Cooperation in A Dynamical Adjustment of Duopoly Game with Incomplete Information and Time Delay

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Abstract: The mechanism of tit-for-tat allows cooperation in repeated games. This mechanism is adopted to build the cooperation between the two firms. In this work, we study the Cournot cooperative model combined with time delay. Delayed dynamics is built for such a process and analysis of local stability is mathematically done for it. Its boundary equilibria are proved to be unstable and the conditions for local stability of its unique interior equilibrium are obtained by Schur-Cohn Criterion. The result shows that the tit-for-tat strategy may lead to the cooperation of them, but the stability of adjustment system is influenced by the parameters. To show how the delayed system evolves and what influence the model parameters including the delay weight (a memory parameter) have on the system stability, numerical simulations are done for different kinds of dynamical behaviors such as bifurcation diagram, phase portrait, and stability region. It is demonstrated that a proper delay weight to the memory plays an important role in expanding the stability region and delaying the occurrence of complex behaviors such as bifurcation and chaos. It shows that a proper delay weight may make the firms more likely to achieve cooperation.

Keywords: Cooperation; Duopoly game; Incomplete information; Time delay; Numerical simulation

1 Introduction

An oligopolistic market is a kind of market mode between monopoly and perfect competition. It refers to the case that majority of some products are controlled by a few large firms. The few firms manufacture the same or homogeneous products and they must take into account all the information in the market and the actions of the competitors [1]. Game theory has been diffusely utilized for oligopolistic markets thanks to its ability to consider strategic interactions among firms. In a classic Cournot model, each firm is assumed to have naive expectation and thus guesses that the opponents’ output remains at the same level as in the previous period and then chooses a optimal production strategy in the current period. Rather than the naive expectation, a so called boundedly rationality based on players’ marginal profits has received great attention in recent years [2–5]. The basic solution in a dynamical Cournot model with this kind of boundedly rationality is Nash equilibrium. The adjust dynamics to get the Nash equilibrium and the stability are studied by these works [6–10].

But just as what Nash equilibrium reveals, Nash equilibrium reflects individual rationality but it may violate collective rationality-Nash equilibrium of the duopoly game may be not Pareto Optimality. Pareto Optimality describes the state that society can not further organize production or consumption to increase the satisfaction of someone while the welfare of others can not be reduced at the same time. Pareto improvement is a change that makes at least one person better without making anyone worse off. On the one hand, Pareto optimality is a state where there is no room for Pareto improvement; on the other hand, Pareto improvement is a path and method to reach Pareto Optimality.

The Prisoner’s Dilemma shows that, there are some contradictions between individual rationality and collective rationality, and the correct choice based on individual rationality will reduce everybody’s welfare. In other words, Pareto improvement cannot be carried on and Pareto Optimality cannot be realized by personal interest’s maximization. The main question which the Prisoner’s Dilemma poses is whether a cooperative behaviour can emerge among rational and self-interested players whenever there is no formal agreement [11].
About standard game models with Prisoner’s Dilemma, theoretical, and experimental studies have indicated several ways by which the cooperative solution can emerge [12, 13]. The results of single and multiple Prisoner’s Dilemma are not the same. If the Prisoner’s Dilemma model is in one short game, the Nash equilibrium solution can be obtained by personal interest’s maximization. But if it is in a repeated game, cooperation can be established through a system of punishments and rewards. The continual cooperation may be the equilibrium solution to the repeated Prisoner’s Dilemma [14, 15]. This is the so-called folk theorem. Axelrod [16] has also demonstrated that the best behaviour allowing the achievement of cooperation in repeated games is the tit-for-tat conduct, consisting in doing what the opponent did in previous move.

So there are cases where coercion is not a necessary condition for reaching a cooperative solution. In real economical markets we can truly observe that competitors are often able to achieve the cooperation. Ding [17] studied that how firms get bigger profits by adjusting their own outputs. The paper [17] is about Cournot game for the competition of output, and the paper has studied two strategies of output adjustment under incomplete information: the tit-for-tat strategy and the tit-for-tat strategy with cooperative intention. Although each firm cannot obtain the competitor’s complete information, each firm completely knows about his own output and profit. Each firm can compare his own profit with the cooperative profit, and hence decreases his output as a reward for opponent if he earns more profit than the cooperative profit otherwise he will increase his output as a penalty if he earns less profit than the cooperative profit [17]. The paper [17] shows the cooperation may be achieved, but the stability of the adjustment system is sensitive to the parameters, and the Pareto Optimality cannot be assured.

