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Stability Analysis of a Competing Fish Populations Model with the Presence of a Predator

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Abstract: In this work we propose to define and study a biological model of three fish populations which are not only in competition but also prey-predator so this model combines between competition model and prey-predator model; we have two competing fish populations and the third is a predator. The existence of the steady states and its stability are studied using eigenvalue analysis and Routh Hurwitz criterion. Some numerical simulations are carried out to illustrate the theoretical part.

Keywords: Biological model; Prey-predator model; Preservation of the biodiversity; Routh Hurwitz criterion

1 Introduction

Many scientists from around the world have provided works on the management of marine fish species and have developed means to prevent and manage the exploitation of marine resources. Among these means one finds the mathematical modeling.

Many mathematical models have been developed to describe the dynamics of fisheries, one can refer for example to [1, 2]. In [1] the authors briefly review the bases and limitations of the models currently used in fisheries management. Let us add that there exist other works where the authors studied the dynamics of the fish populations assuming that these last ones are in competition like [3–5]. In these articles, the authors have assumed the existence of competing fish populations and they have studied the biological models as well as the bio-economic models and for the existence of the equilibrium states and their stability they are studied using eigenvalue analysis. Concerning the prey-predator model one can cite for example [6] and [7]. In [6], the authors consider the combined harvesting of prey-predator system in which both the prey and the predator fish population obey the law of logistic growth and some preys avoid predation by hiding; the dynamical behavior of both the exploited and unexploited systems is examined. In [7], the authors have presented a model that merges a model of competition and a model of prey-predator of three fish populations; they have assumed that on the one hand, the evolution of the first and second fish population is described by a density dependent model taking into account the competition between fish populations which compete with each other for space or food; on the other hand, the evolution of the second and third fish populations is described by a Lotka-Volterra model; the existence of the steady states and its stability are studied using eigenvalue analysis. In [8], the authors have focused thier study on incorporating Holling type III ratio-dependent functional response and two time delays into the predator-prey system with a transmissible disease spreading among the predator population.

An other example is [9] in which the authors have defined a multispecies harvesting model based on Lokta-Volterra dynamics with two competing fish populations which are affected not only by harvesting but also by the presence of a predator, the third fish population. But they have not making a case study in respect of a specific prey-predator community, they have opted for the logistic growth function for both the prey fish populations and for simplicity, they have assumed that the feeding rate of the predator fish population increase linearly with prey density.

In this work, we propose to define a biological model of three fish populations. This model combines a model of competition and a model of prey-predator. The first fish population grows according to a logistic equation and competes with second one for space or food and it is a prey of the third one; the second fish population grows according to a logistic equation and competes with first one for space or food and it is a prey of the third one too and the third fish population

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grows according to a logistic equation and it is the predator of the first and the second one. The chief aim of this paper is to study the existence of steady states and their stability.

This paper is organized as follows. In Section 2, we define the biological model of the three fish populations which is given in the form of a system of ordinary differential equations. In Section 3, we determine the equilibrium states and we study their stability using eigenvalue analysis and Routh Hurwitz Criterion. In Section 4, numerical simulations are carried out to verify the analytical results, and we give a conclusion and some perspectives in Section 5.

2 The mathematical model

We consider three fish populations. Let x, y and z be the density of fish populations. Let r_1, r_2 and r_3 be the growth rates of the first, second and third fish population respectively and K_1, K_2, and K_3 be the environmental carrying capacity of the first, second and third fish population respectively. Let c_{ij} be the coefficient of competition between the fish population i and fish population j and it represents the influence of specie j on species i. Let α and β be the predation rate coefficients of the third fish population, $\delta_1(\alpha, \beta)$ and $\delta_2(\alpha, \beta)$ be the maximum predator conversion rates. In what follow we denote $\delta_1(\alpha, \beta)$ by δ_1 and $\delta_2(\alpha, \beta)$ by δ_2. To assure the existence of the three fish populations and their stability the following expressions must be hold $r_i > c_{ij}K_j, \forall i, j = 1, 2$ with $i \neq j$ and $r_3 > \max(\delta_1 K_1, \delta_2 K_2)$.

