f,g\right)$$

and

$$\overline{\tau}_p^{L^*}\left(W\left[f\right],W\left[g\right]\right)=\left(\frac{1+k_1-k_1\delta\left(\infty;f\right)}{1+k_2-k_2\delta\left(\infty;g\right)}\right)^{\frac{1}{\rho_g}}\cdot\overline{\tau}_p^{L^*}\left(f,g\right),$$

where $W\left(f\right)=W\left(a_1,a_2,\cdots,a_{k_1},f\right)$ *and* $W\left(g\right)=W\left(a_1,a_2,\cdots,a_{k_2},g\right)$.

Lemma 6 *[11] If f be meromorphic and g be entire then for all sufficiently large values of r,*

$$T_{f\circ g}\left(r\right)\leq\left\{1+o\left(1\right)\right\}\frac{T_g\left(r\right)}{\log M_g\left(r\right)}T_f\left(M_g\left(r\right)\right)\ .$$

Lemma 7 *[10] Let f be meromorphic and g be entire and suppose that $0<\mu<\rho_g\leq\infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}\left(r\right)\geq T_f\left(\exp\left(r^{\mu}\right)\right)\ .$$

Lemma 8 *[12] Let f be meromorphic and g be entire such that $0<\rho_g<\infty$ and $0<\lambda_f$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r)>T_g\left(\exp\left(r^{\mu}\right)\right),$$

where $0<\mu<\rho_g$.

Lemma 9 *[13] Let f be a meromorphic function and g be an entire function such that $\lambda_g<\mu<\infty$ and $0<\lambda_f\leq\rho_f<\infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r)<T_f\left(\exp\left(r^{\mu}\right)\right)\ .$$

Lemma 10 *[13] Let f be a meromorphic function of finite order and g be an entire function such that $0<\lambda_g<\mu<\infty$. Then for a sequence of values of r tending to infinity,*

$$T_{f\circ g}(r)<T_g\left(\exp\left(r^{\mu}\right)\right)\ .$$

Lemma 11 *[14] Let f be an entire function which satisfies the Property (A), $\beta>0$, $\delta>1$ and $\alpha>2$. Then*

$$\beta T_f\left(r\right)<T_f\left(\alpha r^{\delta}\right).$$

3 Main results

In this section, we present the main results of the paper.

Theorem 12 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with* $\sum\limits_{a\neq\infty}\delta\left(a;h\right)+\delta\left(\infty;h\right)=2$. *Also let g be an entire function and* $0<\rho_p^{L^*}\left(f,h\right)<\rho_g$, $\sigma_p^{L^*}\left(f,h\right)>0$ *where p is any positive integer. If* $\exp^{[p-1]}L\left(\exp\left(re^{L(r)}\right)^{\beta}\right)=o\left(\left[r\exp^{[p]}L\left(r\right)\right]^{\beta}\right)$ $(r\rightarrow\infty)$ *for any $\beta>0$, then*

$$\varliminf_{r\rightarrow\infty}\frac{\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)}{T_{W[h]}^{-1}T_{W[f]}\left(r\right)}\geq\frac{\lambda_p^{L^*}\left(f,h\right)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}\left(f,h\right)}\ .$$

Proof. From the definition of relative ${}_pL^*$- type of meromorphic function and in view of Lemma 4, we obtain for all sufficiently large values of r that

$$T_{W[h]}^{-1}T_{W[f]}(r) \leq \left(\sigma_p^{L^*}(W[f],W[h])+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(W[f],W[h])},$$
$$i.e.,\ T_{W[h]}^{-1}T_{W[f]}(r) \leq \left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)}. \tag{1}$$

As $0<\rho_p^{L^*}(f,h)<\rho_g$, we obtain in view of Lemma 7 for a sequence of values of r tending to infinity that

$$\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)\geq\log T_h^{-1}T_f\left(\exp\left(re^{L(r)}\right)^{\rho_p^{L^*}(f,h)}\right),$$

$$i.e.,\quad \log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)\ \geq\ \left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\left[\left[re^{L(r)}\right]^{\rho_p^{L^*}(f,h)}+\exp^{[p-1]}L\left(\exp\left(re^{L(r)}\right)^{\rho_p^{L^*}(f,h)}\right)\right].$$

Therefore from (1) and above, it follows for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)}{T_{W[h]}^{-1}T_{W[f]}(r)}\geq\frac{\left(\lambda_p^{L^*}(f,h)-\varepsilon\right)\left[\left[re^{L(r)}\right]^{\rho_p^{L^*}(f,h)}+\exp^{[p-1]}L\left(\exp\left(re^{L(r)}\right)^{\rho_p^{L^*}(f,h)}\right)\right]}{\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)}}.$$

Since $\lim\limits_{r\to\infty}\frac{\exp^{[p-1]}L\left(\exp\left(re^{L(r)}\right)^{\rho_p^{L^*}(f,h)}\right)}{\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)}}=0$ as $\exp^{[p-1]}L\left(\exp\left(re^{L(r)}\right)^{\beta}\right)=o\left(\left[r\exp^{[p]}L(r)\right]^{\beta}\right)$ $(r\to\infty)$ for any $\alpha>0$, we obtain from above that

$$\overline{\lim_{r\to\infty}}\frac{\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)}{T_{W[h]}^{-1}T_{W[f]}(r)}\geq\frac{\lambda_p^{L^*}(f,h)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)}.$$

Thus the theorem follows. ■

Now using the concept of the growth indicator $\overline{\tau}_p^{L^*}(f,h)$ of a meromorphic function f, we may state the following theorem without its proof since it can be carried out in the line of Theorem 12 and in view of Lemma 5.