In the work on the dynamic Cournot model, time delay has also attracted the attention of many scholars. To maximize benefits, enterprises will make a more accurate analysis of the current situation by referring to the previous output information and profit information in the process of output adjustment. The work in [18–22] studied the dynamical Cournot game with delayed bounded rationality, and found that time delay can increase the system stability and delay the occurrence of complex behaviors. In the work [17], every producer only considers competitor’s present behaviour and respond according to the competitor’s behaviour immediately.

But in real economical markets, players with cooperative intention may be lenient with opponents. Every player will take steps according to the opponents’ more previous behaviours rather than one action. Each player wants to achieve cooperation by using smoothed adjustment strategies. In this work, we reconsider the duopoly model in Ding et al [17]. It is different from the paper [17] that this work considers not only the competitor’s present behaviour but also the previous behaviours. We set time delay for the cooperative model discussed in [17], every player is assumed to make decision by depending the market previous states. This work studies whether Pareto Optimality can be achieved and the effects of time delay on the stability of Pareto Optimality.

2 The model

Let us consider a market where two firms produce a homogeneous good. Production decisions are taken at discrete time periods $t = 1, 2, 3, \ldots \ldots$. The quantity of output by each firm at time $t$ is denoted by $q_{i,t} (i = 1, 2)$. The cost of production $C_{i,t}$ is a linear function of the output:

$$C_{i,t} = c_i q_{i,t}. \quad (1)$$

We also assume a linear inverse demand function:

$$p_t = a - bQ_t, \quad (2)$$

where $a, b > 0, a > c_i, Q_t = \sum_i q_{i,t}$.

Then at time $t$, the profit of each firm, $\pi_{i,t}$, is equal to:

$$\pi_{i,t} = (a - bQ_t - c_i)q_{i,t}, i = 1, 2. \quad (3)$$

The solving of the cooperative profit has been introduced in duopoly game theory. The cooperative profit means the profit which is solved by maximizing the sum of all firms profit. We consider the symmetrical case: $c_1 = c_2 = c$, then the total profit $\pi$ is equal to:

$$\pi = \pi_1 + \pi_2 = (a - bQ - c_1)q_1 + (a - bQ - c_2)q_2 = (a - c)(q_1 + q_2) - b(q_1 + q_2)^2.$$

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Let
\[
\begin{align*}
\frac{\partial x}{\partial q_1} &= a - c - 2b(q_1 + q_2) = 0, \\
\frac{\partial x}{\partial q_2} &= a - c - 2b(q_1 + q_2) = 0.
\end{align*}
\]
(4)

To calculate the cooperative profits of each producer, we set \(\frac{\partial x}{\partial q_i} = 0\), \(i = 1, 2\). Then we get the output of the cooperative profit \(q_1 = q_2 = q_c = \frac{(a-c)}{2b}\), the cooperative profit \(\pi_1 = \pi_2 = \pi_c = \frac{(a-c)^2}{4}\).

In the work [17], the Cournot model is studied with the tit-for-tat conduct. The tit-for-tat strategy is the best behaviour allowing the achievement of cooperation in repeated games [16]. Its characteristic is that every player consists in doing what the opponent did in previous move, last time you cooperated, I would cooperate this time, last time you did not cooperate, I would not cooperate this time. The paper [17] is about cooperation under the incomplete information, and the model in work [17] is based on the assumption that the firms compare their own profits with the cooperative profit. Although each producer cannot obtain the competitor’s complete information, he completely knows about his own output and profit. The firm can compare his profit \(\pi_{i, t}\) at time \(t\) with the cooperative profit \(\pi_c\), which is Pareto Optimality. While his own profit is more than the cooperative profit \((\pi_c - \pi_{i, t} < 0)\), he extrapolates that the competitor is cooperative, then he will properly reduce his output in order to continue the cooperation as a reward. Otherwise, if \(\pi_c - \pi_{i, t} > 0\), firm \(i\) cannot realize the cooperative profit, and extrapolates that the competitor is not cooperative, then he will increase his output as penalty. Based on these thoughts, a dynamic equation with outputs adjustment is built as follows:
\[
\begin{align*}
q_1(t+1) &= q_1(t) + u_1[\pi_c - (a - bq - c)q_1(t)], \\
q_2(t+1) &= q_2(t) + u_2[\pi_c - (a - bq - c)q_2(t)].
\end{align*}
\]
(5)

