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The Regularity of Lax-Oleinik Operator

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Abstract: Suppose $L : TM \to \mathbb{R}$ is a generalized Tonelli Lagrangian. For any $t > 0$, given $x, y \in M$, let $A_t(x, y)$ be the fundamental solution of $L(x, v)$, and for any contunious function $u : M \to \mathbb{R}$, we can define the Lax-Oleinik operator as follow: $T_t^- u(x) = \inf_y \{u(y) + A_t(y, x)\}$. Lax-Oleinik operator can be seen as a generalized Lasry-Lions approximation using the kernel determined by the fundamental solution with respect to Tonelli Lagrangian. This operator can be also derived from the Moreau-Yosida approximations in convex analysis. In weak KAM (Kolmogorov-Arnold-Moser) theory, using Lax-Oleinik operator, the viscosity solution of associate Hamilton-Jacobi equation can be obtained as the fixed point of semi-group $\{T_t^-\}$. Moreover, for propagation of singularities, through the analysis of Lax-Oleinik operator, an intrinsic proof of the existence of generalized characteristic has been given. In this paper, we list the regularities of fundamental solution at first, and then, using those regularities, we show the connection between the regularity of fundamental solution and of Lax-Oleinik operator. Furthermore, we obtained the $C^{1,1}$ regularity of Lax-Oleinik operator.

Keywords: Lasry-Lions regularization; Lax-Oleinik operator; Hamilton-Jacobi equations; Weak KAM theory

1 Introduction

In the last two or three decades, remarkable progresses in Hamiltonian systems were achieved by Mather's theory in Lagrangian formalism [1, 2], and Fathi's Weak KAM theory in Hamiltonian formalism [3, 4]. Both these theories succeeded in the analysis of some hard dynamical problems such as Arnold diffusion. The Weak KAM theory connect the viscosity solution of Hamilton-Jacobi equation with action function of the associated Lagrangian and the Lax-Oleinik operators are important tools in it. For the propagation of singularities, through the analysis of Lax-Oleinik operator, an intrinsic proof of the existence of generalized characteristic has been given [5].

In [6] the relations among Lasry-Lions regularization, Lax-Oleinik operators (or inf/sup-convolution) and generalized characteristics have been exploited. Lax-Oleinik operator can be seen as a generalized Lasry-Lions approximation using the kernel determined by the fundamental solution with respect to Tonelli Lagrangian [7]. In the present work, we want to show the relation between the regularity of fundamental solution and its of Lax-Oleinik operators.

This paper was organized as follows: In section 2, we briefly review some necessary definitions. In section 3, some properties and conclusions of the fundamental solution have been listed without detail proof. In section 4, we show the connection between the regularity of fundamental solution and of Lax-Oleinik operator and then we obtain the locally $C^{1,1}$ regularity of $T_t^- u(x)$.

2 Preliminaries and definitions

2.1 Tonelli lagrangian

Throughout this paper, we consider the generalized Tonelli Lagrangian.

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Definition 1 *A function* $L : TM \to \mathbb{R}$ *is called a* generalized Tonelli Lagrangian if L is a function of class C^2 that satisfies the following conditions:

(L1) **Uniform convexity:** *There exists a nonincreasing function* $\nu : [0, +\infty) \to (0, +\infty)$ *such that*

$$L_{vv}(x, v) \geqslant \nu(|v|)I$$

for all $(x, v) \in \mathbb{R}^n \times \mathbb{R}^n$.

(L2) **Growth condition:** *There exist two superlinear functions* $\theta, \bar{\theta} : [0, +\infty) \to [0, +\infty)$ *and a constant* $c_0 > 0$ *such that*

$$\bar{\theta}(|v|) \geqslant L(x, v) \geqslant \theta(|v|) - c_0 \quad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

(L3) **Uniform regularity:** *There exists a nondecrease function* $K : [0, +\infty) \to [0, +\infty)$ *such that for every multi-index* $|\alpha| = 1, 2$,

$$|D^\alpha L(x, v)| \leqslant K(|v|) \quad \forall (x, v) \in \mathbb{R}^n \times \mathbb{R}^n.$$

If L is a generalized Tonelli Lagrangian, the associated Hamiltonian H is the Fenchel-Legendre dual of L defined by

$$H(x, p) = \sup_{v \in \mathbb{R}^n} \{< p, v > -L(x, v)\} \quad (x, p) \in \mathbb{R}^n \times \mathbb{R}^n. \tag{1}$$