Theorem 13 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum\limits_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function and $0<\rho_p^{L^*}(f,h)<\rho_g$, $\overline{\tau}_p^{L^*}(f,h)>0$ where p is any positive integer. If $\exp^{[p-1]}L\left(\exp\left(re^{L(r)}\right)^{\beta}\right)=o\left(\left[r\exp^{[p]}L(r)\right]^{\beta}\right)$ $(r\to\infty)$ for any $\beta>0$, then*

$$\overline{\lim_{r\to\infty}}\frac{\log T_h^{-1}T_{f\circ g}\left(re^{L(r)}\right)}{T_{W[h]}^{-1}T_{W[f]}(r)}\geq\frac{\lambda_p^{L^*}(f,h)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h)}.$$

Theorem 14 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum\limits_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)=\rho_p^{L^*}(g)$, (ii) $\sigma_p^{L^*}(g)<\infty$, and (iii) $\overline{\sigma}_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) if $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, then

$$\overline{\lim_{r\to\infty}}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(f,h)\cdot\sigma_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)},$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ *then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \rho_p^{L^*}(f,h) \ .$$

Proof. Let us consider that $\alpha > 2$ and $\delta \to 1^+$ in Lemma 11. Since $T_h^{-1}(r)$ is an increasing function of r, it follows from Lemma 6, Lemma 11, and the inequality $T_g(r) \leq \log M_g(r)$ [1] for all sufficiently large values of r that

$$\begin{aligned} T_h^{-1}T_{f\circ g}(r) &\leqslant T_h^{-1}\left[\{1+o(1)\} T_f(M_g(r))\right], \\ i.e.,\ T_h^{-1}T_{f\circ g}(r) &\leqslant \alpha\left[T_h^{-1}T_f(M_g(r))\right], \\ i.e.,\ \log T_h^{-1}T_{f\circ g}(r) &\leqslant \log T_h^{-1}T_f(M_g(r)) + O(1), \end{aligned}$$

$$i.e.,\ \log T_h^{-1}T_{f\circ g}(r) \leqslant \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\log M_g(r) + \exp^{[p-1]} L(M_g(r))\right) + O(1),$$

$$\begin{aligned} i.e.,\ \log T_h^{-1}T_{f\circ g}(r) \leqslant & \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\sigma_p^{L^*}(g)+\varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(g)} \\ & + \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\exp^{[p-1]} L(M_g(r)) + O(1), \end{aligned}$$

In view of condition (ii), we obtain from above for all sufficiently large values of r that

$$\begin{aligned} \log T_h^{-1}T_{f\circ g}(r) \leqslant & \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\sigma_p^{L^*}(g)+\varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,h)} \\ & + \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\exp^{[p-1]} L(M_g(r)) + O(1) \ . \end{aligned} \tag{2}$$

Again from the definition of relative ${}_pL^*$-lower type we get in view of Lemma 4, for all sufficiently large values of r that

$$T_{W[h]}^{-1}T_{W[f]}(r) \geq \left(\overline{\sigma}_p^{L^*}(W[f],W[h]) - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(W[f],W[h])},$$

$$i.e.,\ T_{W[h]}^{-1}T_{W[f]}(r) \geq \left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h) - \varepsilon\right)\left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,h)},$$

$$i.e.,\ \left[r\exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,h)} \leq \frac{T_{W[h]}^{-1}T_{W[f]}(r)}{\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h) - \varepsilon\right)} \ . \tag{3}$$

Now from (2) and (3) , it follows for all sufficiently large values of r that

$$\begin{aligned} & \log T_h^{-1}T_{f\circ g}(r) \leqslant \\ & \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\sigma_p^{L^*}(g)+\varepsilon\right)\frac{T_{W[h]}^{-1}T_{W[f]}(r)}{\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h) - \varepsilon\right)} \\ & \qquad + \left(\rho_p^{L^*}(f,h)+\varepsilon\right)\exp^{[p-1]} L(M_g(r)) + O(1), \end{aligned}$$

$$\begin{aligned} ie.,\ \frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq & \frac{O(1)}{T_{W[h]}^{-1}T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \\ & + \frac{\frac{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\sigma_p^{L^*}(g)+\varepsilon\right)}{\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h) - \varepsilon\right)}}{1 + \frac{\exp^{[p-1]} L(M_g(r))}{T_{W[h]}^{-1}T_{W[f]}(r)}} + \frac{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)}{1 + \frac{T_{W[h]}^{-1}T_{W[f]}(r)}{\exp^{[p-1]} L(M_g(r))}} \ . \end{aligned} \tag{4}$$