This is a simple adjustment mechanism, producers do not need to know the relevant information of their opponents, so it is an adjustment strategy with incomplete information. The adjustment equation (5) is not only simple in form but also incorporates the idea of tit-for-tat strategy discussed in the infinite repeated Prisoner’s Dilemma game model. The paper [17] analyses the question that if the adjustment equation with tit-for-tat strategy idea finally can make the Cournot game obtain the Pareto optimal equilibrium of the two parties’ cooperation.

Through numerical simulation, the work [17] shows that under the tit-for-tat strategy, the cooperation may be achieved, but the stability of the adjustment system is sensitive to the parameters, and the Pareto Optimality cannot be assured. The stability of the adjustment system is sensitive to the parameters, and the Pareto Optimality cannot be assured. The adjustment equation (5) is not only simple in form but also incorporates the idea of tit-for-tat strategy discussed in the infinite repeated Prisoner’s Dilemma game model. The paper [17] analyses the question that if the adjustment equation with tit-for-tat strategy idea finally can make the Cournot game obtain the Pareto optimal equilibrium of the two parties’ cooperation.

As we mentioned above, the model in [17] only takes into consideration into the player’s \(\pi_c - \pi_{i, t}\) at time \(t\) and not considers previous datas, the previous information such as:
\[
\pi_c - \pi_{i, t}, \pi_c - \pi_{i, t-1}, ..., \pi_c - \pi_{i, t-T}.
\]

Players’ strategic adjustment may be more tolerant or smoother, they want to make more steady decisions by considering more historical information. This adjustment mechanism may have more possible to realize cooperation. So we improve the model and add the time delay to the model.

Then the dynamical adjustment for player \(i\) at time \(t + 1\) is given by
\[
q_i(t + 1) = q_i(t) + \alpha_i(q_i(t)) \sum_{L=0}^{T} \omega_L^i (\pi_c - \pi_{i, t-1-L}),
\]
(6)
where \(\omega_L^i\) is the weight coefficient assigned to period \(t - L\), \(\omega_L^i > 0\), \(\sum_{L=0}^{T} \omega_L^i = 1\), and \(\alpha_i(q_i(t))\) describes the adjustment rate.

For simplicity, we consider a duopoly game \((n = 2)\) with one step delay \((T = 1)\), and assume a linear form of adjustment rate \([2, 3, 6]: \alpha_i(q_i(t)) = v_i q_i(t) v_i\) is a positive constant representing the relative adjustment rate, and consider a symmetric case: \(\omega_0^i = \omega_1^i = \omega_2^i = \omega\). Then dynamics (6) takes its form as
\[
\begin{align*}
q_1(t + 1) &= q_1(t) + v_1 q_1(t) \omega (\pi_c - \pi_{i, t-1}) + (1 - \omega) (\pi_c - \pi_{i, t-1}), \\
q_2(t + 1) &= q_2(t) + v_2 q_2(t) \omega (\pi_c - \pi_{i, t}) + (1 - \omega) (\pi_c - \pi_{i, t-1}),
\end{align*}
\]
(7)
where \(0 \leq \omega \leq 1\) is a weight coefficient assigned to the non-delayed period \(t\) and \(1 - \omega\) is assigned to the delayed period \(t - 1\). We refer to the parameter \(\omega\) as the delay weight of the model. It is a non-delay case if \(\omega = 1\) and a full delay case if \(\omega = 0\). The case \(0 < \omega < 1\) means smoothed adjustment strategies are adopted by the decision makers.

