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Study on Geometric Evolution Properties of Planar Closed Curve Flow

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Abstract: In this paper, we use the governing equations of the curve geometric variables to explore the geometric evolution characteristics of the plane curve flow, and obtain the description and characterization of the related geometric quantity properties. The overall evolution law and characteristics of the plane closed curve flow are obtained by the distance from the outer point of the curve to the curve. The finite result of the overall evolution velocity of the plane simple closed curve is obtained. By the geometric estimation of the curvature of the plane closed convex curve, the boundedness of the curvature of the plane simple closed convex curve, the monotonicity of the curvature velocity in the time interval and the convexity of the curvature in the time interval are obtained.

Keywords: Plane curve flow; Curvature evolution; Geometric variable; Global evolution; Geometric estimation; Concavity

1 Introduction

Based on the needs of physical phenomena and real-world problems, the curves and surfaces are affected by external forces to produce the corresponding flow properties, which are getting more and more attention and research. Among them, the curve shrinkage flow is one of its typical problems. In 1984, when Gage [1] investigated the Plane curve contraction flow

$$\begin{cases} \dfrac{\partial X}{\partial t} = k(u,t)N(k(u,t)), \\ X(u,0) = X_0(u,0), \end{cases} \tag{1}$$

where κ is the Gaussian curvature at the point (u,t) on the curve and N is the corresponding internal normal vector. They obtained a geodesic equation as follows: When the initial curve is a convex flat simple closed curve, the curve flow (1) will always remain convex during the evolution and shrink into points in a limited time. In 1986, Gage [2, 3] discussed the plane preservation area curve flow

$$\begin{cases} \dfrac{\partial X}{\partial t} = (k(u,t) - \dfrac{2\pi}{L})N(u,t), \\ X(u,0) = X_0(u,0), \end{cases} \tag{2}$$

where κ is the Gaussian curvature at the point (u,t) on the curve and N is the corresponding internal normal vector. They obtained a geodesic equation as follows: If the initial curve is convex, the curve flow will remain convex during the evolution until it eventually becomes a circle of radius $\sqrt{\frac{A_0}{\pi}}$. In 2002, White [4] discussed the average curvature flow which describes the properties of curvature flow evolution, smoothness, arc length expansion, and collision-free flow (the two initially disjoint curves must remain disjoint). In 2006, evolution of curves on a surface driven by the geodesic curvature and external force was studied by Mikula [5], several computational examples of evolution of surface curves driven by the geodesic curvature and external force on various surfaces are presented in this article. Also discuss a link between the geodesic flow and the edge detection problem arising from the image segmentation theory. In 2013, a new area-preserving curvature flow of planar closed-convex curves was studied by Wang et al. [6]

$$\begin{cases} \dfrac{\partial X}{\partial t} = (\kappa^n(u,t) - \dfrac{1}{L}\oint \kappa^n(u,t)\,ds)N(u,t), \\ X(s,0) = X_0(u,0). \end{cases} \tag{3}$$

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In 2014, Centro–affine curvature flows on centrally symmetric convex curves was discussed by Ivaki and Mohammad. The author show that, under any p-contracting flow, the evolving curves shrink to a point in finite time and the only homothetic solutions of the flow are ellipses centered at the origin [7].

For the study of curve flow, similar work such as [8–11]. This paper mainly considers the following simple closed curve telescopic flow $X(s,t) : S^1 \to R^2$, where T is the unit tangent vector, N is the unit internal normal vector, and X_0 is the plane simple closed curve given by $t = 0$. Let

$$g(u,t) := \left|\frac{\partial X}{\partial t}\right| = [x_s^2 + y_s^2]^{1/2}$$

denote the length along the curve. The arc-length parameter $s(u,t)$ is then defined as

$$s(u,t) := \int_0^u g(\xi,t)\, d\xi.$$

Tangent, normal, curvature, angle, arc length, and area are defined by standard methods, i.e.,

$$T := \frac{\partial X}{\partial s} = \frac{1}{g}\frac{\partial X}{\partial u}, \qquad \kappa := \left|\frac{\partial T}{\partial s}\right| = \frac{1}{g}\left|\frac{\partial T}{\partial u}\right|,$$

$$N := \frac{1}{\kappa}\frac{\partial T}{\partial u}, \qquad \theta := \angle(T,x),$$

$$L(u,t) := \int_0^{2\pi} g(u,t)\, du, \qquad A(u,t) := \frac{1}{2}\int_0^{2\pi}(xdx - ydy) = \frac{1}{2}\langle X, N\rangle\, ds,$$

denote the total absolute Gaussian curvature,

$$\bar{\kappa}(u,t) := \int_0^{2\pi} |\kappa(u,t)|\, g(u,t)\, du.$$

The evolution equation of the metric g, angle θ, arc length L, and area A, curvature κ corresponding to the curve flow (1.3) [3]:

$$\frac{\partial g}{\partial t} = \alpha_u - \beta\kappa, \tag{4}$$

$$\frac{\partial \theta}{\partial t} = \frac{1}{g}[\alpha_u + \alpha\kappa], \tag{5}$$

$$\frac{\partial L}{\partial t} = -\int_0^{2\pi} \beta\kappa du, \tag{6}$$

$$\frac{\partial A}{\partial t} = -\int_0^{2\pi} \beta g du, \tag{7}$$

$$\frac{\partial \kappa}{\partial t} = \kappa^2\frac{\partial^2\beta}{\partial\theta^2} + \beta\kappa^2. \tag{8}$$