The mathematical model is

$$\begin{cases} \frac{dx(t)}{dt} = r_1 x(t)\left(1 - \frac{x(t)}{K_1}\right) - c_{12}x(t)y(t) - \alpha x(t)z(t), \\ \frac{dy(t)}{dt} = r_2 y(t)\left(1 - \frac{y(t)}{K_2}\right) - c_{21}x(t)y(t) - \beta y(t)z(t), \\ \frac{dz(t)}{dt} = r_3 z(t)\left(1 - \frac{z(t)}{K_3}\right) + \delta_1 x(t)z(t) + \delta_2 y(t)z(t). \end{cases} \tag{1}$$

Let $X(t) = (x(t), y(t), z(t))$ be the solution of the system (1). Then all the solutions of the system (1) are nonnegative. To prove it we recall that by [10] the system of equation

$$\dot{x} = f(x_1, x_2, \ldots, x_n) \text{ with } x(t = 0) = x_0,$$

is a positive system if and only if

$$\forall i = 1, .., n;\ \dot{x}_i = f_i(x_1 \geq 0, \ldots, x_i = 0, \ldots, x_n \geq 0) \geq 0.$$

In our case, for $x = 0, y, z \geq 0$, we have $\frac{dx}{dt} \geq 0$. And for $y = 0, x, z \geq 0$, we have $\frac{dy}{dt} \geq 0$. Also for $z = 0, x, y \geq 0$, we have $\frac{dz}{dt} \geq 0$. Therefore, all the solutions of the system (1) are nonnegative.

Theorem 1 *All the solutions of system (1) which start in $\mathbb{R}^3_+$ are uniformly bounded.*

Proof. We define the function

$$V = \delta_1 \beta x + \delta_2 \alpha y + \beta \alpha z.$$

The time derivative along a solution of (1) is

$$\frac{dV}{dt} = \delta_1 \beta r_1 x\left(1 - \frac{x}{K_1}\right) + \delta_2 \alpha r_2 y\left(1 - \frac{y}{K_2}\right) + \beta \alpha r_3 z\left(1 - \frac{z}{K_3}\right) - \delta_1 \beta c_{12} xy - \delta_2 \alpha c_{21} xy.$$

For each $\gamma > 0$, we have

$$\begin{aligned} \frac{dV}{dt} + \gamma V &= \delta_1 \beta r_1 x\left(1 - \frac{x}{K_1}\right) + \delta_2 \alpha r_2 y\left(1 - \frac{y}{K_2}\right) + \beta \alpha r_3 z\left(1 - \frac{z}{K_3}\right) \\ &\quad + \gamma \delta_1 \beta x + \gamma \delta_2 \alpha y + \gamma \beta \alpha z - \delta_1 \beta c_{12} xy - \delta_2 \alpha c_{21} xy \\ &\leq \delta_1 \beta x\left[r_1\left(1 - \frac{x}{K_1}\right) + \gamma\right] + \delta_2 \alpha y\left[r_2\left(1 - \frac{y}{K_2}\right) + \gamma\right] + \beta \alpha z\left[r_3\left(1 - \frac{z}{K_3}\right) + \gamma\right] \\ &= -\frac{\delta_1 \beta r_1}{K_1} x^2 + \delta_1 \beta (r_1 + \gamma) x - \frac{\delta_2 \alpha r_2}{K_2} y^2 + \delta_2 \alpha (r_2 + \gamma) y - \frac{\beta \alpha r_3}{K_3} z^2 + \beta \alpha (r_3 + \gamma) z. \end{aligned}$$

It is easy to show that

$$\begin{cases} -\frac{\delta_1\beta r_1}{K_1}x^2+\delta_1\beta(r_1+\gamma)x-\frac{K_1\delta_1\beta}{4r_1}(r_1+\gamma)^2\le 0,\\ -\frac{\delta_2\alpha r_2}{K_2}y^2+\delta_2\alpha(r_2+\gamma)y-\frac{K_2\delta_2\alpha}{4r_2}(r_2+\gamma)^2\le 0,\\ -\frac{\beta\alpha r_3}{K_3}z^2+\beta\alpha(r_3+\gamma)z-\frac{K_3\delta_3\alpha}{4r_3}(r_3+\gamma)^2\le 0. \end{cases}$$