If $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{W[h]}^{-1} T_{W[f]}(r)\right\}$ then from (4) we get that

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\left(\rho_p^{L^*}(f,h)+\varepsilon\right)\left(\sigma_p^{L^*}(g)+\varepsilon\right)}{\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)-\varepsilon\right)} .$$

Since $\varepsilon(>0)$ is arbitrary, it follows from above that

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_p^{L^*}(f,h)\cdot\sigma_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)} .$$

Thus the first part of the theorem follows.

Since $\varepsilon(>0)$ is arbitrary, and if $T_{W[h]}^{-1} T_{W[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then from (4) it follows that

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \rho_p^{L^*}(f,h) .$$

Thus the second part of the theorem is established. ■

Theorem 15 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\lambda_p^{L^*}(f,h)<\infty$, (ii) $\rho_p^{L^*}(f,h)=\rho_p^{L^*}(g)$, (iii) $\sigma_p^{L^*}(g)<\infty$, and (iv) $\overline{\sigma}_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) if $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{W[h]}^{-1} T_{W[f]}(r)\right\}$, then

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\lambda_p^{L^*}(f,h)\cdot\sigma_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \overline{\sigma}_p^{L^*}(f,h)}$$

and (b) if $T_{W[h]}^{-1} T_{W[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \lambda_p^{L^*}(f,h) .$$

Theorem 16 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)=\rho_p^{L^*}(g)$, (ii) $\sigma_p^{L^*}(g)<\infty$, and (iii) $\sigma_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) if $\exp^{[p-1]} L(M_g(r)) = o\left\{T_{W[h]}^{-1} T_{W[f]}(r)\right\}$, then

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \frac{\rho_p^{L^*}(f,h)\cdot\sigma_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h)}$$

and (b) if $T_{W[h]}^{-1} T_{W[f]}(r) = o\left\{\exp^{[p-1]} L(M_g(r))\right\}$ then

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r) + \exp^{[p-1]} L(M_g(r))} \leq \rho_p^{L^*}(f,h) .$$

Theorem 17 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)=\rho_p^{L^*}(g)$, (ii) $\overline{\sigma}_p^{L^*}(g)<\infty$, and (iii) $\overline{\sigma}_p^{L^*}(f,h)>$ where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(f,h)\cdot\overline{\sigma}_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\rho_p^{L^*}(f,h)\ .$$

We omit the proof of the above three theorems as those can be carried out in the line of Theorem 14.

Similarly using the concept of the growth indicator $\tau_p^{L^*}(f,h)$ and $\overline{\tau}_p^{L^*}(g)$ we may state the subsequent four theorems without their proofs since those can be carried out in view of Lemma 5 and in the line of Theorem 14, Theorem 15, Theorem 16 and Theorem 17 respectively .

Theorem 18 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)<\infty$, (ii) $\lambda_p^{L^*}(f,h)=\lambda_p^{L^*}(g)$, (iii) $\overline{\tau}_p^{L^*}(g)<\infty$, and (iv) $\tau_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varlimsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(f,h)\cdot\overline{\tau}_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varlimsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\rho_p^{L^*}(f,h)\ .$$

Theorem 19 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\lambda_p^{L^*}(f,h)=\lambda_p^{L^*}(g)$, (ii) $\overline{\tau}_p^{L^*}(g)<\infty$, and (iii) $\tau_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\lambda_p^{L^*}(f,h)\cdot\overline{\tau}_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\lambda_p^{L^*}(f,h)\ .$$

Theorem 20 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)<\infty$, (ii) $\lambda_p^{L^*}(f,h)=\lambda_p^{L^*}(g)$, (iii) $\overline{\tau}_p^{L^*}(g)<\infty$, and (iv) $\overline{\tau}_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(f,h)\cdot\overline{\tau}_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\tau}_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\rho_p^{L^*}(f,h)\ .$$

Theorem 21 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)<\infty$, (ii) $\lambda_p^{L^*}(f,h)=\lambda_p^{L^*}(g)$, (iii) $\tau_p^{L^*}(g)<\infty$, and (iv) $\tau_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(f,h)\cdot\tau_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\rho_p^{L^*}(f,h)\ .$$

Analogously we state the following four theorems under some different conditions which can also be carried out using the same technique of Theorem 14 and therefore their proofs are omitted.