2 Geometric properties of arc length, area, and total absolute Gaussian curvature

From Eqs.(7)-(9), it can be seen that the evolution governing equation of L, A, κ does not depend on the component of X_t on the tangent line. In addition, the deformation of the curve flow is constrained by the local geometric properties of the curve, i.e., β should be a function of curvature [3]. Therefore, we consider the case of $\alpha(u,t) = 0$. Assume that $X_t = X(.,t)$ is a C^2-classical solution on some interval $[0,t')(t' < \infty)$, i.e.,

$$\begin{cases} \dfrac{\partial X}{\partial t} = \beta(\kappa(u,t))N(u,t), \\ X(s,0) = X_0(u,0). \end{cases} \tag{9}$$

Suppose β is a positive smooth function of κ for $|\kappa| \le C$ and $\frac{\partial^n\beta}{\partial\kappa^n}$, $n = 1, \cdots, 4$. Let $a = inf\frac{|\beta_\kappa(u,0)|}{\kappa(u,0)}$, $b = sup\frac{|\beta_\kappa(u,0)|}{\kappa(u,0)}$. Obviously $a \le b$; $b = inf\frac{\beta(\kappa(u,0))}{\kappa(u,0)}$, $d = inf\frac{\beta(\kappa(u,0))}{\kappa(u,0)}$, $c \le d$.

Lemma 1 $\partial_{\kappa}^{i}\beta_{\kappa}(\kappa(u,t))\ i=1,\cdots,4$ *is bounded, for* $\forall(\theta,t)\in[0,t')(t'<\infty)$.

Proof. We have $\beta_{\kappa}(\kappa(u,t))\in C^4(\Omega)$, where $\Omega=\{\kappa||\kappa|\le C\}$. Obviously, Ω is a bounded closed region. There is a constant $\tilde{C}$ so that $|\beta_{\kappa}(\kappa(u,t))|\le\tilde{C}$. In the same way, the boundedness of $\partial_{\kappa}^{i}\beta_{\kappa}(\kappa(u,t))\ i=1,\cdots,4$ can be obtained. ∎

Lemma 2 *Let* $\kappa_{min}(\theta,t)=inf\kappa(\theta,t),\ 0\le\theta\le2\pi$*, then,* $\kappa_{min}(\theta,t)$ *is non-decreasing in interval* $[0,t')(t'<\infty)$*, for* $\forall\theta\in(0,2\pi)$.

Proof. We have,

$$\frac{\partial\kappa}{\partial t}=\frac{\partial^2\beta}{\partial\kappa^2}\frac{\partial\kappa}{\partial\theta}^2k^2+\frac{\partial\beta}{\partial\kappa}\frac{\partial^2\kappa}{\partial theta^2}\kappa^2+\beta\kappa^2. \tag{10}$$

The counter proof method assumes that $\kappa_{min}(\theta,t)$ is a increase function in the interval $[0,t')(t'<\infty)$, for $\forall\theta\in(0,2\pi)$. That is, there is t, so that $\kappa_{min}(\theta,t)\le\kappa_{min}(\theta,0)$. So, there is $\varepsilon>0$, so that $\kappa_{min}(t)=\kappa_{min}(0)-\varepsilon$. Let $t_0=\inf\{t|\kappa_{min}(t)=\kappa_{min}(0)-\varepsilon\}$, by the continuity of κ, we can obtain the minimum value at the point (θ_0,t_0) where

$$\frac{\partial\kappa}{\partial t}\le0,\qquad\frac{\partial^2\kappa}{\partial\theta^2}\ge0,\qquad\kappa(\theta_0,t_0)>0.$$

This is a contradiction with Eq.(2), then $\kappa_{min}(\theta,t)$ is non-decreasing on interval $[0,t')(t'<\infty)$ for $\forall\theta\in(0,2\pi)$. ∎

Theorem 3 *If* $\kappa(\theta,0)>0$*, then* $\kappa(\theta,t)>0$*, for* $\forall(\theta,t)\in(0,2\pi)\times[0,t')(t'<\infty)$.

Proof. From Lemma 2, we have $\kappa_{min}(\theta,t)$ is non-decreasing in interval $[0,t')(t'<\infty)$, for $\forall\theta\in(0,2\pi)$, then $\kappa(\theta,t)\ge\kappa_{min}(\theta,t)\ge\kappa_{min}(\theta,0)>0$, i.e., If the curve has a convex initial condition, then the curve remains convex during the evolution process. ∎

Lemma 4 *Assume* $X(u,t)$ *is the solution of , and interval* $[0,t')(t'<\infty)$*, then,*

$$-2\pi M\le L_t\le-2\pi m.$$

Proof. β is a positive smooth function of κ for $|\kappa|\le C$,so $m\le\beta(\kappa)\le M$, then,

$$-M\int_0^{2\pi}\kappa du\le\int_0^{2\pi}\beta\kappa du\le-m\int_0^{2\pi}\kappa du.$$

That is,

$$-2\pi M\le L_t\le-2\pi m.$$

∎

Theorem 5 *Assume* $X(u,t)$ *a family of solutions with a convex initial value curve (10), if* $\beta_{\kappa}\le a\kappa$*, then* L_t *monotonically decreases in time interval* $[0,t')(t'<\infty)$.

Proof.

$$\begin{aligned}\frac{\partial^2L}{\partial t^2}=&-\frac{\partial}{\partial t}\int_0^{2\pi}\beta\kappa g du,\\=&-\int_0^{2\pi}(\beta_t\kappa g+\beta\kappa_t g+\beta\kappa g_t)du\\=&-\int_0^{L(t)}\beta_{\kappa}(\beta_{ss}+\beta\kappa^2)\kappa s-\int_0^{L(t)}\beta(\beta_{ss}+\beta\kappa^2)\kappa s-\int_0^{L(t)}\beta^2\kappa^2ds\\=&-\beta_{\kappa}\beta_s\kappa|_0^{L(t)}+\int_0^{L(t)}\beta_s\beta_{\kappa_s}kds+\int_0^{L(t)}\beta_s^2ds-\int_0^{L(t)}\beta_k\beta\kappa^3ds\\&-\beta\beta_s|_0^{L(t)}+\int_0^{L(t)}\beta_s^2ds\\=&\int_0^{L(t)}\beta_{\kappa}\kappa_s\beta_{\kappa_s}kds+2\int_0^{L(t)}\beta_s^2ds-\int_0^{L(t)}\beta_k\beta\kappa^3ds,\end{aligned}$$

