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# Numerical Solution and Stability Analysis of Special Multistep Methods for Fifth Order Differential Equations

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**Abstract:** We have discussed the regions of absolute stability of fifth order boundary value problems in this paper. Some methods derived in this paper are applied to solve a fifth order boundary value problem, hence establishing the superiority of the methods. Numerical results are given to illustrate the efficiency of our methods and compared with exact solution.

**Keywords:** Numerical differentiation; Initial value problem; Boundary value problem; Absolute stability, Multistep methods

## 1 Introduction

Using the mathematical modeling several problems in natural sciences, medicine, business management, engineering, fluid mechanics, elasticity, heat transfer, and chemistry can be transformed into ordinary or partial differential equations. We can employ the numerical methods to obtain an approximate solution which is near to accurate solution.

Fifth order boundary value problems arise in the mathematical modelling of the viscoelastic flows and other branches of mathematical, physical, and engineering sciences, see [1–6], and the references therein. Several numerical methods for solving fifth order boundary value problems have been suggested and investigated. Noor and Mohyud-Din [3, 4] used variational iteration method and variational iteration decomposition method for solving fifth order and other higher-order boundary value problems. He [7] developed the variational iteration method (VIM) for solving nonlinear boundary value problems. Agarwal [8] presented the conditions for the uniqueness and existence of the solutions of such type of problems. In [9], Khan investigated the solution of the fifth order boundary value problem using the finite difference method. Wazwaz, in [6], presented the numerical solution of fifth order BVP using the Adomian decomposition method and its modified form. Chawla and Katti [10], introduced the numerical methods for the solution of higher order differential equations. Further information can be had from [11, 12]. Inspired and motivated by the ongoing research in this area, we apply the special multistep methods for solving the fifth order boundary value problems. It is worth mentioning that our proposed technique can handle any boundary value problem with a set of boundary conditions.

An extensive study of single step and multistep methods for the initial and boundary value problems of ordinary differential equations have been made by several researchers and a detailed treatment of the subject has been provided by many authors like Froberg [13], Gear [14], Gragg and Statter [15], P. Henrici [16], and J. D. Lambert [17]. Special multistep methods based on numerical integration and methods based on numerical differentiation for solving first-order differential equations have been derived in Peter Henrici [16] and Gear [14]. The methods based on numerical differentiation of first order differential equations have been shown to be stiffly stable by Gear [18] and high order stiffly stable methods were considered by Jain [19]. The motivation for the work carried out in this chapter arises from the methods based on numerical differentiation for the first order differential equations. Special multistep methods based on numerical integration for the solution of the special second order differential equations have been derived in Henrici [16] and Special multistep

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methods based on numerical differentiation for solving the initial value problem have been derived in Rama Chandra Rao [5]. In Henrici [16], the methods based on Numerical Integration have been derived by integrating $y'' = f(x, y)$ twice and replacing the function $f(x, y)$ by an interpolating polynomial. Special multistep methods have been derived by replacing $y(x)$ on the left hand side of $F(x, y, y^{'}, y^{''}, y^{'''}, y^{iv}, y^{v}) = 0$ by an interpolating polynomial and differentiating it five times. We have investigated a class of implicit and explicit methods. It is found that the implicit and explicit methods have order $(k-4)$. Some local truncation errors are provided. The regions of absolute stability of the methods are derived. Numerical tests of the performance of the methods are established by solving differential equation and compared with the exact solution.

# 2 General linear multistep methods for special fifth-order differential equations

The special fourth order differential equation

$$F(x, y, y^{'}, y^{''}, y^{'''}, y^{iv}, y^{v}) = 0, \tag{1}$$

frequently occurs in a number of applications of science and engineering. A general linear multistep method of step number $k$ for the numerical solution of Eq.(1) is given by

$$y_{n+1} = \Sigma_{j=1}^{k} a_j y_{n+1-j} + h^5 \Sigma_{j=0}^{k} b_j y_{n+1-j}, \tag{2}$$

where $a_j, b_j$ are constants and '$h$' is the step length.

Introducing the polynomials

$$\rho(\xi) = \xi^k - \Sigma_{j=1}^{k} a_j \xi^{k-j} \text{ and } \sigma(\xi) = \Sigma_{j=1}^{k} b_j \xi^{k-j}, \tag{3}$$

Eq.(2) can be written as:

$$\rho(E) y_{n-k+1} - h^5 \sigma(E) y_{n-k+1}^{v} = 0, \text{ where } E(y_n) = y_{n+1}. \tag{4}$$

Applying (4) to $y^v = \lambda y$, we get the characteristic equation

$$\rho(\xi) - \bar{h}\sigma(\xi) = 0, \text{ where } \bar{h} = \lambda h^5. \tag{5}$$

The roots $\xi_i$ of the characteristic equation (5) and $\bar{h}$ are in general, complex and the region of absolute stability is defined to be the region of the complex $\bar{h}-$plane such that the roots of the characteristic equation (5) lie within the unit circle whenever $\bar{h}$ lies in the interior of the region. If we denote the region of absolute stability of $R$ and its boundary by $\partial R$, then the locus of $\partial R$ is given by

$$\bar{h}(\theta) = \rho(e^{i\theta})/\sigma(e^{i\theta}), 0 \leq \theta \leq 2\pi. \tag{6}$$

# 3 Derivation of the methods

Let $p(x)$ be the backward difference interpolating polynomial of $y(x)$ at $(k+1)$ abscissas $x_{n+1}, x_n, \cdots, x_{n-k+1}$. Then $p(x)$ is given by

$$p(x) = \Sigma_{m=0}^{k} (-1)^m \binom{-s}{m} \nabla^m y_{n+1}, \qquad s = \frac{(x - x_{n+1})}{h}. \tag{7}$$

Differentiating Eq.(7) five times with respect to $x$, we get

$$p^v(x) = \left(\frac{1}{h^5}\right) \sum_{m=0}^{k} \frac{d^5}{ds^5} \left[ (-1^m) \binom{-s}{m} \right] \nabla^m y_{n+1}.$$

Replacing $y^v(x)$ by $p^v(x)$ in Eq.(5.1) and putting $x = x_{n+1-r}$ i.e., $s = -r$, we get,

$$\sum_{m=0}^{k} \delta_{r,m} \nabla^m y_{n+1} = h^5 f_{n+1-r}, \tag{8}$$

$$\text{where } \delta_{r,m} = \frac{d^5}{ds^5} \left[ (-1^m) \binom{-s}{m} \right]. \tag{9}$$

# 4 Generating function for the coefficients $\delta_{r,m}$

We define

$$D_{r,t} = \sum_{m=0}^{\infty} \delta_{r,m} t^m. \tag{10}$$

Substituting $\delta_{r,m}$ from (9) in (10) and simplifying, we get

$$D_{r,t} = \sum_{m=0}^{\infty} \delta_{r,m} t^m = -(1-t)^{-s}\big[log(1-t)\big]^5,$$

$$\therefore \sum_{m=0}^{\infty} \delta_{r,m} t^m = -(1-t)^{r}\big[log(1-t)\big]^5 \quad \text{at} \quad s=-r. \tag{11}$$

# 5 Implicit methods

Taking $r = 0$ in (8), a class of implicit methods can be attained which are given by

$$\sum_{m=0}^{k} \delta_{0,m} \nabla^m y_{n+1} = h^5 f_{n+1}. \tag{12}$$

From the above Eq.(12) it follows that $\delta_{0,m}$ is the coefficient of $t^m$ in the expansion of $-\big[log(1-t)\big]^5$ in powers of $t$. The coefficients $\delta_{0,m}$ are shown in the Table 1.

Differences in (12) are expressed in terms of function values.

After simplification, the Eq.(12) will turn out into the form

$$\sum_{j=0}^{k} a_j y_{n+1-j} = h^5 f_{n+1}. \tag{13}$$

The coefficients $a_j$ are shown in Table 2.