Theorem 22 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)<\infty$, (ii) $\lambda_p^{L^*}(f,h)=\rho_p^{L^*}(g)$, (iii) $\sigma_p^{L^*}(g)<\infty$, and (iv) $\tau_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varlimsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(f,h)\cdot\sigma_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varlimsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\rho_p^{L^*}(f,h)\ .$$

Theorem 23 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\lambda_p^{L^*}(f,h)=\rho_p^{L^*}(g)$, (ii) $\sigma_p^{L^*}(g)<\infty$, and (iii) $\tau_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\lambda_p^{L^*}(f,h)\cdot\sigma_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\tau_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\lambda_p^{L^*}(f,h)\ .$$

Theorem 24 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)=\lambda_p^{L^*}(g)$, (ii) $\overline{\tau}_p^{L^*}(g)<\infty$, and (iii) $\overline{\sigma}_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varlimsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\rho_p^{L^*}(f,h)\cdot\overline{\tau}_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varlimsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\rho_p^{L^*}(f,h)\ .$$

Theorem 25 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function, h satisfy the Property (A) and (i) $\rho_p^{L^*}(f,h)=\lambda_p^{L^*}(g)$, (ii) $\overline{\tau}_p^{L^*}(g)<\infty$, and (iii) $\overline{\sigma}_p^{L^*}(f,h)>0$, where p is any positive integer.*
(a) *if* $\exp^{[p-1]}L(M_g(r))=o\left\{T_{W[h]}^{-1}T_{W[f]}(r)\right\}$, *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\frac{\lambda_p^{L^*}(f,h)\cdot\overline{\tau}_p^{L^*}(g)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}$$

and (b) *if* $T_{W[h]}^{-1}T_{W[f]}(r)=o\left\{\exp^{[p-1]}L(M_g(r))\right\}$ *then*

$$\varliminf_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)+\exp^{[p-1]}L(M_g(r))}\leq\lambda_p^{L^*}(f,h)\ .$$

Theorem 26 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite order with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$. Also let g be an entire function and $0<\lambda_p^{L^*}(f,h)\leq\rho_p^{L^*}(f,h)<\infty$ where p is any positive integer. Then*

$$\varlimsup_{r\to\infty}\frac{\log T_h^{-1}T_{f\circ g}(r)}{\log T_{W[h]}^{-1}T_{W[f]}(r)(\exp r^{\mu})}\geq\frac{\lambda_p^{L^*}(f,h)}{\rho_p^{L^*}(f,h)},$$

where $0<\mu<\rho_g\leq\infty$.

Proof. In view of Lemma 7, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &\geq \log T_h^{-1} T_f(\exp r^\mu), \\ i.e., \ \log T_h^{-1} T_{f \circ g}(r) &\geq \left(\lambda_p^{L^*}(f,h) - \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]. \end{aligned} \tag{5}$$

Also in view of Lemma 4, and for any arbitrary $\varepsilon (> 0)$, it follows for all sufficiently large values of r that

$$\log T_{W[h]}^{-1} T_{W[f]}(\exp r^\mu) \leq \left(\rho_p^{L^*}(W[f], W[h]) + \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right],$$

$$i.e., \ \log T_{W[h]}^{-1} T_{W[f]}(\exp r^\mu) \leq \left(\rho_p^{L^*}(f,h) + \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]. \tag{6}$$

Now from (5) and (6), we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{W[h]}^{-1} T_{W[f]}(\exp r^\mu)} \geq \frac{\left(\lambda_p^{L^*}(f,h) - \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]}{\left(\rho_p^{L^*}(f,h) + \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]}.$$

Since $\varepsilon (> 0)$ is arbitrary, it follows from above that

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{W[h]}^{-1} T_{W[f]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(f,h)}{\rho_p^{L^*}(f,h)}.$$

Thus the theorem follows. ■

Theorem 27 *Let f be a meromorphic function, g be a transcendental entire having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite order with* $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$. *Also let* $0 < \lambda_f$ *and* $0 < \lambda_p^{L^*}(g,h) \leq \rho_p^{L^*}(g,h) < \infty$ *where p is any positive integer. Then*

$$\overline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{W[h]}^{-1} T_{W[g]}(\exp r^\mu)} \geq \frac{\lambda_p^{L^*}(g,h)}{\rho_p^{L^*}(g,h)},$$

where $0 < \mu < \rho_g$.

We omit the proof of the above theorem as it can be carried out in the line of Theorem 26 and with the help of Lemma 8.

Theorem 28 *Let f be a transcendental meromorphic function having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite order with* $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$. *Also let g be an entire function and* $0 < \lambda_p^{L^*}(f,h) \leq \rho_p^{L^*}(f,h) < \infty$ *where p is any positive integer. Then*

$$\underline{\lim_{r\to\infty}} \frac{\log T_h^{-1} T_{f \circ g}(r)}{\log T_{W[h]}^{-1} T_{W[f]}(\exp r^\mu)} \leq \frac{\rho_p^{L^*}(f,h)}{\lambda_p^{L^*}(f,h)},$$

where $\lambda_g < \mu < \infty$.