we have $\beta > 0, \beta_\kappa > 0$, then,

$$\frac{\partial^2 L}{\partial t^2} \leq \int_0^{L(t)} (\beta_\kappa \kappa_s \beta_{\kappa_s} \kappa + 2\beta_\kappa^2 \kappa_s^2) ds$$
$$= \int_0^{L(t)} \beta_\kappa^2 \kappa_s^2 (\frac{\beta_{\kappa_s} \kappa}{\beta_\kappa \kappa_s} + 2) ds.$$

From Theorem 3, it can be seen that when the initial curve is a convex curve, the curve will remain convex all the time, i.e., $\kappa > 0$. The following applies the counter-evidence method,assume $\beta_\kappa \leq a\kappa$ and $\frac{\beta_{\kappa_s}\kappa}{\beta_\kappa \kappa_s} + 2 \geq 0$. $\beta_\kappa \geq C_1 \kappa$ is obtained by solving $\frac{\beta_{\kappa_s}\kappa}{\beta_\kappa \kappa_s} + 2 \geq 0$, so that $\beta_\kappa(\kappa(s, 0)) \geq C_1 \kappa(s, 0)$; so, $C_1 \geq b$, i.e., when $\frac{\beta_{\kappa_s}\kappa}{\beta_\kappa \kappa_s} + 2 \geq 0$,$\beta_\kappa \geq C_1 \kappa (C_1 \geq b)$. This contradicts $\beta_\kappa \leq a\kappa$. So if $\beta_\kappa \leq a\kappa$, then $\frac{\beta_{\kappa_s}\kappa}{\beta_\kappa \kappa_s} + 2 \geq 0$, that is L_t monotonously decreases in interval $[0, t')(t' < \infty)$, and the velocity of curve length is decreasing in interval $[0, t')(t' < \infty)$. ■

Lemma 6 *Assume $X(u, t)$ is the solution of (10), and interval $[0, t')(t' < \infty)$, then,*

$$-ML(t) \leq A_t \leq -mL(t).$$

Proof. We have

$$\frac{\partial A}{\partial t} = \int_0 2\pi \beta g du,$$

then,

$$-M \int_0 2\pi g du \leq A_t \leq -m \int_0 2\pi g du.$$

That is,

$$-ML(t) \leq A_t \leq -mL(t).$$

■

Theorem 7 *Assume $X(u, t)$ are a family of solutions with a convex initial value curve (10), if $\beta_\kappa \leq d\kappa$. Then A_t monotonically decreases in time interval $[0, t')(t' < \infty)$.*

Proof.

$$\frac{\partial^2 A}{\partial t^2} = \frac{\partial}{\partial t} \int_0^{L(t)} \beta - \int_0^{L(t)} \beta_\kappa \beta \kappa^2 ds + \int_0^{L(t)} \beta^2 \kappa ds,$$

since $\beta > 0$, $\beta_\kappa > 0$, $\beta_{\kappa\kappa} > 0$, then,

$$\frac{\partial^2 A}{\partial t^2} \leq \int_0^{L(t)} \beta^2 \kappa ds - \frac{1}{2} \int_0^{L(t)} \beta_\kappa \beta \kappa^2 ds$$
$$= \int_0^{L(t)} (\beta^2 \kappa^2)(\frac{1}{\kappa} - \frac{\beta_\kappa}{\beta}) ds.$$

Similar to Theorem 5, use the counter-evidence method. If $\beta > d\kappa$, then A_t monotonically decreases in the interval $[0, t')(t' < \infty)$, that is, the rate of change of the area is decreasing in the interval $[0, t')(t' < \infty)$. ■

Lemma 8 *Assume $X(u, t)$ are a family of solutions with a convex initial value curve (10), then,*

$$\bar{\kappa}(t) = \bar{\kappa}(0) = 2\pi.$$

Proof. For convex curves, we have,

$$\frac{\partial \bar{\kappa}}{\partial t} = \frac{\partial}{\partial t} \int_0^{2\pi} \kappa g du = \int_0^{2\pi} (\kappa_t g + \kappa g_t) du$$
$$= \int_0^{L(t)} (\beta_{ss} + \beta \kappa^2) ds - \int_0^{L(t)} \beta \kappa^2 ds$$
$$= \beta_s \mid_0^{2\pi} = 0.$$

So,

$$\bar{\kappa}(t) = \bar{\kappa}(0) = 2\pi.$$

■

Lemma 9 *Assume $X(u,t)$ is the solution of (10), and interval $[0,t')(t'<\infty)$, then,*

$$\bar{\kappa}(t) \geq \bar{\kappa}(0).$$

Proof. Define,

$$\hat{q}(t) := \int_0^{2\pi} q(\kappa(u,t))g(u,t)du,$$

where q is the piecewise smooth convex approximation of $f(x)=|x|$ given by

$$q(x) = \begin{cases} |x| & if \quad x \geq \frac{1}{n}, \\ \frac{1}{2n} + \frac{n}{2}x^2 & if \quad x \leq \frac{1}{n}, \end{cases}$$

then,

$$\begin{aligned} q_t(t) &= \int_0^{2\pi} q_\kappa(\kappa)\kappa_t g du + \int_0^{2\pi} q_\kappa(\kappa) g_t du \\ &= \int_0^{L(t)} q_\kappa \beta_{ss} ds + \int_0^{2\pi} q_\kappa \beta \kappa^2 du - \int_0^{2\pi} q(\kappa)\beta\kappa g du \\ &= q_\kappa(\kappa)\beta_s \mid_0^{L(t)} - \int_0^{L(t)} q_{\kappa\kappa}\beta_s\kappa_s ds - \int_0^{2\pi} [q(\kappa) - \kappa q_\kappa(\kappa)](\beta\kappa g) du \\ &= -\int_0^{2\pi} q_{\kappa\kappa}\beta_\kappa(\kappa_s)^2 ds - \int_0^{2\pi} [q(\kappa) - \kappa q_\kappa(\kappa)](\beta\kappa g) du. \end{aligned}$$