Then we obtain a nonlinear dynamical system of output adjustment for the duopoly game with one step delay on \(\pi_c - \pi_{a,t}^j\): 

\[
\begin{align*}
q_1(t + 1) &= q_1(t) + v_1 q_1(t) \left[ \omega (\pi_c - (a - b(q_1(t) + q_2(t)) - c)q_1(t) ) + (1 - \omega)(\pi_c - \right. \\
&\left. \quad (a - b(q_1(t - 1) + q_2(t - 1)) - c)q_1(t - 1)) \right], \\
q_2(t + 1) &= q_2(t) + v_2 q_2(t) \left[ \omega (\pi_c - (a - b(q_1(t) + q_2(t)) - c)q_2(t) ) + (1 - \omega)(\pi_c - \\
&\quad (a - b(q_1(t - 1) + q_2(t - 1)) - c)q_2(t - 1)) \right].
\end{align*}
\]

(8)

In order to study the stability of system (8), we rewrite it as a four-dimensional system in the following form (by setting \(q_3(t) = q_1(t - 1), q_4(t) = q_2(t - 1)\):

\[
\begin{align*}
q_1(t + 1) &= q_1(t) + v_1 q_1(t) \left[ \omega (\pi_c - (a - b(q_1(t) + q_2(t)) - c)q_1(t) ) + (1 - \omega)(\pi_c - \\
&\quad (a - b(q_3(t) + q_4(t)) - c)q_3(t)) \right], \\
q_2(t + 1) &= q_2(t) + v_2 q_2(t) \left[ \omega (\pi_c - (a - b(q_1(t) + q_2(t)) - c)q_2(t) ) + (1 - \omega)(\pi_c - \\
&\quad (a - b(q_3(t) + q_4(t)) - c)q_4(t)) \right], \\
q_3(t + 1) &= q_1(t), \\
q_4(t + 1) &= q_2(t).
\end{align*}
\]

(9)

Let \(q_i(t + 1) = q_i(t) (i = 1, \cdots, 4)\) in system (9), then we obtain four nonnegative equilibrium points:

\[E_0 = (0, 0, 0, 0), E_1 = \left(0, \frac{a-c}{4b}, \frac{a-c}{4b}, 0\right), E_2 = \left(\frac{a-c}{4b}, 0, \frac{a-c}{4b}, 0\right), E^* = (q_1^*, q_2^*, q_3^*, q_4^*),\]

where

\[q_1^* = q_2^* = q_3^* = q_4^* = \frac{a-c}{4b} = q_c.\]

(10)

\(E_0, E_1, E_2\) are three boundary equilibria which are on the boundary of the strategy space set \(S = \{(q_1, q_2, q_3, q_4) | q_1 \geq 0, q_2 \geq 0, q_3 \geq 0, q_4 \geq 0\}\), and \(E^*\) is a unique interior equilibrium. Since \(q_1^* = q_2^* = q_c, E^*\) is Pareto Optimal state. Cooperation can be achieved if \(E^*\) is stable.

3 Stability of equilibrium

The stability of an equilibrium point \(Q = (q_1, q_2, q_3, q_4)\) for system (9) is determined by the eigenvalues of the Jacobian matrix \(J\) at \(Q\), which takes the form as:

\[
J(q_1, q_2, q_3, q_4) = \begin{pmatrix}
a_{11} & b v_1 q_1^2 & v_1 q_1 (1-\omega) (a-c) + b (q_1 + q_4) + b q_1 & b v_1 q_1 (1-\omega) \\
b v_2 q_2 & a_{22} & b v_2 q_2 (1-\omega) & 0 \\
v_2 (1-\omega) (c-a+b(q_1+q_4)+b q_1) & 0 & v_2 q_2 (1-\omega) (c-a+b(q_1+q_4)+b q_1) & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

(11)

where

\[
a_{11} = 1 + v_1 \omega q_1 (c-a+b(q_1+q_2)+b q_1) + v_1 (\omega) (\pi_c + c q_1 - q_1 (a-b(q_1+q_2))) + (1-\omega) (\pi_c + c q_1 - q_3 (a-b(q_3+q_4))), \\
a_{22} = 1 + v_2 \omega q_2 (c-a+b(q_1+q_2)+b q_2) + v_2 (\omega) (\pi_c + c q_2 - q_2 (a-b(q_1+q_2))) + (1-\omega) (\pi_c + c q_4 - q_4 (a-b(q_3+q_4))).
\]

An equilibrium \(Q\) will be locally asymptotically stable if all the eigenvalues (real or complex) of the Jacobian matrix \(J(Q)\) lie inside the unit disk, i.e. \(|\lambda| < 1\) holds for any eigenvalue \(\lambda\) of \(J(Q)\). An equilibrium \(Q\) will be unstable if there is an eigenvalue \(\lambda\) of \(J(Q)\) such that \(|\lambda| > 1\).