The Hamiltonian H is called a *generalized Tonelli Hamiltonian* if L is a generalized Tonelli Lagrangian

The convex conjugate of a superlinear function θ is defined as

$$\theta^*(s) = \sup_{r \geqslant 0} \{rs - \theta(r)\} \quad \forall s \geqslant 0. \tag{2}$$

In view of the superlinear growth of θ, it is clear that θ^* is well defined and satisfies

$$\theta(r) + \theta^*(s) \geqslant rs \quad \forall r, s \geqslant 0. \tag{3}$$

2.2 Lax-Oleinik operators

For any $t > 0$, given $x, y \in M$, we consider the class of arcs

$$\Gamma^t_{x,y} = \{\xi \in W^{1,1}([0, t]; \mathbb{R}^n) : \xi(0) = x, \xi(t) = y\},$$

where $W^{1,1}([a, b]; \mathbb{R}^n)$ denotes the space of all absolutely continuous $\mathbb{R}^n$-valued functions on $[a, b]$, where $-\infty < a < b < +\infty$. We define

$$A_t(x, y) = \min_{\xi \in \Gamma^t_{x,y}} \int_0^t L(\xi(s), \dot{\xi}(s)) ds.$$

In the PDE literature, $A_t(x, y)$ is called the *fundamental solution* of the equation

$$H(x, Du(x)) = 0.$$

For any $t > 0$ and any function $u : M \to [-\infty, +\infty]$, the *Lax-Oleinik operators* $T^\pm_t u : M \to [-\infty, +\infty]$ are defined as follow:

$$T^+_t u(x) = \sup_{y \in M} \{u(y) - A_t(x, y)\}, \quad x \in M, \tag{4}$$

$$T^-_t u(x) = \inf_{y \in M} \{u(y) + A_t(y, x)\}, \quad x \in M. \tag{5}$$

3 Regularity of the fundamental solution $A_t(x,y)$

Lemma 1 *Suppose L is a generalized Tonelli Lagrangian satisfied (L1)-(L3) and u be a Lipschitz function on $\mathbb{R}^n$. Then the infimum in (5) is attained for every $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$. Moreover, there exists a constant $\lambda > 0$, depending only on $Lip(u)$, such that, for any $(t,x) \in \mathbb{R}_+ \times \mathbb{R}^n$ and any minimum point $y_{t,x}$, we have*

$$|y_{t,x} - x| \leqslant \lambda t. \tag{6}$$

This solution has been proved in [8] for the case of $T_t^+ u(x)$, now we prove the case of $T_t^- u(x)$ similarly.
Proof. Let $\psi_t^x(y) = u(y) + A_t(y,x)$, $k_u = \mathrm{Lip}(u) + 1$. Then,

$$\begin{aligned} A_t(y,x) \geqslant & \inf_{\xi \in \Gamma_{y,x}^t} \int_0^t \theta(|\dot{\xi}(s)|)ds - c_0 t \\ \geqslant & \inf_{\xi \in \Gamma_{y,x}^t} k_u \int_0^t |\dot{\xi}(s)|ds - (\theta^*(k_u) + c_0)t \\ \geqslant & k_u|x-y| - (\theta^*(k_u) + c_0)t, \end{aligned}$$

where c_0 is the constant in assumption (L2). Therefore

$$\begin{aligned} \psi_t^x(y) - \psi_t^x(x) =& u(y) - u(x) + A_t(y,x) - A_t(x,x) \\ \geqslant & -\mathrm{Lip}(u)|x-y| + k_u|x-y| - (\theta^*(k_u) + c_0)t - tL(x,0) \\ \geqslant & |x-y| - (\theta^*(k_u) + c_0 + \bar{\theta}(0))t. \end{aligned}$$

Now, taking $\lambda = \theta^*(k_u) + c_0 + \bar{\theta}(0)$, it follows that

$$\Lambda_t^x := \{y : \psi_t^x(y) \leqslant \psi_t^x(x)\} \subset \overline{B}(x, \lambda t). \tag{7}$$

So, Λ_t^x is compact and the infimum in $T_t^- u(x)$ is indeed a minimum, moreover (6) is a consequence of (7). ■

The semiconcavity and convexity of the fundamental solution $A_t(x,y)$ has been investigated in [8]. We just state the conclusions and the details of proof can be seen in [8].