Since $\beta_\kappa > 0$, and convexity of q requires $q_{\kappa\kappa} < 0$, we have

$$q_t(t) \leq -\int_0^{2\pi} [q(\kappa) - \kappa q_\kappa(\kappa)](\beta\kappa g) du.$$

Note that,

$$\bar{\kappa}(t) \leq \hat{q}(t),$$

so that a bound on $\hat{q}$ is a bound on $\bar{\kappa}$. Moreover, we have that

$$0 \leq q(x) - xq'(x) \leq \begin{cases} 0 & if \quad x \geq \frac{1}{n}, \\ \frac{1}{2n} & if \quad x \leq \frac{1}{n}. \end{cases}$$

Now since $\beta(\kappa)\kappa \leq M$, and $[q(\kappa) - xq_\kappa(\kappa)] \geq 0$, then,

$$\begin{aligned} q'(t) &\geq -M \int_0^{2\pi} [q(\kappa) - xq_\kappa(\kappa)]\kappa g du \\ &\geq -M\frac{1}{2n}\int_0^{2\pi} \kappa g du \\ &\geq -\frac{M}{2n}2\pi = -\frac{\pi M}{n}. \end{aligned}$$

Therefore, $\frac{\hat{q}'(t)}{\hat{q}(t)} \geq -\frac{\pi M}{n\bar{\kappa}(t)}$, i.e., $\hat{q}'(t) \geq e^{-\frac{M}{2n}\int_0^t \frac{L(t)}{\bar{\kappa}(t)}dt'}\hat{q}(0)$ as $n \to \infty$, so that,

$$\hat{q}(t) \to \bar{\kappa}(t), \qquad \hat{q}(t) \geq \hat{q}(0),$$

so $\bar{\kappa}(t) \geq \bar{\kappa}(0)$. ■

3 Discussion on the increase and decrease of curvature speed

Lemma 10 *For* $\forall\theta \in (0, 2\pi)$*, there is a constant* $\tilde{C}$*, and in the interval* $[0, t')(t' < \infty)$*,* $\kappa_\theta(\theta, t)$ *grows exponentially at most.*

Proof.

$$\begin{aligned}\frac{\partial^2\kappa}{\partial\theta\partial t} &= \frac{\partial}{\partial\theta}(\beta_{\theta\theta}\kappa^2 + \beta\kappa^2)\\ &= \beta_{\theta\theta\theta}\kappa^2 + 2\beta_{\theta\theta}\kappa\kappa_\theta + \beta_\theta\kappa^2 + 2\beta\kappa\kappa_\theta\\ &= \beta_{\kappa\kappa\kappa}(\kappa_\theta)^3\kappa^2 + 3\beta_{\kappa\kappa}\kappa^2\kappa_\theta\kappa_{\theta\theta} + \beta_{\kappa\kappa}\kappa(\kappa_\theta)^3 + \beta_\kappa\kappa^2\kappa_{\theta\theta\theta}\\ &+ 2\beta_\kappa\kappa\kappa_\theta\kappa_{\theta\theta} + \beta_\kappa\kappa^2\kappa_\theta + 2\beta\kappa\kappa_\theta,\end{aligned} \tag{11}$$

let $w(\theta, t) = e^{\mu t}\kappa_\theta(\theta, t)$, then

$$\begin{aligned}\frac{\partial w}{\partial t} &= (\mu + \beta_\kappa\kappa^2 + 2\beta\kappa)w + (\beta_{\kappa\kappa\kappa}\kappa^2 e^{-2\mu t} + 2\beta_{\kappa\kappa}\kappa e^{-2\mu t})w^3\\ &+ (3\beta_{\kappa\kappa}\kappa^2 e^{-\mu t} + 2\beta_\kappa\kappa e^{-\mu t})w_\theta w + \beta_\kappa\kappa^2 w_{\theta\theta},\end{aligned} \tag{12}$$

when $\kappa_\theta(\theta, 0) \le 0$. Suppose $w(\theta, t)$ is a decreasing function in interval $[0, t')(t' < \infty)$, for $\forall\theta \in (0, 2\pi)$, let $w_{\min}(t) = \inf\{w(\theta, t) : 0 \le \theta \le 2\pi\}$. Suppose there is a η, where $\eta < w_{\min}(\theta, 0) = \kappa_{\theta\min}(\theta, 0) \le 0$, there is a certain point t, so that $w_{\min}(t) = \eta$, assume $t^* = \inf t : w_{\min} = \eta$, then, the continuity of w assures that this minimum η is achieved for the first time at (θ^*, t^*), and at this point

$$\frac{\partial w}{\partial t} \le 0, \qquad \frac{\partial^2 w}{\partial\theta^2} \ge 0, \qquad \frac{\partial w}{\partial\theta} = 0, \qquad and \qquad w = \eta < 0.$$