Then

$$\begin{cases} -\frac{\delta_1\beta r_1}{K_1}x^2+\delta_1\beta(r_1+\gamma)x\le\frac{K_1\delta_1\beta}{4r_1}(r_1+\gamma)^2,\\ -\frac{\delta_2\alpha r_2}{K_2}y^2+\delta_2\alpha(r_2+\gamma)y\le\frac{K_2\delta_2\alpha}{4r_2}(r_2+\gamma)^2,\\ -\frac{\beta\alpha r_3}{K_3}z^2+\beta\alpha(r_3+\gamma)z\le\frac{K_3\delta_3\alpha}{4r_3}(r_3+\gamma)^2. \end{cases}$$

Therefore, we can deduce that

$$\frac{dV}{dt}+\gamma V \quad\le\quad \frac{K_1\delta_1\beta}{4r_1}(r_1+\gamma)^2+\frac{K_2\delta_2\alpha}{4r_2}(r_2+\gamma)^2+\frac{K_3\alpha\beta}{4r_3}(r_3+\gamma)^2.$$

So, the right-hand side is positive, then it is bounded for all $(x,y,z)\in\mathbb{R}^3_+$. Therefore, we find a $\nu>0$ with $\frac{dV}{dt}+\gamma V<\nu$. Using the theory of differential inequality [11], we obtain

$$0\le V(x,y,z)\le\frac{\nu}{\gamma}+\left[V(x(0),y(0),z(0))-\frac{\nu}{\gamma}\right]e^{-\gamma t},$$

which upon letting $t\longrightarrow\infty$, yields $0\le V\le\frac{\nu}{\gamma}$.

Then, we have

$$B=\left\{(x,y,z)\in\mathbb{R}^3_+ : V<\frac{\nu}{\gamma}+\epsilon,\ \text{for any } \epsilon>0\right\},$$

where B is the region in which all the solutions of system of Eq.(1) that start in $\mathbb{R}^3_+$ are confined. ■

3 Equilibrium analysis

The steady state solutions are the solutions of the equations

$$\begin{cases} r_1x\left(1-\frac{x}{K_1}\right)-c_{12}xy-\alpha xz=0,\\ r_2y\left(1-\frac{y}{K_2}\right)-c_{21}xy-\beta yz=0,\\ r_3z\left(1-\frac{z}{K_3}\right)+\delta_1xz+\delta_2yz=0. \end{cases} \tag{2}$$

This system of equations has eight solutions

$P_1(0,0,0)$, $P_2(K_1,0,0)$, $P_3(0,K_2,0)$, $P_4(0,0,K_3)$, $P_5(x_5^*,y_5^*,0)$, $P_6(x_6^*,0,z_6^*)$, $P_7(0,y_7^*,z_7^*)$, $P_8(x^*,y^*,z^*)$.

$$\begin{cases} x_5^*=\dfrac{K_1r_2\left(r_1-c_{12}K_2\right)}{r_1r_2-c_{12}c_{21}K_1K_2}\\ y_5^*=\dfrac{K_2r_1\left(r_2-c_{21}K_1\right)}{r_1r_2-c_{12}c_{21}K_1K_2} \end{cases}$$

$$\begin{cases} x_6^*=\dfrac{K_1r_3\left(r_1-\alpha K_3\right)}{r_1r_3+\alpha\delta_1K_1K_3}\\ z_6^*=\dfrac{K_3r_1\left(r_3+\delta_1K_1\right)}{r_1r_3+\alpha\delta_1K_1K_3} \end{cases}$$

$$\begin{cases} y_7^*=\dfrac{K_2r_3\left(r_2-\beta K_3\right)}{r_3r_2+\beta\delta_2K_2K_3}\\ z_7^*=\dfrac{K_3r_2\left(r_3+\delta_2K_2\right)}{r_3r_2+\beta\delta_2K_2K_3} \end{cases}$$