Proof. In view of Lemma 9, we obtain for a sequence of values of r tending to infinity that

$$\begin{aligned} \log T_h^{-1} T_{f \circ g}(r) &< \log T_h^{-1} T_f(\exp r^\mu), \\ i.e., \ \log T_h^{-1} T_{f \circ g}(r) &< \left(\rho_p^{L^*}(f,h) + \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right]. \end{aligned} \tag{7}$$

Also in view of Lemma 4, and for any arbitrary $\varepsilon (> 0)$, it follows for all sufficiently large values of r that

$$\log T_{W[h]}^{-1} T_{W[f]}(\exp r^\mu) \geq \left(\lambda_p^{L^*}(W[f], W[h]) - \varepsilon\right)\left[r^\mu + \exp^{[p-1]} L(\exp r^\mu)\right],$$

$$i.e.,\ \log T_{W[h]}^{-1} T_{W[f]} (\exp r^{\mu}) \geq$$
$$\left(\lambda_p^{L^*}(f,h) - \varepsilon\right) \left[r^{\mu} + \exp^{[p-1]} L(\exp r^{\mu})\right] . \tag{8}$$

Now from (7) and (8) , we get for a sequence of values of r tending to infinity that

$$\frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{W[h]}^{-1} T_{W[f]} (\exp r^{\mu})} < \frac{\left(\rho_p^{L^*}(f,h) + \varepsilon\right) \left[r^{\mu} + \exp^{[p-1]} L(\exp r^{\mu})\right]}{\left(\lambda_p^{L^*}(f,h) - \varepsilon\right) \left[r^{\mu} + \exp^{[p-1]} L(\exp r^{\mu})\right]} .$$

Since $\varepsilon\,(> 0)$ is arbitrary, it follows from above that

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{W[h]}^{-1} T_{W[f]} (\exp r^{\mu})} \leq \frac{\rho_p^{L^*}(f,h)}{\lambda_p^{L^*}(f,h)} .$$

Thus the theorem follows. ■

Now we state the following theorem without its proof as it can be carried out in the line of above theorem and with the help of Lemma 10:

Theorem 29 *Let f be a meromorphic function, g be a transcendental entire having the maximum deficiency sum and h be a transcendental entire function with regular growth and non zero finite order with $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$. Also let $0 < \lambda_f$ and $0 < \lambda_p^{L^*}(g,h) \leq \rho_p^{L^*}(g,h) < \infty$ where p is any positive integer. Then*

$$\varliminf_{r\to\infty} \frac{\log T_h^{-1} T_{f\circ g}(r)}{\log T_{W[h]}^{-1} T_{W[g]} (\exp r^{\mu})} \leq \frac{\rho_p^{L^*}(g,h)}{\lambda_p^{L^*}(g,h)},$$

where $0 < \lambda_g < \mu < \infty$.

Theorem 30 *Let f be a transcendental meromorphic function having the maximum deficiency sum . Also let h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty} \delta(a;h) + \delta(\infty;h) = 2$ and g be any entire function such that $0 < \overline{\sigma}_p^{L^*}(f\circ g,h) \leq \sigma_p^{L^*}(f\circ g,h) < \infty$, $0 < \overline{\sigma}_p^{L^*}(f,h) \leq \sigma_p^{L^*}(f,h) < \infty$ and $\rho_p^{L^*}(f\circ g,h)$ $= \rho_p^{L^*}(f,h)$ where p is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{A\cdot \sigma_p^{L^*}(f,h)} \leq \varliminf_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r)} \leq \min\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{A\cdot \overline{\sigma}_p^{L^*}(f,h)}, \frac{\sigma_p^{L^*}(f\circ g,h)}{A\cdot \sigma_p^{L^*}(f,h)}\right\}$$
$$\leq \max\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{A\cdot \overline{\sigma}_p^{L^*}(f,h)}, \frac{\sigma_p^{L^*}(f\circ g,h)}{A\cdot \sigma_p^{L^*}(f,h)}\right\} \leq \varlimsup_{r\to\infty} \frac{T_h^{-1} T_{f\circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)}{A\cdot \overline{\sigma}_p^{L^*}(f,h)},$$

where $A = \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}$.

Proof. From the definitions of $\sigma_p^{L^*}(W[f],W[h])$, $\overline{\sigma}_p^{L^*}(f\circ g,h)$ and in view of Lemma 9, we have for arbitrary positive ε and for all sufficiently large values of r that

$$T_h^{-1} T_{f\circ g}(r) \geq \left(\overline{\sigma}_p^{L^*}(f\circ g,h) - \varepsilon\right) \left[r \exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f\circ g,h)} , \tag{9}$$

and

$$T_{W[h]}^{-1} T_{W[f]}(r) \leq \left(\sigma_p^{L^*}(W[f],W[h]) + \varepsilon\right) \left[r \exp^{[p]} L(r)\right]^{\rho_p^{L^*}(W[f],W[h])} ,$$
$$i.e.,\ T_{W[h]}^{-1} T_{W[f]}(r) \leq$$
$$\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}} \cdot \sigma_p^{L^*}(f,h) + \varepsilon\right) \left[r \exp^{[p]} L(r)\right]^{\rho_p^{L^*}(f,h)} . \tag{10}$$