From Lemma 1, the boundedness of $\partial^i_\kappa\beta_\kappa(\kappa(u, t))\ i = 1, \cdots, 4$ and the boundedness of curvature κ, we can obtain $\frac{\partial w}{\partial t} \ge \mu\eta + C_2\eta + C_3\eta^3$, where $C_2 = \sup(\beta_\kappa\kappa^2 + 2\beta\kappa)$, $C_3 = \sup(\beta_{\kappa\kappa\kappa}\kappa^2 + 2\beta_{\kappa\kappa}\kappa)$, let $\mu \le -C_2 - C_3\eta^2$, this contradicts from (13), i.e., for $\forall(\theta, t) \in (0, 2\pi) \times [0, t')(t' < \infty)$, so that $w_{\min}\min(\theta, t) \ge w_{\min}(\theta, 0)$, that is $\kappa_\theta(\theta, t) \ge \kappa_{\theta\min}(\theta, 0)e^{-\mu t}$, then obtain $\kappa_\theta(\theta, t) \ge \kappa_{\theta\min}(\theta, 0)$. When $\kappa_\theta(\theta, 0) \le 0$, suppose $w(\theta, t)$ is a decreasing function in interval $[0, t')(t' < \infty)$, for $\forall\theta \in (0, 2\pi)$, let $w_{\min}(t) = \inf\{w(\theta, t) : 0 \le \theta \le 2\pi\}$. Suppose there is a η, where $\eta < w_{\min}(\theta, 0) = \kappa_{\theta\min}(\theta, 0) \le 0$, there is a certain point t, so that $w_{\min}(t) = \eta$, assume $t^* = \inf t : w_{\min} = \eta$, then, the continuity of W assures that this minimum η is achieved for the first time at (θ^*, t^*), and at this point

$$\frac{\partial w}{\partial t} \le 0, \qquad \frac{\partial^2 w}{\partial\theta^2} \ge 0, \qquad \frac{\partial w}{\partial\theta} = 0, \qquad and \qquad w = \eta < 0.$$

From Lemma 1, the boundedness of $\partial^i_\kappa\beta_\kappa(\kappa(u, t))\ i = 1, \cdots, 4$ and the boundedness of curvature κ, we can obtain $\frac{\partial w}{\partial t} \ge \mu\eta + C_2\eta + C_3\eta^3$, where $C_2 = \sup(\beta_\kappa\kappa^2 + 2\beta\kappa)$, $C_3 = \sup(\beta_{\kappa\kappa\kappa}\kappa^2 + 2\beta_{\kappa\kappa}\kappa)$, let $\mu \le -C_2 - C_3\eta^2$, this contradicts from (13), i.e., for $\forall(\theta, t) \in (0, 2\pi) \times [0, t')(t' < \infty)$, so that $w_{\min}(\theta, t) \ge w_{\min}(\theta, 0)$, that is $\kappa_\theta(\theta, t) \ge \kappa_{\theta\min}(\theta, 0)e^{-\mu t}$, then obtain $\kappa_\theta(\theta, t) \ge \kappa_{\theta\min}(\theta, 0)$. When $\kappa_\theta(\theta, 0) \ge 0$, suppose $w(\theta, t)$ is a increasing function in interval $[0, t')(t' < \infty)$, for $\forall\theta \in (0, 2\pi)$, let $w_{\max}(t) = \sup\{w(\theta, t) : 0 \le \theta \le 2\pi\}$, suppose there is a ξ, where $\xi \ge w_{\max}(0) = \kappa\max(0) \ge 0$, there is a certain point t, so that $w\max(t) = \xi$, assume $\tilde{t} = \inf t : w_{\max} = \xi$, then, the continuity of w assures that this minimum ξ is achieved for the first time at $(\tilde{\theta}, \tilde{t})$, and at this point

$$\frac{\partial w}{\partial t} \ge 0, \qquad \frac{\partial^2 w}{\partial\theta^2} \le 0, \qquad \frac{\partial w}{\partial\theta} = 0, \qquad and \qquad w = \xi > 0.$$

From Lemma 1, the boundedness of $\partial^i_\kappa\beta_\kappa(\kappa(u, t))\ i = 1, \cdots, 4$ and the boundedness of curvature κ, we can obtain $\frac{\partial w}{\partial t} \ge \mu\eta + C_2'\xi + C_3'\xi^3$, where $C_2' = \sup(\beta_\kappa\kappa^2 + 2\beta\kappa)$,$C_3' = \sup(\beta_{\kappa\kappa\kappa}\kappa^2 + 2\beta_{\kappa\kappa}\kappa)$, let $\mu \le -C_2' - C_3'\xi^2$, this contradicts from (12), i.e., for $\forall(\theta, t) \in (0, 2\pi) \times [0, t')(t' < \infty)$, so that $w_{\max}(\theta, t) \le w_{\max}(\theta, 0)$, that is $\kappa_\theta(\theta, t) \le \kappa_{\theta\max}(\theta, 0)e^{-\mu t'}$. This means that κ_θ grows exponentially at most and remains bounded for a limited time interval. ■

Theorem 11 *When* $\kappa_\theta(\theta, 0) > 0$*, for* $\theta \in (0, 2\pi)$*,* $\kappa_\theta(\theta, t)$ *is non-decreasing in interval* $[0, t')(t' < \infty)$*. When* $\kappa_\theta(\theta, 0) < 0$*,* $\kappa_\theta(\theta, t)$ *is non-increasing in interval* $[0, t')(t' < \infty)$*.*

Proof. The counter-evidence method, when $\kappa_\theta(\theta,0)<0$, for $\theta\in(0,2\pi)$, $\kappa_\theta(\theta,t)$ is an increasing function in the interval $[0,t')(t'<\infty)$. That is, there is a δ, where $\delta\geq\kappa_\theta(\theta,0)$, so that there is a certain point t, assume $t_1=\inf\{t:\kappa_\theta(\theta,t)=\delta\}$, from the continuity of κ_θ assures that this minimum δ is achieved for the first time at (θ_1,t_1), and at this point

$$\frac{\partial\kappa_\theta}{\partial t}\geq 0,\qquad \frac{\partial^3\kappa}{\partial\theta^3}\leq 0,\qquad \frac{\partial^2\kappa}{\partial\theta^2}=0,\qquad and\qquad \kappa_\theta=\delta<0.$$