The local truncation error of the formula (13) is given by

$$LTE = \delta_{0,k+1} h^{k+1} y^{k+1}(\eta). \tag{14}$$

Table 1: Coefficients of $\delta_{0,m}$; $m = 0(1)11$.

| **M** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta_{0,m}$ | 0 | 0 | 0 | 0 | 0 | -1 | $\frac{-5}{2}$ | $\frac{-25}{6}$ | $\frac{-35}{6}$ | $\frac{-5021}{720}$ | $\frac{-285}{32}$ | $\frac{-31063}{3024}$ |

It follows that the $k$-step method (14) has the order $k-4$, which is absolutely stable. For the method (13), we have

$$\rho(\xi) = \sum_{j=0}^{k} a_j \xi^{k-j} \text{and} \sigma(\xi) = \xi^k. \tag{15}$$

The regions of absolute stability of the method for $k = 5$ (1) 11 are shown in Figs. 1 and 2 (taking real part on $x-$axis and imaginary part on $y-$axis). The region of absolute stability is the region lying outside the boundary.

Table 2: Coefficients of $\alpha_j; j = 0(1)k, k = 5(1)11$.

| K \ J | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **5** | -1 | 5 | -10 | 10 | -5 | 1 | | | | | | |
| **6** | $\frac{-7}{2}$ | $\frac{40}{2}$ | $\frac{-95}{2}$ | $\frac{120}{2}$ | $\frac{-85}{2}$ | $\frac{32}{2}$ | $\frac{-5}{2}$ | | | | | |
| **7** | $\frac{-46}{2}$ | $\frac{295}{2}$ | $\frac{-810}{2}$ | $\frac{1235}{2}$ | $\frac{-1130}{2}$ | $\frac{621}{2}$ | $\frac{-190}{2}$ | $\frac{25}{2}$ | | | | |
| **8** | $\frac{-81}{2}$ | $\frac{575}{2}$ | $\frac{-1790}{2}$ | $\frac{3195}{2}$ | $\frac{-3580}{2}$ | $\frac{2581}{2}$ | $\frac{-1170}{2}$ | $\frac{305}{2}$ | $\frac{-35}{2}$ | | | |
| **9** | $\frac{-14741}{720}$ | $\frac{114189}{720}$ | $\frac{-395556}{720}$ | $\frac{805164}{720}$ | $\frac{-1062246}{720}$ | $\frac{942366}{720}$ | $\frac{-562164}{720}$ | $\frac{217356}{720}$ | $\frac{-49389}{720}$ | $\frac{5021}{720}$ | | |
| **10** | $\frac{-42307}{1440}$ | $\frac{356628}{1440}$ | $\frac{-1368237}{1440}$ | $\frac{3149328}{1440}$ | $\frac{-4817742}{1440}$ | $\frac{5116632}{1440}$ | $\frac{-3817578}{1440}$ | $\frac{1973712}{1440}$ | $\frac{-675903}{1440}$ | $\frac{138292}{1440}$ | $\frac{-12825}{1440}$ | |
| **11** | $\frac{-1199077}{30240}$ | $\frac{10906118}{30240}$ | $\frac{-45817627}{30240}$ | $\frac{117389838}{30240}$ | $\frac{-203680482}{30240}$ | $\frac{250960332}{30240}$ | $\frac{-223680198}{30240}$ | $\frac{143955852}{30240}$ | $\frac{-65447913}{30240}$ | $\frac{19988782}{30240}$ | $\frac{-3686255}{30240}$ | $\frac{310630}{30240}$ |

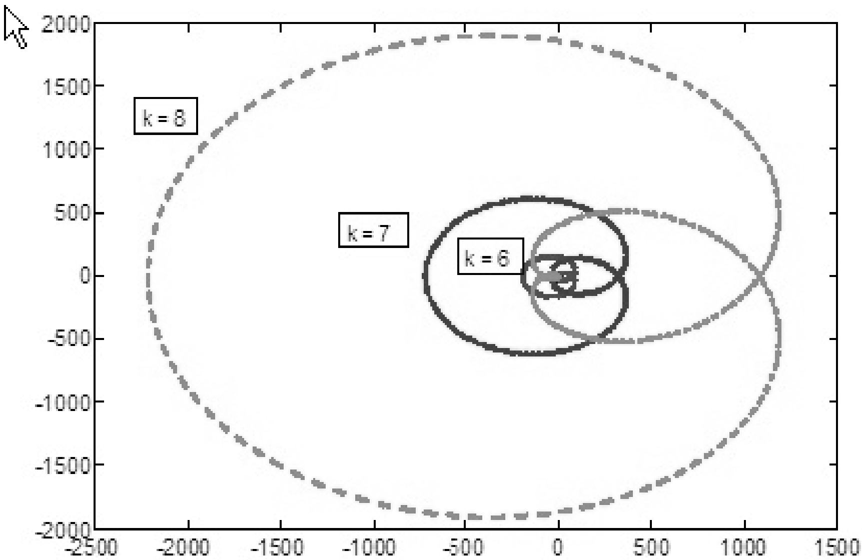


Figure 1: The region of absolute stability of the method (13) for $k = 5, 6, 7$ and 8.

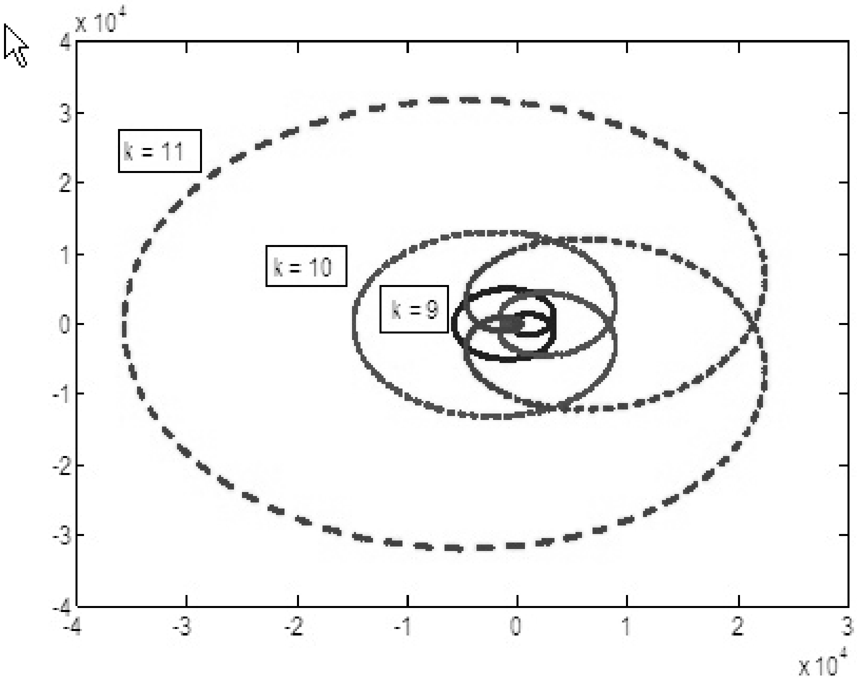


Figure 2: The region of absolute stability of the method (13) for $k = 9, 10$ and 11.

# 6 Explicit methods

Choosing $r = 1$ in (8), we obtain a class of explicit methods given by

$$\sum_{m=0}^{k} \delta_{1,m} \nabla^{m} y_{n+1} = h^{5} f_{n}. \tag{16}$$

From the Eq.(11), it is clear that $\delta_{1,m}$ is the coefficient of $t^m$ in the expansion of $-(1-t)\big[log(1-t)\big]^5$ in powers of $t$. The coefficients $\delta_{1,m}$ are listed in the Table 3.

Differences in (16) are expressed in terms of function values.

After simplification, the Eq.(16) will turn out into the form

$$\sum_{j=0}^{k} a_j y_{n+1-j} = h^5 f_n. \tag{17}$$

The coefficients $a_j$ are shown in Table 4.

The local truncation error of the formula (17) can be obtained as

$$LTE = \delta_{1,k+1} h^{k+1} y^{k+1}(\eta). \tag{18}$$

It follows that the k-step method (18) has the order $k - 4$, which is absolutely stable.

The regions of absolute stability of the method for $k = 5$ (1) 11 are shown in Figs. 3 and 4 (taking real part on x-axis and imaginary part on y-axis). The region of absolute stability is the region lying outside the boundary.