**Proposition 1** The zero solution \(E_0 = (0, 0, 0, 0)\) is an unstable equilibrium.

**Proof.** From (11), we have the Jacobi matrix

\[
J(E_0) = \begin{pmatrix}
1 + v_1 \pi_c & 0 & 0 & 0 \\
0 & 1 + v_2 \pi_c & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

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which has four eigenvalues $\lambda_1 = 1 + v_1 \pi_c, \lambda_2 = 1 + v_2 \pi_c, \lambda_3 = 0$, and $\lambda_4 = 0$. Since $|\lambda_{1,2}| > 1$, we conclude that the equilibrium $E_0$ is unstable. ■

**Proposition 2** The boundary equilibria $E_1$ and $E_2$ are both unstable equilibrium points.

**Proof.** At the boundary equilibrium point $E_1$, the Jacobian matrix (11) is simplified as

$$J(E_1) = \begin{pmatrix}
\frac{1 + v_1 \pi_c}{166} & 0 & 0 & 0 \\
0 & \frac{1 + v_2 \pi_c}{166} & 0 & 0 \\
0 & 0 & \frac{(a-c)^2}{166} v_2 (1 - \omega) & 0 \\
0 & 0 & 0 & \frac{(a-c)^2}{8b} v_2 (1 - \omega)
\end{pmatrix}.$$  

By calculation we get that $J(E_1)$ has an eigenvalue $\lambda = 1 + v_1 \pi_c$, which means the boundary equilibrium $E_1$ is unstable. The instability of $E_2$ can be shown in a similar way. ■

Since the boundary equilibria are unstable, we focus on the interior equilibrium $E^*$ in the following parts.

It is easy to see that at the interior equilibrium $E^*(q_1^*, q_2^*, q_3^*, q_4^*)$, system (9) must meet the following equilibrium conditions:

$$\begin{cases}
\pi_c - (a - b(q_1^* + q_2^*) - c)q_1^* = 0, \\
\pi_c - (a - b(q_1^* + q_2^*) - c)q_2^* = 0, \\
q_3^* = q_1^*, \\
q_4^* = q_2^*,
\end{cases} \tag{12}$$

from which we also obtain $q_1^* = q_3^* = q_2^* = q_4^*$ as shown in Eq.(10).

Substituting the conditions (12) into the Jacobian matrix (11), we have

$$J(E^*) = \begin{pmatrix}
D_1 & b\omega v_1 q_1^2 & v_1 q_c (1 - \omega) (c - a + 3b q_c) & b v_1 q_2^2 (1 - \omega) \\
D_2 & b\omega v_2 q_2^2 & v_2 q_c (1 - \omega) (c - a + 3b q_c) & v_2 q_c (1 - \omega) (c - a + 3b q_c) \\
0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0
\end{pmatrix},$$

where

$$D_1 = v_1 \omega q_c (c - a + 3b q_c) + 1, \\
D_2 = v_2 \omega q_c (c - a + 3b q_c) + 1.$$  

And by direct calculation we obtain the characteristic polynomial $p(\lambda)$ for $J(E^*)$ as following:

$$p(\lambda) = \lambda^4 + p_1 \lambda^3 + p_2 \lambda^2 + p_3 \lambda + p_4,$$

where

$$p_1 = (v_1 + v_2) q_c (a - c - 3b q_c) - 2, \\
p_2 = (v_1 + v_2) q_c (1 - 2\omega)(a - c - 3b q_c) + v_1 v_2 q_c^2 (8b^2 q_c^2 + (a - c)^2) + 6b (a - c) q_c + 1, \\
p_3 = (v_1 + v_2) q_c (1 - \omega)(c - a + 3b q_c) + 2v_1 v_2 q_c^2 (1 - \omega)(6b( a - c) q_c + 8b^2 q_c^2 + (a - c)^2), \\
p_4 = v_1 v_2 (1 - \omega)^2 q_c^2 (6b (a - c) q_c + 8b^2 q_c^2 + (a - c)^2).$$