This contradicts from (12), i.e., for $\forall(\theta,t)\in(0,2\pi)\times[0,t')(t'<\infty)$, so that $\kappa_\theta(\theta,t)\leq\kappa_\theta(\theta,0)$. Similarly, when $\kappa_\theta(\theta,0)>0$, for $\forall(\theta,t)\in(0,2\pi)\times[0,t')(t'<\infty)$, so that $\kappa_\theta(\theta,t)\geq\kappa_\theta(\theta,0)$. ■

4 Discussion of curvature and convexity

Lemma 12 *For $\forall\theta\in(0,2\pi)$, there is a constant C^*, and in the interval $[0,t')(t'<\infty)$,*

$$|\kappa_{\theta\theta}(\theta,t)|_{L^P(0,2\pi)}\leq C^*, 1\leq P<\infty.$$

Proof. We have

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta &= 4\int_0^{2\pi}(\kappa_{\theta\theta})^3(\beta_{\theta\theta}\kappa^2+\beta\kappa^2)_{\theta\theta}d\theta\\ &= -12\int_0^{2\pi}[(\kappa_{\theta\theta})^2\kappa^2\beta_{\theta\theta\theta}+2\beta_{\theta\theta}\kappa\kappa_\theta(\kappa_{\theta\theta})^2+\beta\theta\kappa^2(\kappa_{\theta\theta})^2\\ &+2\beta\kappa\kappa_\theta(\kappa_{\theta\theta})^2]\kappa_{\theta\theta\theta}d\theta\\ &= -12\int_0^{2\pi}[\beta_{\kappa\kappa\kappa}\kappa^2(\kappa_\theta)^3(\kappa_{\theta\theta})^2\kappa_{\theta\theta\theta}+3\beta_{\kappa\kappa}\kappa^2\kappa_\theta(\kappa_{\theta\theta})^3\kappa_{\theta\theta\theta}\\ &+\beta_\kappa\kappa^2(\kappa_{\theta\theta})^2(\kappa_{\theta\theta\theta})^2+2\beta_{\kappa\kappa}\kappa(\kappa_\theta)^3(\kappa_{\theta\theta})^2\kappa_{\theta\theta\theta}\\ &+2\beta_\kappa\kappa\kappa_\theta(\kappa_{\theta\theta})^3\kappa_{\theta\theta\theta}+\beta_\kappa\kappa^2\kappa_\theta(\kappa_{\theta\theta})^2\kappa_{\theta\theta\theta}+2\beta\kappa\kappa_\theta(\kappa_{\theta\theta})^2\kappa_{\theta\theta\theta}]d\theta,\end{aligned}$$

and from $ab\leq\varepsilon a^2+\frac{1}{4\varepsilon}b^2$, we can obtain

$$\int_0^{2\pi}\beta_{\kappa\kappa\kappa}\kappa^2(\kappa_\theta)^3(\kappa_{\theta\theta})^2\kappa_{\theta\theta\theta}d\theta\leq c_1\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta+c_2\int_0^{2\pi}(\kappa_{\theta\theta\theta})^2d\theta,$$

$$3\int_0^{2\pi}\beta_{\kappa\kappa}\kappa^2\kappa_\theta(\kappa_{\theta\theta})^3\kappa_{\theta\theta\theta}\leq c_3\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta+c_4\int_0^{2\pi}(\kappa_{\theta\theta})^2(\kappa_{\theta\theta\theta})^2d\theta,$$

$$\int_0^{2\pi}\beta_\kappa\kappa^2(\kappa_{\theta\theta})^2(\kappa_{\theta\theta\theta})^2d\theta\leq c_5\int_0^{2\pi}(\kappa_{\theta\theta})^2d\theta+c_6\int_0^{2\pi}(\kappa_{\theta\theta})^2(\kappa_{\theta\theta\theta})^2d\theta,$$

$$2\int_0^{2\pi}\beta_{\kappa\kappa}\kappa(\kappa_\theta)^3(\kappa_{\theta\theta})^2\kappa_{\theta\theta\theta}d\theta\leq c_7\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta+c_8\int_0^{2\pi}(\kappa_{\theta\theta\theta})^2d\theta,$$

$$2\int_0^{2\pi}\beta_\kappa\kappa\kappa_\theta(\kappa_{\theta\theta})^3\kappa_{\theta\theta\theta}\leq c_9\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta+c_{10}\int_0^{2\pi}(\kappa_{\theta\theta})^2(\kappa_{\theta\theta\theta})^2d\theta,$$

$$\int_0^{2\pi}\beta_\kappa\kappa^2\kappa_\theta(\kappa_{\theta\theta})^2\kappa_{\theta\theta\theta}\leq c_{11}\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta+c_{12}\int_0^{2\pi}(\kappa_{\theta\theta})^2(\kappa_{\theta\theta\theta})^2d\theta,$$

$$\beta\kappa\kappa_\theta(\kappa_{\theta\theta})^2\kappa_{\theta\theta\theta}]d\theta\leq c_{13}\int_0^{2\pi}(\kappa_{\theta\theta})^2d\theta+c_{14}\int_0^{2\pi}(\kappa_{\theta\theta})^2(\kappa_{\theta\theta\theta})^2d\theta,$$

and from $\int_0^{2\pi}(\kappa_{\theta\theta})^2d\theta\leq\sqrt[2]{2\pi}(\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta)^{\frac{1}{2}}$, we choose the right $c_1,c_2,\cdots,c_{14}$, we can get

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta &\leq 12[c_1'\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta+c_1'\int_0^{2\pi}(\kappa_{\theta\theta})^2d\theta]\\ &\leq c_1''\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta+c_2''(\int_0^{2\pi}(\kappa_{\theta\theta})^4d\theta)^{\frac{1}{2}}.\end{aligned}$$