Proposition 2 *Suppose L is a Tonelli Lagrangian. Then for any $\lambda > 0$ there exists a constant $C_\lambda > 0$ such that for any $x \in \mathbb{R}^n$, $t \in (0, 2/3)$, $y \in B(x, \lambda t)$, and $(h,z) \in \mathbb{R} \times \mathbb{R}^n$ satisfying $|h| < t/2$ and $|z| < \lambda t$, we have*

$$A_{t+h}(x, y+z) + A_{t-h}(x, y-z) - 2A_t(x,y) \leqslant \frac{C_\lambda}{t}(|h|^2 + |z|^2). \tag{8}$$

Proposition 3 *Suppose L is a Tonelli Lagrangian satisfying (L1)-(L3). Then the following properties are true.*

(a) For any $\lambda > 0$, there exists $t_\lambda > 0$ such that, for any $x \in \mathbb{R}^n$, the function $(t,y) \to A_t(x,y)$ is semi-convex on the cone

$$S_\lambda(x, t_\lambda) = \{(t,y) \in \mathbb{R} \times \mathbb{R}^n : 0 < t < t_\lambda, |y - x| < \lambda t\}. \tag{9}$$

that is, there exists $C > 0$ such that for all $x \in \mathbb{R}^n$, all $(t,y) \in S_\lambda(x, t_\lambda)$, all $h \in [0, t/2)$, and all $z \in B(0, \lambda t)$ we have that

$$A_{t+h}(x, y+z) + A_{t-h}(x, y-z) - 2A_t(x,y) \geqslant -\frac{C}{t}(h^2 + |z|^2).$$

(b) For all $t \in (0, t_\lambda]$, $A_t(x,y)$ is uniformly convex on $B(x, \lambda t)$, that is, there exists a constant $C' > 0$ such that for all $x \in \mathbb{R}^n$, all $y \in B(x, \lambda t)$, and all $z \in B(0, \lambda t)$, we have

$$A_t(x, y+z) + A_t(x, y-z) - 2A_t(x,y) \geqslant \frac{C'}{t}|z|^2.$$

Combining proposition 2 and 3, it is easy to obtain the following solution.

Proposition 4 *Suppose L is a Tonelli Lagrangian, for any $\lambda > 0$, let t_λ be defined as proposition 3. Then, for any $x \in \mathbb{R}^n$, the function $(t,y) \to A_t(x,y)$ and $(t,y) \to A_t(y,x)$ are of class $C_{loc}^{1,1}$ on the cone $S_\lambda(x, t_\lambda)$ defined above (9).*

4 The regularity of the Lax-Oleinik operator

Now, using the regularity of fundamental solution $A_t(x,y)$, we discuss the regularity of the Lax-Oleinik operator.

4.1 Semi-concavity of Lax-Oleinik operator

Proposition 5 *Let $t>0$ and $u: M \to \mathbb{R}$ be a continuous function, then the function, defined with Lax-Oleinik operator, $T_t^- u : M \to \mathbb{R}$ is a semi-concave function.*

Proof. For any $x_1, x_2 \in M$, let y_1, y_2 be the respective minimum in

$$T_t^- u(x) = \inf_{y \in M} \{u(y) + A_t(y,x)\}.$$

For convenience, let $x_0 = \frac{x_1+x_2}{2}$ and y_0 be the corresponding minimum in $T_t^- u(x_0)$, then

$$\begin{aligned} & T_t^- u(x_1) + T_t^- u(x_2) - 2T_t^- u(x_0) \\ =& T_t^- u(x_1) + T_t^- u(x_2) - 2u(y_0) - 2A_t(y_0,x_0) \\ \leqslant & u(y_0) + A_t(y_0,x_1) + u(y_0) + A_t(y_0,x_2) \\ & - 2u(y_0) - 2A_t(y_0,x_0) \\ =& A_t(y_0,x_1) + A_t(y_0,x_2) - 2A_t(y_0,x_0). \end{aligned}$$