Now from (9), (10) and the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, it follows for all sufficiently large values of r that,

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)}\geqslant\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)-\varepsilon}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)+\varepsilon}\,.$$

As $\varepsilon\,(>0)$ is arbitrary , we obtain from above that

$$\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)}\geqslant\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)}\,. \tag{11}$$

Again for a sequence of values of r tending to infinity,

$$T_h^{-1}T_{f\circ g}(r)\leq\left(\overline{\sigma}_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f\circ g,h)} \tag{12}$$

and in view of Lemma 9, it follows for all sufficiently large values of r,

$$T_{W[h]}^{-1}T_{W[f]}(r)\geq\left(\overline{\sigma}_p^{L^*}(W[f],W[h])-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(W[f],W[h])},$$

$$i.e.,\ T_{W[h]}^{-1}T_{W[f]}(r)\geq$$

$$\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)}. \tag{13}$$

Combining (12) and (13) and the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)}\leq\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)+\varepsilon}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)-\varepsilon}\,.$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows from above that

$$\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)}\leq\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}\,. \tag{14}$$

Also in view of Lemma 9, for a sequence of values of r tending to infinity we get that

$$T_{W[h]}^{-1}T_{W[f]}(r)\leq\left(\overline{\sigma}_p^{L^*}(W[f],W[h])+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(W[f],W[h])},$$

$$i.e.,\ T_{W[h]}^{-1}T_{W[f]}(r)\leq$$

$$\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)}. \tag{15}$$

Now from (9), (15) and the condition $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(f,h)$, we obtain for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)}\geq\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)-\varepsilon}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)+\varepsilon}\,.$$

As $\varepsilon\,(>0)$ is arbitrary, we get from above that

$$\varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)}\geq\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)}\,. \tag{16}$$

Also for all sufficiently large values of r,

$$T_h^{-1}T_{f\circ g}(r) \leq \left(\sigma_p^{L^*}(f\circ g,h)+\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f\circ g,h)} . \tag{17}$$

In view of the condition $\rho_p^{L^*}(f\circ g,h) = \rho_p^{L^*}(f,h)$, it follows from (13) and (17) for all sufficiently large values of r that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)+\varepsilon}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)-\varepsilon} .$$

Since $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\overline{\lim_{r\to\infty}}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\overline{\sigma}_p^{L^*}(f,h)} . \tag{18}$$

Again the definition of $\sigma_p^{L^*}(W[f],W[h])$ and in view of Lemma 9, we get for a sequence of values of r tending to infinity that

$$T_{W[h]}^{-1}T_{W[f]}(r) \geq \left(\sigma_p^{L^*}(W[f],W[h])-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(W[f],W[h])},$$

$$i.e.,\ T_{W[h]}^{-1}T_{W[f]}(r) \geq$$

$$\left(\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f,h)} . \tag{19}$$

Now from (17), (19) and the condition $\rho_p^{L^*}(f\circ g,h) = \rho_p^{L^*}(f,h)$, it follows for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)+\varepsilon}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)-\varepsilon} .$$

As $\varepsilon\,(>0)$ is arbitrary, we obtain that

$$\underline{\lim_{r\to\infty}}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)} \leq \frac{\sigma_p^{L^*}(f\circ g,h)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)} . \tag{20}$$

Again for a sequence of values of r tending to infinity that

$$T_h^{-1}T_{f\circ g}(r) \geqslant \left(\sigma_p^{L^*}(f\circ g,h)-\varepsilon\right)\left[r\exp^{[p]}L(r)\right]^{\rho_p^{L^*}(f\circ g,h)} . \tag{21}$$

So combining (10) and (21) and in view of the condition $\rho_p^{L^*}(f\circ g,h) = \rho_p^{L^*}(f,h)$, we get for a sequence of values of r tending to infinity that

$$\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)} \geqslant \frac{\sigma_p^{L^*}(f\circ g,h)-\varepsilon}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)+\varepsilon} .$$

Since $\varepsilon\,(>0)$ is arbitrary, it follows that

$$\overline{\lim_{r\to\infty}}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)} \geqslant \frac{\sigma_p^{L^*}(f\circ g,h)}{\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}\cdot\sigma_p^{L^*}(f,h)} . \tag{22}$$

Thus the theorem follows from (11), (14), (16), (18), (20) and (22). ∎

Next theorem can be carried out in the line of Theorem 30 and therefore we omit its proof.