$$\begin{cases} x^* = & K_1(r_1r_2r_3 + r_1\beta\delta_2K_2K_3 + r_3c_{12}\beta K_2K_3 \\ & -r_2r_3c_{12}K_2 - r_2r_3\alpha K_3 - r_2\alpha\delta_2K_2K_3)/\Delta \\ y^* = & K_2(r_1r_2r_3 + r_2\alpha\delta_1K_1K_3 + r_3\alpha c_{21}K_1K_3 \\ & -r_1r_3\beta K_3 - r_1r_3c_{21}K_1 - r_1\beta\delta_1K_1K_3)/\Delta \\ z^* = & K_3(r_1r_2r_3 + r_1r_2\delta_1K_1 + r_1r_2\delta_2K_2 \\ & -r_1c_{21}\delta_2K_1K_2 - r_2c_{12}\delta_1K_1K_2 - r_3c_{12}c_{21}K_1K_2)/\Delta \\ \Delta = & r_1r_2r_3 + r_1\beta\delta_2K_2K_3 + r_2\alpha\delta_1K_1K_2 \\ & -r_3c_{12}c_{21}K_1K_2 - c_{12}\beta\delta_1K_1K_2K_3 - \alpha c_{21}\delta_2K_1K_2K_3 \end{cases}$$

The system of Eq.(2) has several solutions, but only one of them can give the coexistence of the biomass of the three fish populations, this solution is the point $P_8(x^*, y^*, z^*)$.

The variational matrix of the system (2) is

$$J = \begin{bmatrix} J_{11} & -c_{12}x & -\alpha x \\ -c_{21}y & J_{22} & -\beta y \\ \delta_1 z & \delta_2 z & J_{33} \end{bmatrix},$$

where

$$\begin{cases} J_{11} = r_1\left(1 - \frac{2}{K_1}x\right) - c_{12}y - \alpha z, \\ J_{22} = r_2\left(1 - \frac{2}{K_2}y\right) - c_{21}x - \beta z, \\ J_{33} = r_3\left(1 - \frac{2}{K_3}z\right) + \delta_1 x + \delta_2 y. \end{cases}$$

Lemma 2 *The point* $P_1(0,0,0)$ *is unstable.*

Proof. The variational matrix of the system (2) at the steady state $P_1(0,0,0)$ is

$$J_1 = \begin{bmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{bmatrix}.$$

The eigenvalues of J_1 are $\lambda_1 = r_1$, $\lambda_2 = r_2$ and $\lambda_3 = r_3$. Then, the point $P_1(0,0,0)$ is unstable. ■

Lemma 3 *The point* $P_2(K_1,0,0)$ *is unstable.*

Proof. The variational matrix of the system (2) at the steady state $P_2(K_1,0,0)$ is

$$J_2 = \begin{bmatrix} -r_1 & -c_{12}K_1 & -\alpha K_1 \\ 0 & r_2 - c_{21}K_1 & 0 \\ 0 & 0 & r_3 + \delta_1K_1 \end{bmatrix}.$$

The eigenvalues of J_2 are

$$\begin{cases} \lambda_1 = -r_1, \\ \lambda_2 = r_2 - c_{21}K_1, \\ \lambda_3 = r_3 + \delta_1K_1. \end{cases}$$

Then, the point $P_2(K_1,0,0)$ is unstable. ■

Lemma 4 *The point* $P_3(0,K_2,0)$ *is unstable.*

Proof. The variational matrix of the system (2) at the steady state $P_3(0,K_2,0)$ is

$$J_3 = \begin{bmatrix} r_1 - c_{12}K_2 & 0 & 0 \\ -c_{21}K_2 & -r_2 & -\beta K_2 \\ 0 & 0 & r_3 + \delta_2K_2 \end{bmatrix}.$$

The eigenvalues of J_3 are

$$\begin{cases} \lambda_1 = r_1 - c_{12}K_2, \\ \lambda_2 = -r_2, \\ \lambda_3 = r_3 + \delta_2K_2. \end{cases}$$

Therefore, the point $P_3(0,K_2,0)$ is unstable. ■

Lemma 5 *The point $P_4(0,0,K_3)$ is stable if*

$$\begin{cases} r_1-\alpha K_3<0, \\ r_2-\beta K_3<0, \end{cases}$$

if not, it is unstable.