Table 3: Coefficients of $_{1,m}; m = 0(1)11$.

| **M** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $\delta_{1,m}$ | 0 | 0 | 0 | 0 | 0 | 1 | $\frac{3}{2}$ | $\frac{5}{3}$ | $\frac{5}{3}$ | $\frac{821}{720}$ | $\frac{2783}{1440}$ | $\frac{8261}{6048}$ |

Table 4: Coefficients of $\alpha_j; j = 0(1)k, k = 5(1)11$.

| **K \ J** | **0** | **1** | **2** | **3** | **4** | **5** | **6** | **7** | **8** | **9** | **10** | **11** |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| **5** | 1 | -5 | 10 | -10 | 5 | -1 | | | | | | |
| **6** | $\frac{5}{2}$ | $\frac{-28}{2}$ | $\frac{65}{2}$ | $\frac{-80}{2}$ | $\frac{55}{2}$ | $\frac{-20}{2}$ | $\frac{3}{2}$ | | | | | |
| **7** | $\frac{25}{6}$ | $\frac{-154}{6}$ | $\frac{405}{6}$ | $\frac{-590}{6}$ | $\frac{515}{6}$ | $\frac{-270}{6}$ | $\frac{79}{6}$ | $\frac{-10}{6}$ | | | | |
| **8** | $\frac{35}{6}$ | $\frac{-234}{6}$ | $\frac{685}{6}$ | $\frac{-1150}{6}$ | $\frac{1215}{6}$ | $\frac{-830}{6}$ | $\frac{359}{6}$ | $\frac{-90}{6}$ | $\frac{10}{6}$ | | | |
| **9** | $\frac{5021}{720}$ | $\frac{-35469}{720}$ | $\frac{111756}{720}$ | $\frac{-206964}{720}$ | $\frac{249246}{720}$ | $\frac{-203046}{720}$ | $\frac{112044}{720}$ | $\frac{-40356}{720}$ | $\frac{8589}{720}$ | $\frac{821}{720}$ | | |
| **10** | $\frac{12825}{1440}$ | $\frac{-98768}{1440}$ | $\frac{348747}{1440}$ | $\frac{-747888}{1440}$ | $\frac{1082922}{1440}$ | $\frac{-1107408}{1440}$ | $\frac{808518}{1440}$ | $\frac{-414672}{1440}$ | $\frac{142413}{1440}$ | $\frac{29472}{1440}$ | $\frac{2783}{1440}$ | |
| **11** | $\frac{310630}{30240}$ | $\frac{-2528483}{30240}$ | $\frac{9595462}{30240}$ | $\frac{-22520973}{30240}$ | $\frac{36372012}{30240}$ | $\frac{-42338478}{30240}$ | $\frac{36061788}{30240}$ | $\frac{-22338762}{30240}$ | $\frac{9805998}{30240}$ | $\frac{2890687}{30240}$ | $\frac{512798}{30240}$ | $\frac{41305}{30240}$ |

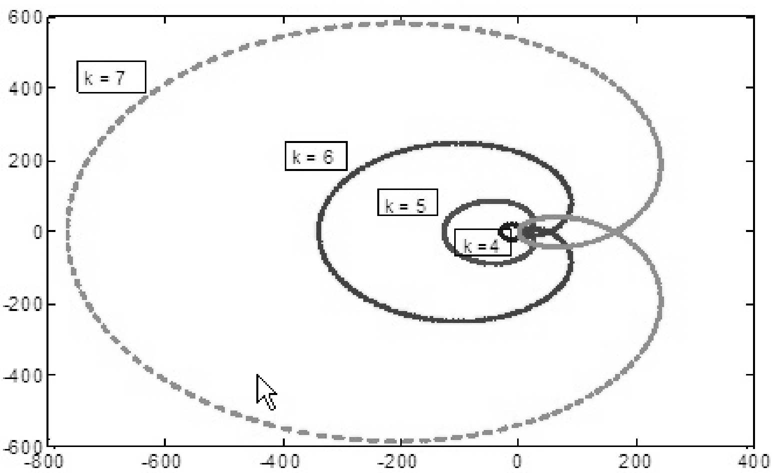


Figure 3: The region of absolute stability of the method (17) for $k = 5, 6, 7$ and $8$.

The implicit fifth order numerical differentiation method (13) derived in this paper with k = 6 is

$$y_{n+1} = \frac{40}{7}y_n - \frac{95}{7}y_{n-1} + \frac{120}{7}y_{n-2} - \frac{85}{7}y_{n-3} + \frac{32}{7}y_{n-4} - \frac{5}{7}y_{n-5} - \frac{2}{7}h^5 f_{n+1}. \tag{19}$$

The implicit sixth order numerical differentiation method (13) derived in this paper with $k = 7$ is

$$\begin{aligned} y_{n+1} = &\frac{295}{46}y_n - \frac{810}{46}y_{n-1} + \frac{1235}{46}y_{n-2} - \frac{1130}{46}y_{n-3} + \frac{621}{46}y_{n-4} - \frac{190}{46}y_{n-5} + \frac{25}{46}y_{n-6} \\ &- \frac{6}{46}h^5 f_{n+1}. \end{aligned} \tag{20}$$

The implicit seventh order numerical differentiation method (13) derived in this paper with k = 8 is

$$\begin{aligned} y_{n+1} = &\frac{575}{81}y_n - \frac{1790}{81}y_{n-1} + \frac{3195}{81}y_{n-2} - \frac{3580}{81}y_{n-3} + \frac{2581}{81}y_{n-4} - \frac{1170}{81}y_{n-5} + \frac{305}{81}y_{n-6} \\ &- \frac{35}{81}y_{n-7} - \frac{6}{81}h^5 f_{n+1}. \end{aligned} \tag{21}$$

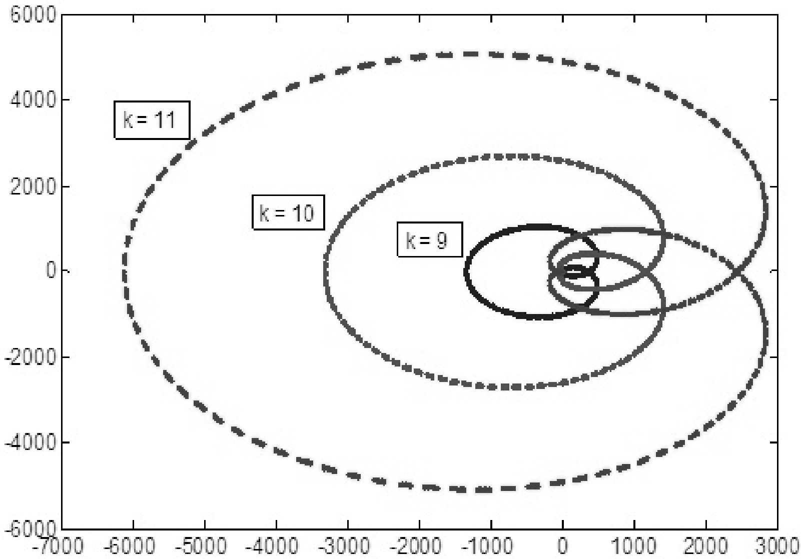


Figure 4: The region of absolute stability of the method (17) for $k = 9, 10$ and $11$.