The semi-concavity of $A_t(x,y)$ lead to the semi-concavity of $T_t^- u(x)$. ■

4.2 Semi-convexity of Lax-Oleinik operator

The semi-convexity of $T_t^- u(x)$ is more complicate than its semi-concavity. Assuming y_i be the minimizer of $T_t^- u(x_i)$, $i=1,2$. For convenience, let $x_0 = \frac{x_1+x_2}{2}$ and $y_0 = \frac{y_1+y_2}{2}$, then

$$\begin{aligned} & T_t^- u(x_1) + T_t^- u(x_2) - 2T_t^- u(x_0) \\ \geqslant & u(y_1) + A_t(y_1,x_1) + u(y_2) + A_t(y_2,x_2) \\ & - 2u(y_0) - 2A_t(y_0,x_0) \\ =& u(y_1) + u(y_2) - 2u(y_0) + A_t(y_1,x_1) + A_t(y_2,x_2) \\ & - 2A_t(y_0,x_0). \end{aligned}$$

Let ξ_i be the minimizer of $A_t(y_i,x_i)$, $\xi_i(0) = y_i$ and $\xi_i(t) = x_i$, $i=1,2$. Let $\xi_0 = \frac{\xi_1+\xi_2}{2}$, then $\xi_0(0) = y_0$ and $\xi_0(t) = x_0$. For the purpose of the semi-convexity, we need the following important lemma.

Lemma 6 *Suppose L is a Tonelli Lagrangian, ξ_i is the minimizer of $A_t(y_i,x_i)(i=1,2)$, then there exists $t_\lambda > 0$ and constants $C, C', C'' > 0$, such that, for all $t \in (0, t_\lambda)$, we have*

$$\|\xi_2 - \xi_1\|^2_{L^\infty(0,t)} \leqslant \frac{C}{t}\left(|y_2-y_1|^2 + |x_2-x_1|^2\right), \tag{10}$$

$$\int_0^t |p_2 - p_1|^2 ds \leqslant \frac{C'}{t}\left(|y_2-y_1|^2 + |x_2-x_1|^2\right), \tag{11}$$

$$\int_0^t |\dot\xi_2 - \dot\xi_1|^2 ds \leqslant \frac{C''}{t}\left(|y_2-y_1|^2 + |x_2-x_1|^2\right). \tag{12}$$

Proof. Since L is a Tonelli Lagrangian, we have $\xi_i(s)(i=1,2)$ are of class C^2. Let $p_i(s) = L_v(\xi_i(s), \dot\xi_i(s))$, then the pair $(\xi_i(s), p_i(s))$ satisfies the Hamiltonian system

$$\begin{cases} \dot\xi_i = H_p(\xi_i, p_i) \\ \dot p_i = -H_x(\xi_i, p_i) \end{cases} \quad s \in [0,t]$$

with $\xi_i(0) = y_i, \xi_i(t) = x_i$. Therefore

$$\frac{1}{2}\frac{d}{ds}|\xi_2 - \xi_1|^2 =< H_p(\xi_2, p_2) - H_p(\xi_1, p_1), \xi_2 - \xi_1 > .$$

Integrating over $[s, t]$, we conclude that

$$\begin{aligned}|x_2 - x_1|^2 - |\xi_2(s) - \xi_1(s)|^2 &\geqslant - C_1 \int_s^t (|\xi_2 - \xi_1|^2 + |p_2 - p_1| \cdot |\xi_2 - \xi_1|)d\tau \\ &\geqslant - C_1 \int_s^t |p_2 - p_1|^2 d\tau - 2C_1 \int_s^t |\xi_2 - \xi_1|^2 d\tau,\end{aligned}$$

where $C_1 > 0$ is an upper bound for $D^2H(x, p)$ on $\{(x, p) : |p| \leqslant \kappa(4\lambda)\}$. Then

$$\|\xi_2 - \xi_1\|^2_{L^\infty(0,t)} \leqslant |x_2 - x_1|^2 + C_1 \int_s^t |p_2 - p_1|^2 d\tau + 2C_1 t\|\xi_2 - \xi_1\|^2_{L^\infty(0,t)}. \tag{13}$$

On the other hand,

$$\begin{aligned}&\frac{d}{ds} < p_2 - p_1, \xi_2 - \xi_1 > \\ &=< p_2 - p_1, H_p(\xi_2, p_2) - H_p(\xi_1, p_1) > - < H_x(\xi_2, p_2) - H_x(\xi_1, p_1), \xi_2 - \xi_1 > .\end{aligned}$$