Proof. The variational matrix of the system (2) at the steady state $P_4(0,0,K_3)$ is

$$J_4=\begin{bmatrix} r_1-\alpha K_3 & 0 & 0 \\ 0 & r_2-\beta K_3 & 0 \\ -\alpha K_3 & \delta_2 K_3 & -r_3 \end{bmatrix},$$

the eigenvalues of J_4 are

$$\begin{cases} \lambda_1=r_1-\alpha K_3, \\ \lambda_2=r_2-\beta K_3, \\ \lambda_3=-r_3. \end{cases}$$

So, the point $P_4(0,0,K_3)$ is stable if

$$\begin{cases} r_1-\alpha K_3<0, \\ r_2-\beta K_3<0, \end{cases}$$

if not, it is unstable. ■

Lemma 6 *The point $P_5(x_5^*,y_5^*,0)$ is unstable.*

Proof. The variational matrix of the system (2) at the steady state $P_5(x_5^*,y_5^*,0)$ is

$$J_5=\begin{bmatrix} -\frac{r_1}{K_1}x_5^* & -c_{12}x_5^* & -\alpha x_5^* \\ -c_{21}y_5^* & -\frac{r_2}{K_2}y_5^* & -\beta y_5^* \\ 0 & 0 & r_3+\delta_1 x_5^*+\delta_2 y_5^* \end{bmatrix},$$

where

$$\begin{cases} x_5^*=\dfrac{K_1 r_2\left(r_1-c_{12}K_2\right)}{r_1 r_2-c_{12}c_{21}K_1K_2}, \\ y_5^*=\dfrac{K_2 r_1\left(r_2-c_{21}K_1\right)}{r_1 r_2-c_{12}c_{21}K_1K_2}. \end{cases}$$

The eigenvalues of J_5 are

$$\begin{cases} \lambda_1=-\frac{1}{2K_1K_2}(r_1x_5^*K_2+r_2y_5^*K_1+\sqrt{A}), \\ \lambda_2=-\frac{1}{2K_1K_2}(r_1x_5^*K_2+r_2y_5^*K_1-\sqrt{A}), \\ \lambda_3=r_3+\delta_1x_5^*+\delta_2y_5^*, \end{cases}$$

where

$$A=r_1^2(x_5^*)^2K_2^2-2r_1r_2x_5^*y_5^*K_1K_2+r_2^2(y_5^*)^2K_1^2+4c_{12}c_{21}x_5^*y_5^*K_1^2K_2^2.$$

One can see that either $\lambda_1>0$ or $\lambda_3>0$. Therefore, the point $P_5(x_5^*,y_5^*,0)$ is unstable. ■

Lemma 7 *The point $P_6(x_6^*,0,z_6^*)$ is unstable.*

Proof. The variational matrix of the system (2) at the steady state $P_6(x_6^*,0,z_6^*)$ is

$$J_6=\begin{bmatrix} -\frac{r_1}{k_1}x_6^* & -c_{12}x_6^* & -\alpha x_6^* \\ 0 & r_2-c_{21}x_6^*-\beta z_6^* & 0 \\ \delta_1 z_6^* & \delta_2 z_6^* & -\frac{r_3}{K_3}z_6^* \end{bmatrix},$$

where

$$\begin{cases} x_6^*=\dfrac{K_1 r_3\left(r_1-\alpha K_3\right)}{r_1 r_3+\alpha\delta_1K_1K_3}, \\ z_6^*=\dfrac{K_3 r_1\left(r_3+\delta_1 K_1\right)}{r_1 r_3+\alpha\delta_1K_1K_3}. \end{cases}$$