The implicit eigth order numerical differentiation method (13) derived in this paper with $k = 9$ is

$$\begin{aligned} y_{n+1} =& \frac{114189}{1474} y_n - \frac{395556}{1474} y_{n-1} + \frac{805164}{1474} y_{n-2} - \frac{1062246}{1474} y_{n-3} + \frac{942366}{1474} y_{n-4} - \frac{562164}{1474} y_{n-5} \\ &+ \frac{217356}{1474} y_{n-6} - \frac{49389}{1474} y_{n-7} + \frac{5021}{1474} y_{n-8} - \frac{720}{1474} h^5 f_{n+1}. \end{aligned} \tag{22}$$

The explicit fifth order numerical differentiation method (17) derived in this paper with k = 6 is

$$y_{n+1} = \frac{28}{5} y_n - \frac{65}{5} y_{n-1} + \frac{80}{5} y_{n-2} - \frac{55}{5} y_{n-3} + \frac{20}{5} y_{n-4} - \frac{3}{5} y_{n-5} + \frac{2}{5} h^5 f_n. \tag{23}$$

The explicit sixth order numerical differentiation method (17) derived in this paper with $k = 7$ is

$$\begin{aligned} y_{n+1} =& \frac{154}{25} y_n - \frac{405}{25} y_{n-1} + \frac{590}{25} y_{n-2} - \frac{515}{25} y_{n-3} + \frac{270}{25} y_{n-4} - \frac{79}{25} y_{n-5} + \frac{10}{25} y_{n-6} \\ &+ \frac{6}{25} h^5 f_n. \end{aligned} \tag{24}$$

The explicit seventh order numerical differentiation method (17) derived in this paper with $k = 8$ is

$$\begin{aligned} y_{n+1} =& \frac{234}{35} y_n - \frac{685}{35} y_{n-1} + \frac{1150}{35} y_{n-2} - \frac{1215}{35} y_{n-3} + \frac{830}{35} y_{n-4} - \frac{359}{35} y_{n-5} \\ &+ \frac{90}{35} y_{n-6} - \frac{10}{35} y_{n-7} + \frac{6}{35} h^5 f_n. \end{aligned} \tag{25}$$

The explicit eighth order numerical differentiation method (17) derived in this paper with $k = 9$ is

$$\begin{aligned} y_{n+1} =& \frac{35469}{5021} y_n - \frac{111756}{5021} y_{n-1} + \frac{206964}{5021} y_{n-2} - \frac{249246}{5021} y_{n-3} + \frac{203046}{5021} y_{n-4} - \frac{112044}{5021} y_{n-5} \\ &+ \frac{40356}{5021} y_{n-6} - \frac{8589}{5021} y_{n-7} + \frac{821}{5021} y_{n-8} + \frac{720}{5021} h^5 f_n. \end{aligned} \tag{26}$$

# 7 Numerical examples

In this section, we have applied the ND methods which are derived in this paper to solve some linear and nonlinear boundary value problems.

Table 5: Solution by the implicit sixth order numerical differentiation (ND) method with $k = 7$ and $h = 0.01$.

| X | Exact Solution | Numerical Solution by Implicit sixth order ND | Absolute Error |
|---|---|---|---|
| 0.10 | 9.500527385052E-02 | 9.500527346539E-02 | 3.851257646126E-10 |
| 0.20 | 1.801028752744E-01 | 1.801028748402E-01 | 4.342650405942E-10 |
| 0.30 | 2.556184927678E-01 | 2.556184922807E-01 | 4.870562286818E-10 |
| 0.40 | 3.222767124880E-01 | 3.222767119443E-01 | 5.437609251757E-10 |
| 0.50 | 3.813777373543E-01 | 3.813777367497E-01 | 6.046118605774E-10 |
| 0.60 | 4.349950924545E-01 | 4.349950917841E-01 | 6.703877453162E-10 |
| 0.70 | 4.861958844101E-01 | 4.861958836685E-01 | 7.416947056527E-10 |
| 0.80 | 5.392853662346E-01 | 5.392853654154E-01 | 8.192814204833E-10 |
| 0.90 | 6.000777875176E-01 | 6.000777866136E-01 | 9.039867743255E-10 |
| 1.00 | 6.761957875984E-01 | 6.761957866014E-01 | 9.969693959277E-10 |
| 1.10 | 7.774009222403E-01 | 7.774009211406E-01 | 1.099667135129E-09 |
| 1.20 | 9.159583080796E-01 | 9.159583068662E-01 | 1.213469991157E-09 |
| 1.30 | 1.107038830160E+00 | 1.107038828820E+00 | 1.340078270573E-09 |
| 1.40 | 1.369162893048E+00 | 1.369162891566E+00 | 1.481503586476E-09 |
| 1.50 | 1.724690312819E+00 | 1.724690311179E+00 | 1.639980817814E-09 |
| 1.60 | 2.200361653808E+00 | 2.200361651990E+00 | 1.817962669293E-09 |
| 1.70 | 2.827897119229E+00 | 2.827897117210E+00 | 2.018527567316E-09 |
| 1.80 | 3.644660018535E+00 | 3.644660016291E+00 | 2.244413987995E-09 |
| 1.90 | 4.694392867379E+00 | 4.694392864879E+00 | 2.499740858752E-09 |
| 2.00 | 6.028035340213E+00 | 6.028035337425E+00 | 2.788313580027E-09 |
| 2.10 | 7.704634604225E+00 | 7.704634601110E+00 | 3.114928759373E-09 |
| 2.20 | 9.792360031030E+00 | 9.792360027546E+00 | 3.484251109853E-09 |
| 2.30 | 1.236963592580E+01 | 1.236963592190E+01 | 3.901464040723E-09 |
| 2.40 | 1.552640775055E+01 | 1.552640774617E+01 | 4.373891471232E-09 |
| 2.50 | 1.936555936940E+01 | 1.936555936449E+01 | 4.908120132541E-09 |
| 2.60 | 2.400450113199E+01 | 2.400450112647E+01 | 5.512973189070E-09 |
| 2.70 | 2.957695116165E+01 | 2.957695115545E+01 | 6.195183033242E-09 |
| 2.80 | 3.623493505710E+01 | 3.623493505014E+01 | 6.963922771774E-09 |
| 2.90 | 4.415103238129E+01 | 4.415103237345E+01 | 7.833421022951E-09 |
| 3.00 | 5.352090183528E+01 | 5.352090182647E+01 | 8.811120721930E-09 |
| 3.10 | 6.456612093892E+01 | 6.456612092900E+01 | 9.918281307364E-09 |
| 3.20 | 7.753738040788E+01 | 7.753738039672E+01 | 1.116059422657E-08 |
| 3.30 | 9.271807828043E+01 | 9.271807826787E+01 | 1.255571646652E-08 |
| 3.40 | 1.104283642623E+02 | 1.104283642482E+02 | 1.412844596871E-08 |
| 3.50 | 1.310296907886E+02 | 1.310296907727E+02 | 1.589299358784E-08 |
| 3.60 | 1.549299340189E+02 | 1.549299340010E+02 | 1.788373538147E-08 |
| 3.70 | 1.825891554638E+02 | 1.825891554436E+02 | 2.011148581005E-08 |
| 3.80 | 2.145260832800E+02 | 2.145260832574E+02 | 2.260276232846E-08 |
| 3.90 | 2.513254015633E+02 | 2.513254015379E+02 | 2.540551236052E-08 |
| 4.00 | 2.936459463250E+02 | 2.936459462965E+02 | 2.854187641788E-08 |

Table 6: Solution by the implicit sixth order numerical differentiation (ND) method with $k = 7$ and $h = 0.02$.