By Gronwall Inequality, we can obtain $\int_0^{2\pi}(\kappa_{\theta\theta})^4 d\theta \leq a_0 + c_2'' \int_0^t (\int_0^{2\pi}(\kappa_{\theta\theta})^4 d\theta)^{\frac{1}{2}} d\xi$, where $a_0 = e^{c_1'' t}\int_0^{2\pi}(\kappa_{\theta\theta}(\theta,0))^4 d\theta$. Counter-evidence, for $\forall Q > 0$, $\exists t_0 \geq 0$, so that $(\int_0^{2\pi}(\kappa_{\theta\theta})^4 d\theta)^{\frac{1}{2}} d\xi > Q$, and since

$$\int_0^{2\pi}(\kappa_{\theta\theta})^4 d\theta \leq a_0 + c_2'' \int_0^{t_0}(\int_0^{2\pi}(\kappa_{\theta\theta})^4 d\theta)^{\frac{1}{2}} d\xi \leq a_0 + c'' Q(t_0),$$

this contradicts the arbitrariness of Q, so Q' exists, so that $|\kappa_{\theta\theta}(\theta,t)|_{L^4(0,2\pi)} \leq Q_0$. Similarly, we can prove that there is Q', so that $|\kappa_{\theta\theta}(\theta,t)|_{L^2(0,2\pi)} \leq Q'$. And by Sobolev embedding theorem,

$$\begin{aligned}\max|\kappa_{\theta\theta}|^2 &= \|\kappa_{\theta\theta}\|^2_{L^\infty} \leq \|\kappa_{\theta\theta}\|^2_{L^2} + \|\kappa_{\theta\theta\theta}\|^2_{L^2} \\ &= \int_0^{2\pi}(\kappa_{\theta\theta})^2 d\theta + \int_0^{2\pi}(\kappa_{\theta\theta\theta})^2 d\theta.\end{aligned}$$

Thus $\kappa_{\theta\theta}$ is bounded. Further, it is possible to introduce the existence of a constant C^*, so that $|\kappa_{\theta\theta}(\theta,t)|_{L^P(0,2\pi)} \leq C^*, 1 \leq P < \infty$. ■

Lemma 13 *Suppose $\kappa_\theta > 0$, for $\forall\theta \in (0,2\pi)$, $\kappa_{\theta\theta}$ grows exponentially at most in the interval $[0,t')(t' < \infty)$, and remains bounded for a limited time interval.*

Proof.

$$\begin{aligned}\frac{\partial^3\kappa}{\partial\theta^2\partial t} &= \frac{\partial}{\partial\theta}(\beta_{\kappa\kappa\kappa}\kappa^2(\kappa_\theta)^3 + 3\beta_{\kappa\kappa}\kappa^2\kappa_\theta\kappa_{\theta\theta} + \beta_\kappa\kappa^2\kappa_{\theta\theta\theta} + 2\beta_{\kappa\kappa}\kappa(\kappa_\theta)^3) \\ &= (\beta_{\kappa\kappa\kappa\kappa}\kappa^2(\kappa_\theta)^4 + 6\beta_{\kappa\kappa\kappa}\kappa^2(\kappa_\theta)^2\kappa_{\theta\theta} + 2\beta_{\kappa\kappa\kappa}\kappa(\kappa_\theta)^4 + 3\beta_{\kappa\kappa}\kappa^2(\kappa_{\theta\theta})^2 \\ &+ 4\beta_{\kappa\kappa}\kappa^2\kappa_\theta\kappa_{\theta\theta\theta} + 14\beta_{\kappa\kappa}\kappa(\kappa_\theta)^2\kappa_{\theta\theta} + \beta_\kappa\kappa_{\theta\theta\theta\theta}\kappa^2 + 4\beta_\kappa\kappa\kappa_\theta\kappa_{\theta\theta\theta} \\ &+ 2\beta_{\kappa\kappa\kappa}\kappa(\kappa_\theta)^4 + 2\beta_{\kappa\kappa}(\kappa_\theta)^4 + 2\beta_\kappa(\kappa_\theta)^2\kappa_{\theta\theta} + 2\beta_\kappa\kappa(\kappa_{\theta\theta})^2 \\ &+ \beta_{\kappa\kappa}(\kappa_\theta)^2\kappa^2 + \beta_\kappa\kappa_{\theta\theta}\kappa^2 + 4\beta_\kappa(\kappa_\theta)^2\kappa + 2\beta(\kappa_\theta)^2 + 2\beta\kappa\kappa_{\theta\theta}.\end{aligned} \tag{13}$$

Let $w(\theta,t) = e^{at}\kappa_{\theta\theta}(\theta,t)$, then

$$\begin{aligned}\frac{\partial w}{\partial t} &= (a + 6\beta_{\kappa\kappa\kappa}\kappa^2(\kappa_\theta)^2 + 14\beta_{\kappa\kappa}\kappa(\kappa_\theta)^2 + 2\beta_\kappa(\kappa_\theta)^2 + \beta_\kappa\kappa^2 + 2\beta\kappa)w \\ &+ (33\beta_{\kappa\kappa}\kappa^2 e^{-at} + 2\beta_\kappa\kappa e^{-at})w^2 + (4\beta_{\kappa\kappa}\kappa^2\kappa_\theta + 4\beta_\kappa\kappa\kappa_\theta)w_\theta \\ &+ \beta_\kappa\kappa^2 w_{\theta\theta} + \beta_{\kappa\kappa\kappa\kappa}\kappa^2(\kappa_\theta)^4 e^{at} + 4\beta_{\kappa\kappa\kappa}\kappa(\kappa_\theta)^4 e^{at} + 22\beta_{\kappa\kappa}(\kappa_\theta)^4 e^{at} \\ &+ 4\beta_\kappa(\kappa_\theta)^2\kappa e^{at} + 2\beta(\kappa_\theta)^2 e^{at}.\end{aligned} \tag{14}$$