Schur-Cohn Criterion (see e.g. [23]) tells that all the roots of the characteristic polynomial $p(\lambda)$ lie inside the unit disk if and only if the following hold:

(i) $p(1) = 1 + p_1 + p_2 + p_3 + p_4 > 0$;
(ii) $(-1)^4 p(-1) = 1 - p_1 + p_2 - p_3 + p_4 > 0$;
(iii) The determinates of the $1 \times 1$ matrices $B_1^{\pm}$ and the $3 \times 3$ matrices $B_2^{\pm}$ are all positive, where

$$B_1^{+} = (1 + p_4), \\
B_1^{-} = (1 - p_4),$$

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In our model, $1 + p_4 > 0$ holds obviously and

$$1 + p_1 + p_2 + p_3 + p_4 = v_1 v_2 q_c^2 (6b(a - c)q_c + 8b^2 q_c^2 + (c - a)^2),$$

which is also positive. So we conclude that the interior equilibrium $E^*$ of system (9) is locally asymptotically stable if

$$1 - p_1 + p_2 - p_3 + p_4 > 0, 1 - p_4^4 > 0,$$

$$\text{Det}(B_3^+) > 0, \text{Det}(B_4^-) > 0,$$

which are equivalent to the following two conditions:

(a) $p_1 - p_2 + p_3 - 1 < p_4 < 1.$

(b) $-1 - p_1 p_3 q_c + p_2^3 + p_2 q_c^2 + p_c^2 < p_2 + p_1 - p_1 p_3 + p_2^3 q_c^4 - p_4^4 < 1 + p_1 p_3 q_c + 2p_2 q_c^4 - p_2^3 - p_2 q_c^2 - p_c^2.$

The inequalities (a) and (b) above define a parameter region of the stability for the system and will be used in the following section to make numerical simulation for the stability region of system (9). If the characteristic polynomial $p(\lambda)$ at $E^*$ meets the stability conditions, the interior equilibrium $E^*$ is stable. So we can achieve Pareto Optimality and realize cooperation at the same time. But the stability of the system (9) at $E^*$ may be affected by the change of the parameters. If the characteristic polynomial $p(\lambda)$ at $E^*$ does not meet the stability conditions, the system (9) may lose stability and produce some complicated dynamical behaviors. In the following section, we will show that the system may display much complicated dynamical behaviors when the model parameters are not located in the stability region.

## 4 Numerical simulations

In this section, numerical simulations by computer are used to show the significant influence of the model parameters, especially the delay weight, on the dynamic features of system (9). We provide numerical evidence for the complicated dynamical behaviors of system (9) and for the great influence of the model parameters on the stability of the system. We display various numerical figures including bifurcation, phase portrait, and stability region. In all these numerical experiments the constant $a, b, c$ are fixed as $a = 12, b = 1, \text{and } c = 4.$

Fig. 1 shows the bifurcation with respect to the adjustment speed $v_1$ (the other adjustment speed $v_2$ is fixed as $v_2 = 0.34$). To show the influence of delay weight $\omega$ on the dynamical behaviors of the system, four cases of $\omega$ are considered. Fig. 1(A) ($\omega = 0.45$) shows that the system keeps stable for $v_1$ taking its value up to nearly 0.11. It shows that the firms cannot achieve the Pareto Optimality when $v_1 > 0.11$. Fig. 1(B) ($\omega = 0.6$) shows that the system keeps stable for $v_1$ taking its value up to nearly 0.27. Under the circumstances, the firms in Cournot game cannot achieve the Pareto optimal equilibrium when $v_1 > 0.27$. We can see from Fig. 1(C) ($\omega = 0.75$) that the system always keeps stable with $v_1$ varying in all the interval considered.