| X | Exact Solution | Numerical Solution by Implicit sixth order ND | Absolute Error |
|---|---|---|---|
| 0.10 | 9.500527385052E-02 | 9.500527385047E-02 | 5.072331443756E-14 |
| 0.20 | 1.801028752744E-01 | 1.801028752745E-01 | 3.438915818776E-14 |
| 0.30 | 2.556184927678E-01 | 2.556184927678E-01 | 2.675637489347E-14 |
| 0.40 | 3.222767124880E-01 | 3.222767124880E-01 | 2.026157019941E-14 |
| 0.50 | 3.813777373543E-01 | 3.813777373544E-01 | 6.300515664748E-14 |
| 0.60 | 4.349950924545E-01 | 4.349950924546E-01 | 4.968248035198E-14 |
| 0.70 | 4.861958844101E-01 | 4.861958844102E-01 | 5.273559366969E-15 |
| 0.80 | 5.392853662346E-01 | 5.392853662346E-01 | 4.696243394164E-14 |
| 0.90 | 6.000777875176E-01 | 6.000777875176E-01 | 2.142730437527E-14 |
| 1.00 | 6.761957875984E-01 | 6.761957875985E-01 | 5.428990590417E-14 |
| 1.10 | 7.774009222403E-01 | 7.774009222403E-01 | 4.185540802837E-14 |
| 1.20 | 9.159583080796E-01 | 9.159583080797E-01 | 3.497202527569E-14 |
| 1.30 | 1.107038830160E+00 | 1.107038830161E+00 | 5.395683899678E-14 |
| 1.40 | 1.369162893048E+00 | 1.369162893048E+00 | 6.261657858886E-14 |
| 1.50 | 1.724690312819E+00 | 1.724690312819E+00 | 4.418687638008E-14 |
| 1.60 | 2.200361653808E+00 | 2.200361653808E+00 | 7.371880883511E-14 |
| 1.70 | 2.827897119229E+00 | 2.827897119229E+00 | 3.819167204711E-14 |
| 1.80 | 3.644660018535E+00 | 3.644660018535E+00 | 1.092459456231E-13 |
| 1.90 | 4.694392867379E+00 | 4.694392867379E+00 | 7.194245199571E-14 |
| 2.00 | 6.028035340213E+00 | 6.028035340213E+00 | 7.016609515631E-14 |
| 2.10 | 7.704634604225E+00 | 7.704634604225E+00 | 1.776356839400E-13 |
| 2.20 | 9.792360031030E+00 | 9.792360031030E+00 | 2.202682480856E-13 |
| 2.30 | 1.236963592580E+01 | 1.236963592580E+01 | 2.557953848736E-13 |
| 2.40 | 1.552640775055E+01 | 1.552640775055E+01 | 2.362554596402E-13 |
| 2.50 | 1.936555936940E+01 | 1.936555936940E+01 | 3.268496584496E-13 |
| 2.60 | 2.400450113199E+01 | 2.400450113199E+01 | 5.897504706809E-13 |
| 2.70 | 2.957695116165E+01 | 2.957695116165E+01 | 4.973799150321E-13 |
| 2.80 | 3.623493505710E+01 | 3.623493505710E+01 | 7.034373084025E-13 |
| 2.90 | 4.415103238129E+01 | 4.415103238129E+01 | 4.263256414561E-14 |
| 3.00 | 5.352090183528E+01 | 5.352090183528E+01 | 1.826094830903E-12 |
| 3.10 | 6.456612093892E+01 | 6.456612093892E+01 | 1.236344360223E-12 |
| 3.20 | 7.753738040788E+01 | 7.753738040788E+01 | 7.389644451905E-13 |
| 3.30 | 9.271807828043E+01 | 9.271807828043E+01 | 1.364242052659E-12 |
| 3.40 | 1.104283642623E+02 | 1.104283642623E+02 | 9.947598300641E-13 |
| 3.50 | 1.310296907886E+02 | 1.310296907886E+02 | 3.097966327914E-12 |
| 3.60 | 1.549299340189E+02 | 1.549299340189E+02 | 2.813749233610E-12 |
| 3.70 | 1.825891554638E+02 | 1.825891554637E+02 | 3.723243935383E-12 |
| 3.80 | 2.145260832800E+02 | 2.145260832800E+02 | 3.268496584496E-12 |
| 3.90 | 2.513254015633E+02 | 2.513254015633E+02 | 2.756905814749E-12 |
| 4.00 | 2.936459463250E+02 | 2.936459463250E+02 | 5.741185304942E-12 |

Table 7: Solution by the explicit sixth order numerical differentiation (ND) method with $k = 7$ and $h = 0.01$.

| X | Exact Solution | Numerical Solution by Implicit sixth order ND | Absolute Error |
|---|---|---|---|
| 0.10 | 9.500527385052E-02 | 9.500527385047E-02 | 5.072331443756E-14 |
| 0.20 | 1.801028752744E-01 | 1.801028752745E-01 | 3.438915818776E-14 |
| 0.30 | 2.556184927678E-01 | 2.556184927678E-01 | 2.675637489347E-14 |
| 0.40 | 3.222767124880E-01 | 3.222767124880E-01 | 2.026157019941E-14 |
| 0.50 | 3.813777373543E-01 | 3.813777373544E-01 | 6.300515664748E-14 |
| 0.60 | 4.349950924545E-01 | 4.349950924546E-01 | 4.968248035198E-14 |
| 0.70 | 4.861958844101E-01 | 4.861958844102E-01 | 5.273559366969E-15 |
| 0.80 | 5.392853662346E-01 | 5.392853662346E-01 | 4.696243394164E-14 |
| 0.90 | 6.000777875176E-01 | 6.000777875176E-01 | 2.142730437527E-14 |
| 1.00 | 6.761957875984E-01 | 6.761957875985E-01 | 5.428990590417E-14 |
| 1.10 | 7.774009222403E-01 | 7.774009222403E-01 | 4.185540802837E-14 |
| 1.20 | 9.159583080796E-01 | 9.159583080797E-01 | 3.497202527569E-14 |
| 1.30 | 1.107038830160E+00 | 1.107038830161E+00 | 5.395683899678E-14 |
| 1.40 | 1.369162893048E+00 | 1.369162893048E+00 | 6.261657858886E-14 |
| 1.50 | 1.724690312819E+00 | 1.724690312819E+00 | 4.418687638008E-14 |
| 1.60 | 2.200361653808E+00 | 2.200361653808E+00 | 7.371880883511E-14 |
| 1.70 | 2.827897119229E+00 | 2.827897119229E+00 | 3.819167204711E-14 |
| 1.80 | 3.644660018535E+00 | 3.644660018535E+00 | 1.092459456231E-13 |
| 1.90 | 4.694392867379E+00 | 4.694392867379E+00 | 7.194245199571E-14 |
| 2.00 | 6.028035340213E+00 | 6.028035340213E+00 | 7.016609515631E-14 |
| 2.10 | 7.704634604225E+00 | 7.704634604225E+00 | 1.776356839400E-13 |
| 2.20 | 9.792360031030E+00 | 9.792360031030E+00 | 2.202682480856E-13 |
| 2.30 | 1.236963592580E+01 | 1.236963592580E+01 | 2.557953848736E-13 |
| 2.40 | 1.552640775055E+01 | 1.552640775055E+01 | 2.362554596402E-13 |
| 2.50 | 1.936555936940E+01 | 1.936555936940E+01 | 3.268496584496E-13 |
| 2.60 | 2.400450113199E+01 | 2.400450113199E+01 | 5.897504706809E-13 |
| 2.70 | 2.957695116165E+01 | 2.957695116165E+01 | 4.973799150321E-13 |
| 2.80 | 3.623493505710E+01 | 3.623493505710E+01 | 7.034373084025E-13 |
| 2.90 | 4.415103238129E+01 | 4.415103238129E+01 | 4.263256414561E-14 |
| 3.00 | 5.352090183528E+01 | 5.352090183528E+01 | 1.826094830903E-12 |
| 3.10 | 6.456612093892E+01 | 6.456612093892E+01 | 1.236344360223E-12 |
| 3.20 | 7.753738040788E+01 | 7.753738040788E+01 | 7.389644451905E-13 |
| 3.30 | 9.271807828043E+01 | 9.271807828043E+01 | 1.364242052659E-12 |
| 3.40 | 1.104283642623E+02 | 1.104283642623E+02 | 9.947598300641E-13 |
| 3.50 | 1.310296907886E+02 | 1.310296907886E+02 | 3.097966327914E-12 |
| 3.60 | 1.549299340189E+02 | 1.549299340189E+02 | 2.813749233610E-12 |
| 3.70 | 1.825891554638E+02 | 1.825891554637E+02 | 3.723243935383E-12 |
| 3.80 | 2.145260832800E+02 | 2.145260832800E+02 | 3.268496584496E-12 |
| 3.90 | 2.513254015633E+02 | 2.513254015633E+02 | 2.756905814749E-12 |
| 4.00 | 2.936459463250E+02 | 2.936459463250E+02 | 5.741185304942E-12 |

Table 8: Solution by the explicit sixth order numerical differentiation (ND) method with $k = 7$ and $h = 0.02$.