Theorem 31 *Let g be a transcendental entire having the maximum deficiency sum . Also let h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ and f be any meromorphic function such that $0<\overline{\sigma}_p^{L^*}(f\circ g,h)\leq\sigma_p^{L^*}(f\circ g,h)<\infty$, $0<\overline{\sigma}_p^{L^*}(g,h)\leq\sigma_p^{L^*}(g,h)<\infty$ and $\rho_p^{L^*}(f\circ g,h)=\rho_p^{L^*}(g,h)$ where p is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{B\cdot\sigma_p^{L^*}(g,h)}\leq\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[g]}(r)}\leq\min\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{B\cdot\overline{\sigma}_p^{L^*}(g,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{B\cdot\sigma_p^{L^*}(g,h)}\right\}$$
$$\leq\max\left\{\frac{\overline{\sigma}_p^{L^*}(f\circ g,h)}{B\cdot\overline{\sigma}_p^{L^*}(g,h)},\frac{\sigma_p^{L^*}(f\circ g,h)}{B\cdot\sigma_p^{L^*}(g,h)}\right\}\leq\varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[g]}(r)}\leq\frac{\sigma_p^{L^*}(f\circ g,h)}{B\cdot\overline{\sigma}_p^{L^*}(g,h)},$$

where $B=\left(\frac{1+k_1-k_1\delta(\infty;g)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}$.

Now in the line of Theorem 30 and Theorem 31, and with the help of Lemma 10 one can easily prove the following two theorems using the notion of relative ${}_pL^*$- weak type and therefore their proofs are omitted.

Theorem 32 *Let f be a transcendental meromorphic function having the maximum deficiency sum . Also let h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ and g be any entire function such that $0<\tau_p^{L^*}(f\circ g,h)\leq\overline{\tau}_p^{L^*}(f\circ g,h)<\infty$, $0<\tau_p^{L^*}(f,h)\leq\overline{\tau}_p^{L^*}(f,h)<\infty$ and $\lambda_p^{L^*}(f\circ g,h)=\lambda_p^{L^*}(f,h)$ where p is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f\circ g,h)}{A\cdot\overline{\tau}_p^{L^*}(f,h)}\leq\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)}\leq\min\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{A\cdot\tau_p^{L^*}(f,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{A\cdot\overline{\tau}_p^{L^*}(f,h)}\right\}$$
$$\leq\max\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{A\cdot\tau_p^{L^*}(f,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{A\cdot\overline{\tau}_p^{L^*}(f,h)}\right\}\leq\varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[f]}(r)}\leq\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{A\cdot\tau_p^{L^*}(f,h)},$$

where $A=\left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}$.

Theorem 33 *Let g be a transcendental entire having the maximum deficiency sum . Also let h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ and f be any meromorphic function such that $0<\tau_p^{L^*}(f\circ g,h)\leq\overline{\tau}_p^{L^*}(f\circ g,h)<\infty$, $0<\tau_p^{L^*}(g,h)\leq\overline{\tau}_p^{L^*}(g,h)<\infty$ and $\lambda_p^{L^*}(f\circ g,h)=\lambda\rho_p^{L^*}(g,h)$ where p is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f\circ g,h)}{B\cdot\overline{\tau}_p^{L^*}(g,h)}\leq\varliminf_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[g]}(r)}\leq\min\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{B\cdot\tau_p^{L^*}(g,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{B\cdot\overline{\tau}_p^{L^*}(g,h)}\right\}$$
$$\leq\max\left\{\frac{\tau_p^{L^*}(f\circ g,h)}{B\cdot\tau_p^{L^*}(g,h)},\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{B\cdot\overline{\tau}_p^{L^*}(g,h)}\right\}\leq\varlimsup_{r\to\infty}\frac{T_h^{-1}T_{f\circ g}(r)}{T_{W[h]}^{-1}T_{W[g]}(r)}\leq\frac{\overline{\tau}_p^{L^*}(f\circ g,h)}{B\cdot\tau_p^{L^*}(g,h)},$$

where $B=\left(\frac{1+k_1-k_1\delta(\infty;g)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}$.

We may now state the following theorems without their proofs based on relative ${}_pL^*$- type and relative ${}_pL^*$- weak type because those can easily be carried out with the help of Lemma 9 and Lemma 10:

Theorem 34 *Let f be a transcendental meromorphic function having the maximum deficiency sum. Also let h be a transcendental entire function with regular growth and non zero finite type with $\sum_{a\neq\infty}\delta(a;h)+\delta(\infty;h)=2$ and g*

be any entire function such that $0 < \overline{\sigma}_p^{L^*}(f \circ g, h) \le \sigma_p^{L^*}(f \circ g, h) < \infty$, $0 < \tau_p^{L^*}(f, h) \le \overline{\tau}_p^{L^*}(f, h) < \infty$ *and* $\rho_p^{L^*}(f \circ g, h) = \lambda_p^{L^*}(f, h)$ *where p is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)} \le \underline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r)} \le \min\left\{\frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)}, \frac{\sigma_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)}\right\}$$
$$\le \max\left\{\frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)}, \frac{\sigma_p^{L^*}(f \circ g, h)}{A \cdot \overline{\tau}_p^{L^*}(f, h)}\right\} \le \overline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r)} \le \frac{\sigma_p^{L^*}(f \circ g, h)}{A \cdot \tau_p^{L^*}(f, h)},$$

where $A = \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}$.