Since $\kappa_\theta(\theta,0) > 0$, from Theorem 11, $\kappa_\theta(\theta,t) > \kappa_\theta(\theta,0) > 0$, for $\forall(\theta,t) \in (0,2\pi)\times[0,t')(t' < \infty)$. When $\kappa_\theta(\theta,0) > 0$, $w(\theta,t)$ is a decreasing function in the interval $[0,t')(t' < \infty)$, for $\forall\theta \in (0,2\pi)$. Let $w_{\min}(t) = \inf w(t) : 0 \leq \theta \leq 2\pi$, suppose there is a λ, where $\lambda < w_{\min}(0) = \kappa_{\theta\theta}(0) \leq 0$, then $\exists t$, so that $w_{\min}(\theta,t) = \lambda$, suppose $t_0 = \inf\{t : w_{\min} = \lambda\}$, from the continuity of w assures that this minimum λ is achieved for the first time at (θ_0,t_0), and at this point

$$\frac{\partial w}{\partial t} \leq 0, \qquad \frac{\partial^2 w}{\partial\theta^2} \geq 0, \qquad \frac{\partial w}{\partial\theta} = 0, \qquad and \qquad w = \lambda < 0.$$

From Lemma 1, the boundedness of $\partial^i_\kappa\beta_\kappa(\kappa(u,t))$ $i = 1,\cdots,4$ and the boundedness of curvature κ and κ_θ, we can obtain $\frac{\partial w}{\partial t} \geq a\lambda + C_4\lambda + C_5\lambda^2$, where $C_4 = \sup(6\beta_{\kappa\kappa\kappa}\kappa^2(\kappa_\theta)^2 + 14\beta_{\kappa\kappa}\kappa(\kappa_\theta)^2 + 2\beta_\kappa(\kappa_\theta)^2 + \beta_\kappa\kappa^2 + 2\beta\kappa)$, $C_5 = \sup(33\beta_{\kappa\kappa}\kappa^2 e^{-at} + 2\beta_\kappa\kappa e^{-at})$. Let $a \leq -C_4 - C_5\lambda$, this contradicts from (15), i.e., for $\forall(\theta,t) \in (0,2\pi)\times[0,t')(t' < \infty)$, so that $w_{\min}(\theta,t) \geq w_{\min}(\theta,0)$, that is $\kappa_{\theta\theta}(\theta,t) \geq \kappa_{\theta\theta\,\min}(\theta,0)e^{-\mu t}$, then obtain $\kappa_{\theta\theta}(\theta,t) \geq \kappa_{\theta\theta\,\min}(\theta,0)$. Similarly, we can obtain when $\kappa_{\theta\theta}(\theta,0) \geq 0$, $\kappa_{\theta\theta}(\theta,t) \leq \kappa_{\theta\theta}(\theta,0)e^{-at'}$, this means that $\kappa_{\theta\theta}$ grows exponentially at most and remains bounded for a limited time interval. ■

Lemma 14 *Suppose $\kappa_\theta(\theta,0) > 0$, when $\kappa_{\theta\theta}(\theta,0) > 0$, for $\theta \in (0,2\pi)$, $\kappa_{\theta\theta}(\theta,t)$ is non-decreasing in interval $[0,t')(t' < \infty)$. When $\kappa_{\theta\theta}(\theta,0) < 0$, $\kappa_\theta(\theta,t)$ is non-increasing in interval $[0,t')(t' < \infty)$.*

Proof. Since $\kappa_\theta(\theta,0)>0$, from Theorem 11, $\kappa_\theta(\theta,t)>\kappa_\theta(\theta,0)>0$, for $\forall(\theta,t)\in(0,2\pi)\times[0,t')(t'<\infty)$. The counter-evidence method, when $\kappa_{\theta\theta}(\theta,0)<0$, for $\theta\in(0,2\pi)$, $\kappa_\theta(\theta,t)$ is an increasing function in the interval $[0,t')(t'<\infty)$. That is, there is a b, where $b\geq\kappa_{\theta\theta}(\theta,0)$, so that there is a certain point t, suppose $t_2=\inf\{t:\kappa_{\theta\theta}(\theta,t)=b\}$, from the continuity of κ_θ assures that this minimum δ is achieved for the first time at (θ_2,t_2), and at this point

$$\frac{\partial\kappa_{\theta\theta}}{\partial t}\geq 0,\qquad \frac{\partial^4\kappa}{\partial\theta^4}\leq 0,\qquad \frac{\partial^3\kappa}{\partial\theta^3}=0,\qquad and \qquad \kappa_{\theta\theta}=b<0.$$

This contradicts from (14), i.e., for $\forall(\theta,t)\in(0,2\pi)\times[0,t')(t'<\infty)$, so that $\kappa_{\theta\theta}(\theta,t)\leq\kappa_{\theta\theta}(\theta,0)$. Similarly, when $\kappa_{\theta\theta}(\theta,0)>0$, for $\forall(\theta,t)\in(0,2\pi)\times[0,t')(t'<\infty)$, so that $\kappa_{\theta\theta}(\theta,t)\geq\kappa_{\theta\theta}(\theta,0)$.

Theorem 15 *When $\kappa_\theta(\theta,0)>0$, for $\forall(\theta,t)\in(0,2\pi)\times[0,t')(t'<\infty)$, when $\kappa_\theta(\theta,0)>0$, $\kappa(\theta,t)$ is a convex function; when $\kappa_\theta(\theta,0)<0$, $\kappa(\theta,t)$ is a concave function.*

Proof: It can be proved by Theorem 11 and Lemma 14 . ■

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