| X | Exact Solution | Numerical Solution by Implicit sixth order ND | Absolute Error |
|---|---|---|---|
| 0.10 | 9.500527385052E-02 | 9.500527385056E-02 | 3.923250613269E-14 |
| 0.20 | 1.801028752744E-01 | 1.801028752745E-01 | 4.976574707882E-14 |
| 0.30 | 2.556184927678E-01 | 2.556184927678E-01 | 7.982503547055E-14 |
| 0.40 | 3.222767124880E-01 | 3.222767124882E-01 | 1.831867990632E-13 |
| 0.50 | 3.813777373543E-01 | 3.813777373544E-01 | 1.329492071989E-13 |
| 0.60 | 4.349950924545E-01 | 4.349950924548E-01 | 2.604583215771E-13 |
| 0.70 | 4.861958844101E-01 | 4.861958844105E-01 | 3.225752998048E-13 |
| 0.80 | 5.392853662346E-01 | 5.392853662350E-01 | 3.457234498683E-13 |
| 0.90 | 6.000777875176E-01 | 6.000777875181E-01 | 4.938272013533E-13 |
| 1.00 | 6.761957875984E-01 | 6.761957875990E-01 | 5.414557691097E-13 |
| 1.10 | 7.774009222403E-01 | 7.774009222408E-01 | 5.597744490160E-13 |
| 1.20 | 9.159583080796E-01 | 9.159583080804E-01 | 7.263079027098E-13 |
| 1.30 | 1.107038830160E+00 | 1.107038830161E+00 | 7.471800955727E-13 |
| 1.40 | 1.369162893048E+00 | 1.369162893048E+00 | 8.899547765395E-13 |
| 1.50 | 1.724690312819E+00 | 1.724690312820E+00 | 1.000977079002E-12 |
| 1.60 | 2.200361653808E+00 | 2.200361653809E+00 | 1.135980198796E-12 |
| 1.70 | 2.827897119229E+00 | 2.827897119230E+00 | 1.322053577724E-12 |
| 1.80 | 3.644660018535E+00 | 3.644660018537E+00 | 1.348254841105E-12 |
| 1.90 | 4.694392867379E+00 | 4.694392867381E+00 | 1.650235503803E-12 |
| 2.00 | 6.028035340213E+00 | 6.028035340215E+00 | 1.785238623597E-12 |
| 2.10 | 7.704634604225E+00 | 7.704634604227E+00 | 1.973532448574E-12 |
| 2.20 | 9.792360031030E+00 | 9.792360031032E+00 | 2.147615418835E-12 |
| 2.30 | 1.236963592580E+01 | 1.236963592580E+01 | 2.630784479152E-12 |
| 2.40 | 1.552640775055E+01 | 1.552640775055E+01 | 2.682298827494E-12 |
| 2.50 | 1.936555936940E+01 | 1.936555936940E+01 | 3.016253913302E-12 |
| 2.60 | 2.400450113199E+01 | 2.400450113199E+01 | 3.371525281182E-12 |
| 2.70 | 2.957695116165E+01 | 2.957695116165E+01 | 3.794298208959E-12 |
| 2.80 | 3.623493505710E+01 | 3.623493505711E+01 | 4.689582056017E-12 |
| 2.90 | 4.415103238129E+01 | 4.415103238129E+01 | 4.554578936222E-12 |
| 3.00 | 5.352090183528E+01 | 5.352090183528E+01 | 5.400124791777E-12 |
| 3.10 | 6.456612093892E+01 | 6.456612093893E+01 | 6.593836587854E-12 |
| 3.20 | 7.753738040788E+01 | 7.753738040789E+01 | 6.480149750132E-12 |
| 3.30 | 9.271807828043E+01 | 9.271807828044E+01 | 8.171241461241E-12 |
| 3.40 | 1.104283642623E+02 | 1.104283642623E+02 | 8.242295734817E-12 |
| 3.50 | 1.310296907886E+02 | 1.310296907886E+02 | 9.009681889438E-12 |
| 3.60 | 1.549299340189E+02 | 1.549299340189E+02 | 9.492850949755E-12 |
| 3.70 | 1.825891554638E+02 | 1.825891554638E+02 | 1.145394890045E-11 |
| 3.80 | 2.145260832800E+02 | 2.145260832800E+02 | 1.548983163957E-11 |
| 3.90 | 2.513254015633E+02 | 2.513254015633E+02 | 1.384137249261E-11 |
| 4.00 | 2.936459463250E+02 | 2.936459463251E+02 | 1.608668753761E-11 |

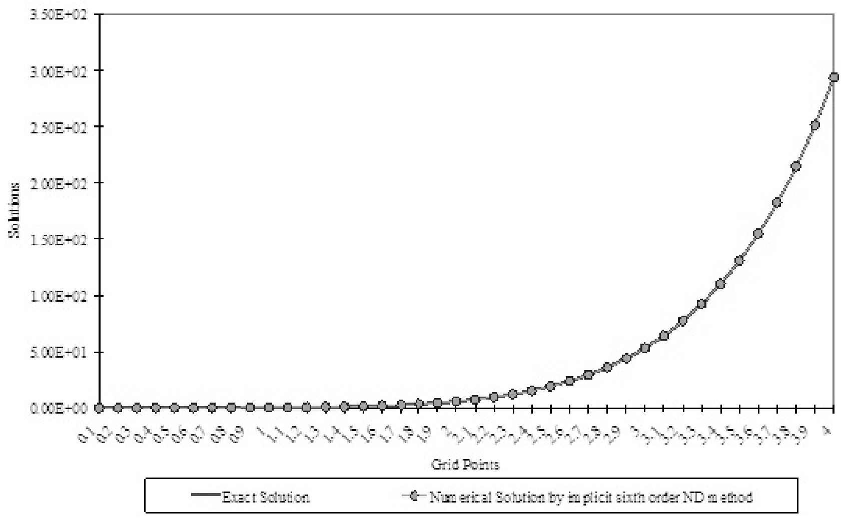


Figure 5: Comparison of Exact Solution and Solution by implicit sixth order ND for the example 1 with $k = 7$ and $h = 0.01$.

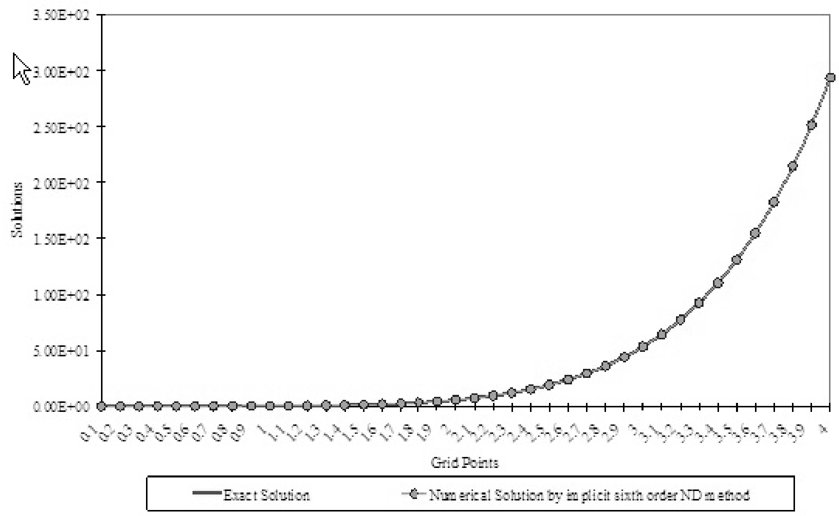


Figure 6: Comparison of Exact Solution and Solution by implicit sixth order ND for the example 1 with k = 7 and h=0.02.

Table 9: Solution by the implicit eighth order numerical differentiation (ND) method with $k = 9$ and $h = 0.01$.

| X | Exact Solution | Numerical Solution by Implicit eighth order ND | Absolute Error |
|---|---|---|---|
| 0.00 | 1.000000000000E+00 | 9.999999999902E-01 | 9.761524921714E-12 |
| 0.10 | 1.105170918076E+00 | 1.105170918065E+00 | 1.078048761372E-11 |
| 0.20 | 1.221402758160E+00 | 1.221402758148E+00 | 1.194422338813E-11 |
| 0.30 | 1.349858807576E+00 | 1.349858807563E+00 | 1.318478659584E-11 |
| 0.40 | 1.491824697641E+00 | 1.491824697627E+00 | 1.455036091613E-11 |
| 0.50 | 1.648721270700E+00 | 1.648721270684E+00 | 1.610067634772E-11 |
| 0.60 | 1.822118800391E+00 | 1.822118800373E+00 | 1.781064185025E-11 |
| 0.70 | 2.013752707470E+00 | 2.013752707451E+00 | 1.968025742372E-11 |
| 0.80 | 2.225540928492E+00 | 2.225540928471E+00 | 2.169775470406E-11 |
| 0.90 | 2.459603111157E+00 | 2.459603111133E+00 | 2.401323584422E-11 |
| 1.00 | 2.718281828459E+00 | 2.718281828432E+00 | 2.655120567852E-11 |