If we set

$$\widehat{H}_{px}(s) = \int_0^1 H_{px}(\lambda\xi_2(s) + (1 - \lambda)\xi_1(s), \lambda p_2(s) + (1 - \lambda)p_1(s))d\lambda,$$

and $\widehat{H}_{pp}$, $\widehat{H}_{xp}$, $\widehat{H}_{xx}$ are defined in a similar way. Then there exist positive constants ν such that

$$\frac{d}{ds} < p_2 - p_1, \xi_2 - \xi_1 >\geqslant \nu|p_2 - p_1|^2 - C_2|\xi_2 - \xi_1|^2. \tag{14}$$

Now, integrating over $[0, t]$, we have

$$\begin{aligned}\nu \int_0^t |p_2 - p_1|^2 ds \leqslant &C_2 \int_0^t |\xi_2 - \xi_1|^2 ds+ < p_2(t) - p_1(t), \xi_2(t) - \xi_1(t) > \\ &- < p_2(0) - p_1(0), \xi_2(0) - \xi_1(0) > .\end{aligned}$$

Since $p_i(s) \in D^+A_t(x_i, \xi_i(s))$ for $s \in [0, t]$, following the semi-concavity of $A_t(x, y)$, we have

$$\begin{aligned}\nu \int_0^t |p_2 - p_1|^2 ds \leqslant &C_2 \int_0^t |\xi_2 - \xi_1|^2 ds + \frac{C_3}{t}|y_2 - y_1|^2 \\ &+ \frac{C_3}{t}|x_2 - x_1|^2.\end{aligned}$$

So,

$$\int_0^t |p_2 - p_1|^2 ds \leqslant \frac{C_2 t}{\nu}\|\xi_2 - \xi_1\|^2_{L^\infty(0,t)} + \frac{C_3}{\nu t}|y_2 - y_1|^2 + \frac{C_3}{\nu t}|x_2 - x_1|^2. \tag{15}$$

Combining (13) and (15), we obtain

$$\|\xi_2 - \xi_1\|^2_{L^\infty(0,t)} \leqslant (\frac{C_2}{\nu} + 2)C_1 t\|\xi_2 - \xi_1\|^2_{L^\infty(0,t)} + (1 + \frac{C_3}{\nu t})|x_2 - x_1|^2 + \frac{C_3}{\nu t}|y_2 - y_1|^2,$$

taking

$$t_\lambda = \min\{1, \frac{\nu}{2C_1(C_2 + 2\nu)}\},$$

we conclude that, for all $t \in (0, t_\lambda)$, there exist a constant C such that

$$\|\xi_2 - \xi_1\|^2_{L^\infty(0,t)} \leqslant \frac{C}{t}|x_2 - x_1|^2 + \frac{C}{t}|y_2 - y_1|^2.$$

Therefore, owing to (15), there exists a constant C' such that

$$\int_0^t |p_2 - p_1|^2 ds \leqslant \frac{C'}{t}|x_2 - x_1|^2 + \frac{C'}{t}|y_2 - y_1|^2.$$

Now, we have proved (10) and (11), finally we will obtain (12) by appealing (10) and (11). In fact,

$$\begin{aligned}
&\int_0^t |\dot\xi_2 - \dot\xi_1|^2 ds = \int_0^t |H_p(\xi_2, p_2) - H_p(\xi_1, p_1)|^2 ds \\
\leqslant& 2\int_0^t |H_p(\xi_2, p_2) - H_p(\xi_2, p_1)|^2 ds + 2\int_0^t |H_p(\xi_2, p_1) - H_p(\xi_1, p_1)|^2 ds \\
\leqslant& 2C_4\left(\int_0^t |p_2 - p_1|^2 ds + \int_0^t |\xi_2 - \xi_1|^2 ds\right).
\end{aligned}$$

So, we obtained (12). ■

Under the conclusion of lemma 6, now we discuss the semiconvexity of $A_t(\cdot,\cdot)$.