So the firms in Cournot game can keep the cooperation with $v_1$ varying in all the interval considered. Fig. 1(D) shows that the bifurcation diagram for the non-delay case $\omega = 1$ converges to the equilibrium as $v_1 < 0.15$ approximately; when $v_1$ increases, the equilibrium point becomes unstable, period-doubling bifurcations appear and finally chaotic behaviors occur. That is to say, the firms in Cournot game cannot achieve cooperation when $v_1 > 0.15$. Comparing these four diagrams, we can see that instability for a case of proper delay (Fig. 1(B)) occurs later than that for the non-delay case (Fig. 1(D)) and than that for a case of excessive delay (Fig. 1(A)).

And we can see that the stability region of the system can be expanded to a very large interval if an intermediate value of the delay weight $\omega$ is set (Fig. 1(C)). So we can get that the system is more likely to achieve cooperation when an intermediate value of the delay weight $\omega$ is set.

From Fig. 1(A) and (B), we can also see that the system loses stability through a Neimark-Sacker bifurcation, whereas in Fig. 1(D) the stability loss is evidently due to a period doubling bifurcation. Two dimensional phase portraits which are associated with Fig. 1(A, B, D) are plotted in Fig. 2, which gives a more detailed description of the trajectories of the system. From Fig. 2(C) (the non-delay case $\omega = 1$), we observe that there are orbits of period 1, 2, \ldots. When $v_1$ takes small values and there are attracting chaotic sets when $v_1$ takes large values. Fig. 2(A) and Fig. 2(B) show that in a relatively high delay case ($\omega$ is small, the weight coefficient $1 - \omega$ assigned to the previous period is large), there may be a Neimark-Sacker bifurcation so that closed invariant curves can be observed when the system loses stability. In Fig. 2(A) and Fig. 2(B) there are also attracting chaotic sets when $v_1$ takes large values.
Figure 1: Bifurcation diagrams with respect to the adjustment speed $v_1$.

Figure 2: Phase portraits for Fig. 1(A, B, D) with various values of $\omega$ and $v_1$. 

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Finally, we use the stability conditions (a) and (b) given at the end of Section 3 to make a numerical analysis of the influence of the delay weight $\omega$ on the stability region for the system. Four cases are considered ($\omega = 0.45, \omega = 0.6, \omega = 0.75, \omega = 1$), and the numerical results for the stability region in the $(v_1, v_2)$ space are plotted in Fig. 6. Comparing the four cases we observe that the stability region for $\omega = 0.75$ is much larger than that for the other cases ($\omega = 0.45, \omega = 0.6, \omega = 1$). So we conclude that an intermediate delay weight can also expand the stability region of the system. That is to say, the firms in the Cournot game have more possibility to achieve Pareto Optimality when the delay weight is an intermediate level.

![Figure 3](image.png)

**Figure 3:** Stability region in $(v_1; v_2)$ plane for different levels of delay weight $\omega$.

### 5 Conclusion

In this work, we reconsider the cooperation model of output adjustment under incomplete information. The work [17] shows that the tit-for-tat strategy may lead to the cooperation of players, but the stability of the adjustment system is sensitive to the parameters. In fact, players with cooperative intention may have a lot of tolerance to allow opponents make mistakes. That is to say, each player will not decide punishments at once for opponents’ one non-cooperation, he will make decisions according to the opponents’ previous behaviours. So in this work, we set time delay for the cooperative model discussed in [17], every player is able to adopt a smoothed strategy by adjusting his output according to the weighted market previous data. We have established the dynamics of players’ output adjustments and done analysis for it. The instability of three boundary equilibria is shown and the conditions for the local stability of the interior equilibrium are obtained by Schur-Cohn Criterion. Through analysis we can get that the Pareto Optimality can be achieved and cooperation may be realized under the stability conditions. But the stability of the adjustment system is affected by the change of the parameters.

Numerical simulations are used to illustrate the complexity of system evolution and the influence of the model parameters on the system stability. It is demonstrated that a proper time delay plays an important role in the dynamical behaviors of the system. In a relatively low delay case or a relatively high delay case the system is possible to lose stability through bifurcation. In that case, the cooperation cannot be achieved. An intermediate delay weight can delay the occurrence of complex behaviors such as bifurcation and chaos, and can expand the stability region for the system. Numerical simulations also show that the firms in the Cournot game are more likely to achieve cooperation when an intermediate value of the delay weight $\omega$ is set.

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References