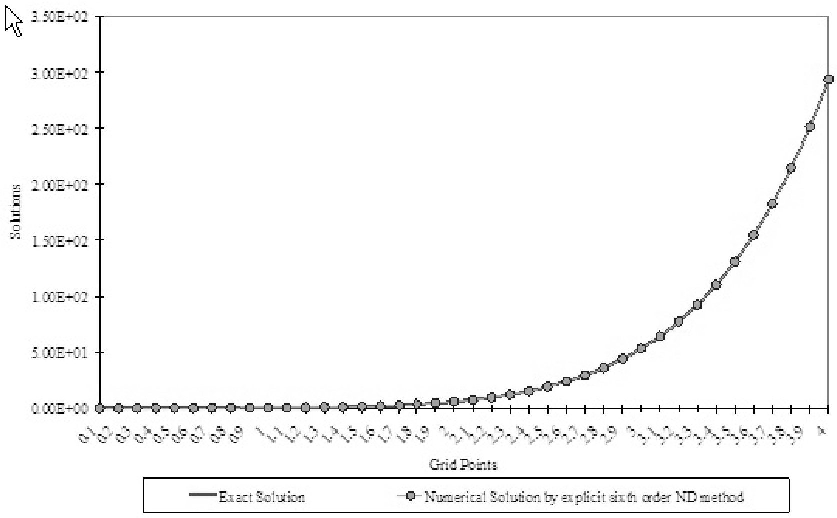


Figure 7: Comparison of Exact Solution and Solution by explicit sixth order ND for the example 1 with $k = 7$ and $h = 0.01$.

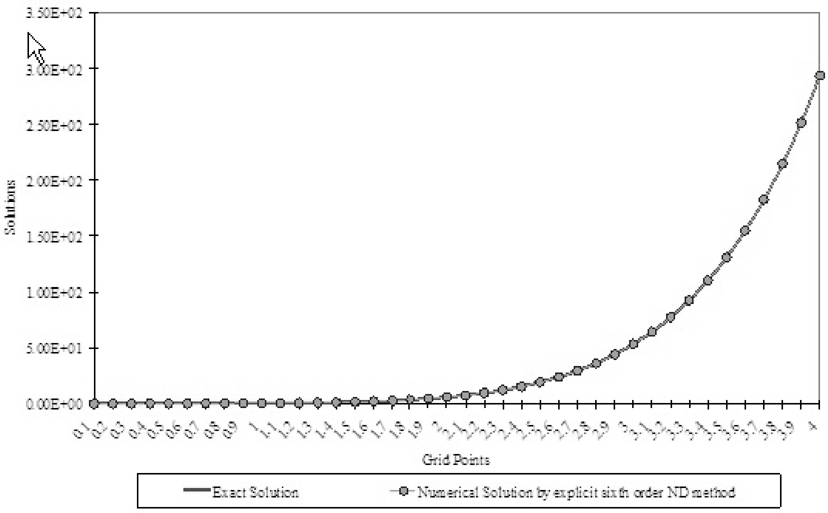


Figure 8: Comparison of Exact Solution and Solution by explicit sixth order ND for the example1 with $k = 7$ and $h = 0.02$.

Table 10: Solution by the implicit eighth order numerical differentiation (ND) method with $k = 9$ and $h = 0.02$.

| X | Exact Solution | Numerical Solution by Implicit eighth order ND | Absolute Error |
|---|---|---|---|
| 0.00 | 1.000000000000E+00 | 9.999999996874E-01 | 3.125977254825E-10 |
| 0.10 | 1.105170918076E+00 | 1.105170917730E+00 | 3.454783126244E-10 |
| 0.20 | 1.221402758160E+00 | 1.221402757778E+00 | 3.818036997671E-10 |
| 0.30 | 1.349858807576E+00 | 1.349858807154E+00 | 4.219560256757E-10 |
| 0.40 | 1.491824697641E+00 | 1.491824697175E+00 | 4.663454067355E-10 |
| 0.50 | 1.648721270700E+00 | 1.648721270185E+00 | 5.153748539044E-10 |
| 0.60 | 1.822118800391E+00 | 1.822118799821E+00 | 5.695948157580E-10 |
| 0.70 | 2.013752707470E+00 | 2.013752706841E+00 | 6.294742505020E-10 |
| 0.80 | 2.225540928492E+00 | 2.225540927797E+00 | 6.956777376388E-10 |
| 0.90 | 2.459603111157E+00 | 2.459603110388E+00 | 7.688560899055E-10 |
| 1.00 | 2.718281828459E+00 | 2.718281827609E+00 | 8.497491599258E-10 |

Table 11: Solution by the explicit eighth order numerical differentiation (ND) method with $k = 94$ and $h = 0.01$.

| X | Exact Solution | Numerical Solution by Implicit eighth order ND | Absolute Error |
|---|---|---|---|
| 0.00 | 1.000000000000E+00 | 1.000000000000E+00 | 1.532107773983E-14 |
| 0.10 | 1.105170918076E+00 | 1.105170918076E+00 | 1.953992523340E-14 |
| 0.20 | 1.221402758160E+00 | 1.221402758160E+00 | 8.215650382226E-15 |
| 0.30 | 1.349858807576E+00 | 1.349858807576E+00 | 7.993605777301E-15 |
| 0.40 | 1.491824697641E+00 | 1.491824697641E+00 | 3.330669073875E-14 |
| 0.50 | 1.648721270700E+00 | 1.648721270700E+00 | 1.731947918415E-14 |
| 0.60 | 1.822118800391E+00 | 1.822118800391E+00 | 7.549516567451E-15 |
| 0.70 | 2.013752707470E+00 | 2.013752707471E+00 | 2.753353101070E-14 |
| 0.80 | 2.225540928492E+00 | 2.225540928493E+00 | 7.727152251391E-14 |
| 0.90 | 2.459603111157E+00 | 2.459603111157E+00 | 3.863576125696E-14 |
| 1.00 | 2.718281828459E+00 | 2.718281828459E+00 | 1.154631945610E-14 |

Table 12: Solution by the explicit eighth order numerical differentiation (ND) method with $k = 9$ and $h = 0.02$.

| X | Exact Solution | Numerical Solution by Implicit eighth order ND | Absolute Error |
|---|---|---|---|
| 0.00 | 1.000000000000E+00 | 1.000000000001E+00 | 6.246114736541E-13 |
| 0.10 | 1.105170918076E+00 | 1.105170918076E+00 | 6.870060076380E-13 |
| 0.20 | 1.221402758160E+00 | 1.221402758161E+00 | 7.582823258190E-13 |
| 0.30 | 1.349858807576E+00 | 1.349858807577E+00 | 8.428813202954E-13 |
| 0.40 | 1.491824697641E+00 | 1.491824697642E+00 | 9.272582701669E-13 |
| 0.50 | 1.648721270700E+00 | 1.648721270701E+00 | 1.021183138050E-12 |
| 0.60 | 1.822118800391E+00 | 1.822118800392E+00 | 1.127098414599E-12 |
| 0.70 | 2.013752707470E+00 | 2.013752707472E+00 | 1.239897073901E-12 |
| 0.80 | 2.225540928492E+00 | 2.225540928494E+00 | 1.388222869991E-12 |
| 0.90 | 2.459603111157E+00 | 2.459603111158E+00 | 1.544986361068E-12 |
| 1.00 | 2.718281828459E+00 | 2.718281828461E+00 | 1.663114090888E-12 |

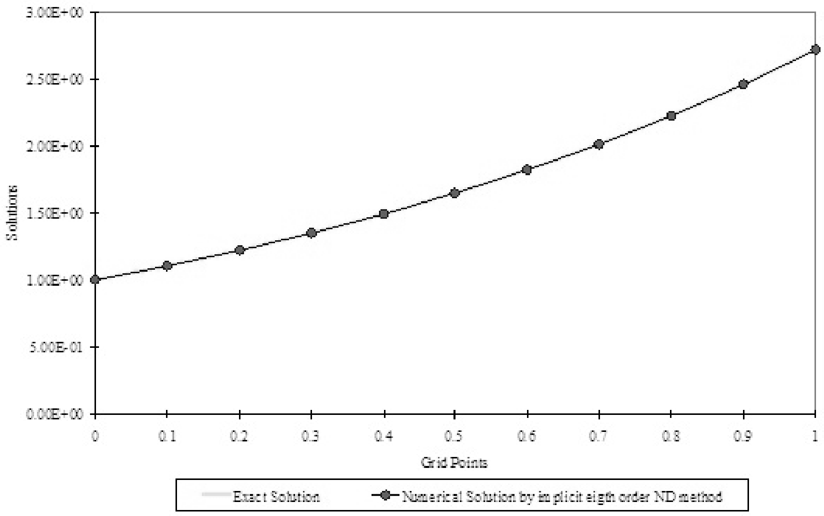


Figure 9: Comparison of Exact Solution and Solution by implicit eighth order ND method for the example 2 with $k = 9$ and $h = 0.01$.