$$\begin{aligned}
&A_t(y_1, x_1) + A_t(y_2, x_2) - 2A_t(y_0, x_0) \\
=&\int_0^t L(\xi_1, \dot\xi_1) + L(\xi_2, \dot\xi_2) ds - 2A_t(x_0, y_0) \\
\geqslant&\int_0^t L(\xi_1, \dot\xi_1) + L(\xi_2, \dot\xi_2) - 2L(\frac{\xi_1 + \xi_2}{2}, \frac{\dot\xi_1 + \dot\xi_2}{2}) ds \\
=&\int_0^t L(\xi_1, \dot\xi_1) - L(\xi_0, \dot\xi_1) ds + \int_0^t L(\xi_2, \dot\xi_2) - L(\xi_0, \dot\xi_2) ds \\
&+\int_0^t L(\xi_0, \dot\xi_1) + L(\xi_0, \dot\xi_2) - 2L(\xi_0, \dot\xi_0) ds.
\end{aligned}$$

Let

$$\begin{aligned}
\mathrm{I} &= \int_0^t L(\xi_1, \dot\xi_1) - L(\frac{\xi_1 + \xi_2}{2}, \dot\xi_1) ds, \\
\mathrm{II} &= \int_0^t L(\xi_2, \dot\xi_2) - L(\frac{\xi_1 + \xi_2}{2}, \dot\xi_2) ds, \\
\mathrm{III} &= \int_0^t L(\frac{\xi_1 + \xi_2}{2}, \dot\xi_1) + L(\frac{\xi_1 + \xi_2}{2}, \dot\xi_2) - 2L(\frac{\xi_1 + \xi_2}{2}, \frac{\dot\xi_1 + \dot\xi_2}{2}),
\end{aligned}$$

then

$$I = \int_0^t \int_0^1 < L_x(\lambda\xi_1 + (1-\lambda)\frac{\xi_1 + \xi_2}{2}, \dot\xi_1), \frac{\xi_1 - \xi_2}{2} > d\lambda ds,$$

$$II = \int_0^t \int_0^1 < L_x(\lambda\xi_2 + (1-\lambda)\frac{\xi_1 + \xi_2}{2}, \dot\xi_2), \frac{\xi_2 - \xi_1}{2} > d\lambda ds.$$

Let

$$\hat L_x(\lambda, s) = L_x(\lambda\xi_1 + (1-\lambda)\frac{\xi_1 + \xi_2}{2}, \dot\xi_1) - L_x(\lambda\xi_2 + (1-\lambda)\frac{\xi_1 + \xi_2}{2}, \dot\xi_2),$$

then

$$\begin{aligned}
I + II &= \int_0^t \int_0^1 < \hat L_x(\lambda, s), \frac{\xi_1 - \xi_2}{2} > d\lambda ds \\
&\geqslant -C_1 \int_0^t (|\xi_1 - \xi_2|^2 + |\xi_1 - \xi_2| \cdot |\dot\xi_1 - \dot\xi_2|) ds,
\end{aligned}$$

$$III \geqslant C_2 \int_0^t |\dot{\xi}_1 - \dot{\xi}_2|^2 ds.$$

Following lemma 6, by combining (10) and (12), we have

$$\begin{aligned} & I + II + III = A_t(y_1, x_1) + A_t(y_2, x_2) - 2A_t(y_0, x_0) \\ \geqslant & -\frac{3C_1 \cdot C}{2t}(|y_2 - y_1|^2 + |x_2 - x_1|^2) - \frac{C_1 \cdot C''}{2t}(|y_2 - y_1|^2 + |x_2 - x_1|^2) \\ & + \frac{C_2 \cdot C''}{t}(|y_2 - y_1|^2 + |x_2 - x_1|^2) \\ \geqslant & -\frac{\overline{C}}{t}(|y_2 - y_1|^2 + |x_2 - x_1|^2). \end{aligned}$$

Then, as $u(x)$ is semiconvex, we have obtained the semiconvexity of $T_t^- u(x)$.

In view of the convexity of $A_t(\cdot, x)$, when the function u is semiconvex, then, there exists a unique maximum point y_x of the formula $u(\cdot) + A_t(\cdot, x)$. Moreover, $D_1 A_t(y_x, x) \in -D^- u(y_x)$.