Theorem 35 *Let f be a transcendental meromorphic function having the maximum deficiency sum . Also let h be a transcendental entire function with regular growth and non zero finite type with* $\sum_{a \neq \infty} \delta(a; h) + \delta(\infty; h) = 2$ *and g be any entire function such that* $0 < \tau_p^{L^*}(f \circ g, h) \le \overline{\tau}_p^{L^*}(f \circ g, h) < \infty$, $0 < \overline{\sigma}_p^{L^*}(f, h) \le \sigma_p^{L^*}(f, h) < \infty$ *and* $\lambda_p^{L^*}(f \circ g, h) = \rho_p^{L^*}(f, h)$ *where p is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \sigma_p^{L^*}(f, h)} \le \underline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r)} \le \min\left\{\frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \overline{\sigma}_p^{L^*}(f, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \sigma_p^{L^*}(f, h)}\right\}$$
$$\le \max\left\{\frac{\tau_p^{L^*}(f \circ g, h)}{A \cdot \overline{\sigma}_p^{L^*}(f, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \sigma_p^{L^*}(f, h)}\right\} \le \overline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{W[h]}^{-1} T_{W[f]}(r)} \le \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{A \cdot \overline{\sigma}_p^{L^*}(f, h)},$$

where $A = \left(\frac{1+k_1-k_1\delta(\infty;f)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}$.

Theorem 36 *Let g be a transcendental entire having the maximum deficiency sum . Also let h be a transcendental entire function with regular growth and non zero finite type with* $\sum_{a \neq \infty} \delta(a; h) + \delta(\infty; h) = 2$ *and f be any meromorphic function such that* $0 < \overline{\sigma}_p^{L^*}(f \circ g, h) \le \sigma_p^{L^*}(f \circ g, h) < \infty$, $0 < \tau_p^{L^*}(g, h) \le \overline{\tau}_p^{L^*}(g, h) < \infty$ *and* $\rho_p^{L^*}(f \circ g, h) = \lambda\rho_p^{L^*}(g, h)$ *where p is any positive integer. Then*

$$\frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{B \cdot \overline{\tau}_p^{L^*}(g, h)} \le \underline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{W[h]}^{-1} T_{W[g]}(r)} \le \min\left\{\frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{B \cdot \tau_p^{L^*}(g, h)}, \frac{\sigma_p^{L^*}(f \circ g, h)}{B \cdot \overline{\tau}_p^{L^*}(g, h)}\right\}$$
$$\le \max\left\{\frac{\overline{\sigma}_p^{L^*}(f \circ g, h)}{B \cdot \tau_p^{L^*}(g, h)}, \frac{\sigma_p^{L^*}(f \circ g, h)}{B \cdot \overline{\tau}_p^{L^*}(g, h)}\right\} \le \overline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{W[h]}^{-1} T_{W[g]}(r)} \le \frac{\sigma_p^{L^*}(f \circ g, h)}{B \cdot \tau_p^{L^*}(g, h)},$$

where $B = \left(\frac{1+k_1-k_1\delta(\infty;g)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}$.

Theorem 37 *Let g be a transcendental entire having the maximum deficiency sum . Also let h be a transcendental entire function with regular growth and non zero finite type with* $\sum_{a \neq \infty} \delta(a; h) + \delta(\infty; h) = 2$ *and f be any meromorphic function such that* $0 < \tau_p^{L^*}(f \circ g, h) \le \overline{\tau}_p^{L^*}(f \circ g, h) < \infty$, $0 < \overline{\sigma}_p^{L^*}(g, h) \le \sigma_p^{L^*}(g, h) < \infty$ *and* $\lambda_p^{L^*}(f \circ g, h) = \rho\rho_p^{L^*}(g, h)$ *where p is any positive integer. Then*

$$\frac{\tau_p^{L^*}(f \circ g, h)}{B \cdot \sigma_p^{L^*}(g, h)} \le \underline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{W[h]}^{-1} T_{W[g]}(r)} \le \min\left\{\frac{\tau_p^{L^*}(f \circ g, h)}{B \cdot \overline{\sigma}_p^{L^*}(g, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{B \cdot \sigma_p^{L^*}(g, h)}\right\}$$
$$\le \max\left\{\frac{\tau_p^{L^*}(f \circ g, h)}{B \cdot \overline{\sigma}_p^{L^*}(g, h)}, \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{B \cdot \sigma_p^{L^*}(g, h)}\right\} \le \overline{\lim}_{r \to \infty} \frac{T_h^{-1} T_{f \circ g}(r)}{T_{W[h]}^{-1} T_{W[g]}(r)} \le \frac{\overline{\tau}_p^{L^*}(f \circ g, h)}{B \cdot \overline{\sigma}_p^{L^*}(g, h)},$$

where $B = \left(\frac{1+k_1-k_1\delta(\infty;g)}{1+k_2-k_2\delta(\infty;h)}\right)^{\frac{1}{\rho_h}}$.

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