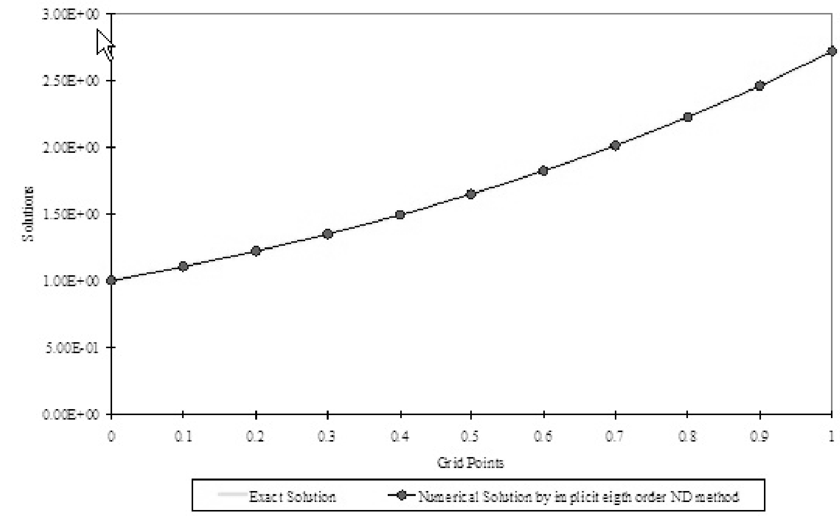


Figure 10: Comparison of Exact Solution and Solution by implicit eighth order ND method for the example 2 with $k = 9$ and $h = 0.02$.

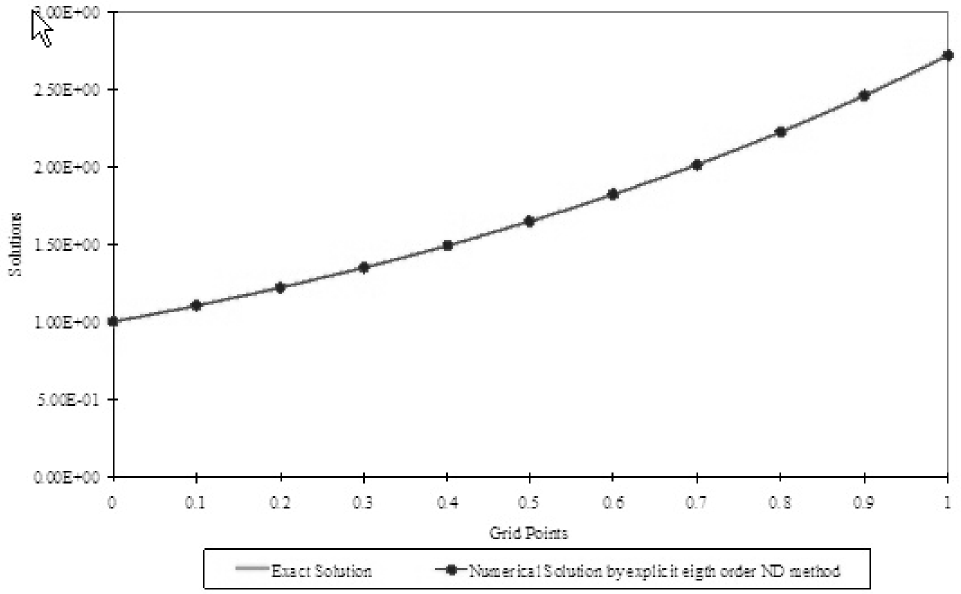


Figure 11: Comparison of Exact Solution and Solution by explicit eighth order ND method for the example 2 with $k = 9$ and $h = 0.01$.

**Example 1**

Consider the fifth order boundary value problem

$$y^{(5)} - y^{'} = 12e^{x} + 7\sin x - 2x, \tag{27}$$

subject to the boundary conditions $y(0) = 0, y^{'}(0) = 1, y^{''}(0) = -1, y^{'''}(0) = 0, y^{iv}(0) = 1$ in the interval [0-4] with $h = 0.01$ and $0.02$.

We have applied the ND methods to solve the above differential equation and the results, i.e., exact solution, the numerical solution and absolute errors at the grid points are presented in the Tables 5 to 8. Also the comparison of exact and approximate solution has been shown graphically in the Figs. 5 to 8.

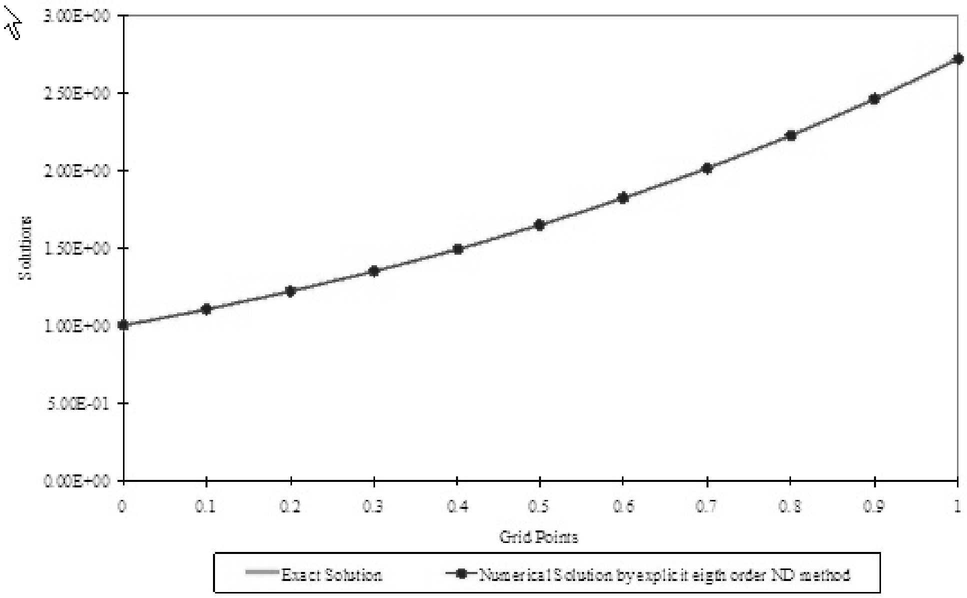


Figure 12: Comparison of Exact Solution and Solution by explicit eighth order ND method for the example 2 with $k = 9$ and $h = 0.02$.

**Example 2**

Consider the fifth order nonlinear boundary value problem $y^{(5)} = e^{-x}y^2$, $0 < x < 1$ subject to the boundary conditions

$$y(0) = 0, y^{'}(0) = 1, y^{''}(0) = -1, y^{'''}(0) = 0, y^{iv}(0) = 1. \tag{28}$$

The solution for the above problem is $y(x) = e^x$.

We have applied the ND methods to solve the above differential equation and the results are tabulated. The exact solution, numerical solution, and absolute errors at the grid points are presented in the Tables 9 to 12. Also the comparison of exact and approximate solution have been shown graphically in the Figs. 9 to 12.

# 8 Discussion and conclusion

In this paper, we have discussed the regions of absolute stability of fifth order boundary value problems. The methods derived in this paper are applied to solve a fifth-order linear and nonlinear boundary value problems, hence establishing the superiority of the methods. The methods based on numerical differentiation are found to be absolutely stable outside some closed boundaries. Numerical results are given to illustrate the efficiency of our methods and compared with exact solution. The absolute errors are very small. Also it is evident that the numerical methods based on the approach we used will require considerably less computational effort.

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