Proposition 7 *Let L be a Tonelli Lagrangian, and $u : M \to \mathbb{R}$ is a semiconvex function. Then there exists t such that the map $\Psi_t : \Omega \to M$ defined by $\Psi_t(x) = y_x$ is Lipschitz, where y_x is the unique maximum point of $u(\cdot) + A_t(\cdot, x)$.*

Proof. Recalling the proposition 3, for any $\lambda > 0$ there exists $t_\lambda > 0$ and for any $t \in (0, t_\lambda)$, $A_t(\cdot, x)$ is uniformly convex in the ball $B(x, \lambda t)$ with constant C'/t. Therefore, for any x, x', $|x - x'| < \lambda t$, we denote by y_x and $y_{x'}$ the unique minimizers of $u(\cdot) - A_t(\cdot, x)$ respectively. So,

$$A_t(y_x, x) - A_t(y_{x'}, x) - D_1 A_t(y_{x'}, x)(y_x - y_{x'}) \geqslant \frac{C'}{t}|y_x - y_{x'}|^2. \tag{16}$$

On the other hand, since y_x is the minimizer corresponding x,

$$u(y_x) + A_t(y_x, x) \leqslant u(y_{x'}) + A_t(y_{x'}, x).$$

Following the semiconvexity of u,we have

$$\begin{aligned} A_t(y_x, x) - A_t(y_{x'}, x) &\leqslant u(y_{x'}) - u(y_x) \\ &\leqslant D_1 A_t(y_{x'}, x')(y_x - y_{x'}) + C_1|y_x - y_{x'}|^2. \end{aligned}$$

So

$$\begin{aligned} \frac{C'}{t}|y_x - y_{x'}|^2 &\leqslant (D_1 A_t(y_{x'}, x') - D_1 A_t(y_{x'}, x))(y_x - y_{x'}) + C_1|y_x - y_{x'}|^2 \\ &\leqslant C_2|x - x'| \cdot |y_x - y_{x'}| + C_1|y_x - y_{x'}|^2. \end{aligned}$$

Let $t_1 = \max t : C'/t - C_1 > 0$, taking $t < t_1$, we have, there exists $\widetilde{C}$ such that

$$|y_x - y_{x'}| \leqslant \widetilde{C}|x - x'|.$$

■

Now, combining the result of semiconvexity of A_t and Proposition 7, we obtain the following theorem.

Proposition 8 *Let $u : M \to \mathbb{R}$ be a semiconvex function, then there exists $t > 0$ such that the Lax-Oleinik operator $T_t^- u : M \to \mathbb{R}$ is a semi-convex function.*

Proof. For any $x_1, x_2 \in M$ and $|x_1 - x_2| < \lambda t$, let y_i be the minimizers of $u(\cdot) + A_t(\cdot, x_i)$ respectively. Set $x_0 = \frac{x_1+x_2}{2}$ and $y_0 = \frac{y_1+y_2}{2}$, then

$$
\begin{aligned}
&T_t^- u(x_1) + T_t^- u(x_2) - 2T_t^- u(x_0) \\
\geqslant& u(y_1) + A_t(y_1, x_1) + u(y_2) + A_t(y_2, x_2) \\
&- 2u(\frac{y_1 + y_2}{2}) - 2A_t(\frac{y_1 + y_2}{2}, \frac{x_1 + x_2}{2}) \\
=& u(y_1) + u(y_2) - 2u(\frac{y_0}{2}) + A_t(y_1, x_1) + A_t(y_2, x_2) - 2A_t(\frac{y_0}{2}, \frac{x_0}{2}) \\
\geqslant& - C_1|y_1 - y_2|^2 - \frac{\widetilde{C}}{t}(|y_1 - y_2|^2 + |x_1 - x_2|^2) \\
\geqslant& \frac{\widetilde{C_1}}{t}|x_1 - x_2|^2.
\end{aligned}
$$

■

Combining Proposition 5 and Proposition 8, we obtain the regularity of $T_t^- u(x)$.

Theorem 9 *Let L be a Tonelli Lagrangian satisfying conditions (L1)-(L3). Suppose $u : M \to \mathbb{R}$ is a continuous semi-convex function and the Lax-Oleinik operator $T_t^- u$ is defined as* (5)*. Then, $T_t^- u(x) : M \to \mathbb{R}$ is a local $C^{1,1}$ function.*

Proof. Since Proposition 5, $T_t^- u(x)$ is semiconcave function. Using Proposition 8, we can obtain the semiconvexity of $T_t^- u(x)$. Therefore, we can draw a conclusion of the $C^{1,1}$ regularity of $T_t^- u(x)$. ■

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