The eigenvalues of J_6 are

$$\begin{cases} \lambda_1 = -\frac{1}{2K_1K_3}(r_1x_6^*K_3 + r_3z_6^*K_1 - \sqrt{B}), \\ \lambda_2 = r_2 - c_{21}x_6^* - \beta z_6^*, \\ \lambda_3 = -\frac{1}{2K_1K_3}(r_1x_6^*K_3 + r_3z_6^*K_1 + \sqrt{B}), \end{cases}$$

where

$$B = r_1^2(x_6^*)^2K_3^2 - 2r_1r_3x_6^*z_6^*K_3K_1 + r_3^2(z_6^*)^2K_1^2 - 4\alpha\delta_1x_6^*z_6^*K_1^2K_3^2.$$

One can see that either $\lambda_1 > 0$ or $\lambda_2 > 0$. Therefore, the point $P_6(x_6^*, 0, z_6^*)$ is unstable. ■

Lemma 8 *The point $P_7(0, y_7^*, z_7^*)$ is unstable.*

Proof. The variational matrix of the system (2) at the steady state $P_7(0, y_7^*, z_7^*)$ is

$$J_7 = \begin{bmatrix} r_1 - c_{12}y_7^* - \alpha z_7^* & 0 & 0 \\ -c_{21}y_7^* & -\frac{r_2}{K_2}y_7^* & -\beta y_7^* \\ \delta_1z_7^* & \delta_2z_7^* & -\frac{r_3}{K_3}z_7^* \end{bmatrix},$$

where

$$\begin{cases} y_7^* = \dfrac{K_2r_3\,(r_2 - \beta K_3)}{r_3r_2 + \beta\delta_2K_2K_3}, \\ z_7^* = \dfrac{K_3r_2\,(r_3 + \delta_2K_2)}{r_3r_2 + \beta\delta_2K_2K_3}. \end{cases}$$

The eigenvalues of J_7 are

$$\begin{cases} \lambda_1 = r_1 - c_{12}y_7^* - \alpha z_7^*, \\ \lambda_2 = -\frac{1}{2K_2K_3}(r_2y_7^*K_3 + r_3z_7^*K_2 - \sqrt{C}), \\ \lambda_3 = -\frac{1}{2K_2K_3}(r_2y_7^*K_3 + r_3z_7^*K_2 + \sqrt{C}), \end{cases}$$

where

$$C = r_2^2(y_7^*)^2K_3^2 + 2r_2r_3y_7^*z_7^*K_3K_2 + K_2^2r_3^2(z_7^*)^2 - 4\beta\delta_2y_7^*z_7^*K_2^2K_3^2.$$

One can see that either $\lambda_1 > 0$ or $\lambda_2 > 0$. Therefore, the point $P_7(0, y_7^*, z_7^*)$ is unstable. ■

Theorem 9 *The point $P_8(x^*, y^*, z^*)$ is stable.*

Proof. The variational matrix of the system (2) at the steady state $P_8(x^*, y^*, z^*)$ is

$$J_8 = \begin{bmatrix} J_{11} & -c_{12}x^* & -\alpha x^* \\ -c_{21}y^* & J_{22} & -\beta y^* \\ \delta_1z^* & \delta_2z^* & J_{33} \end{bmatrix},$$

where

$$\begin{cases} x^* = & K_1(r_1r_2r_3 + r_1\beta\delta_2K_2K_3 + r_3c_{12}\beta K_2K_3 \\ & -r_2r_3c_{12}K_2 - r_2r_3\alpha K_3 - r_2\alpha\delta_2K_2K_3)/\Delta, \\ y^* = & K_2(r_1r_2r_3 + r_2\alpha\delta_1K_1K_3 + r_3\alpha c_{21}K_1K_3 \\ & -r_1r_3\beta K_3 - r_1r_3c_{21}K_1 - r_1\beta\delta_1K_1K_3)/\Delta, \\ z^* = & K_3(r_1r_2r_3 + r_1r_2\delta_1K_1 + r_1r_2\delta_2K_2 \\ & -r_1c_{21}\delta_2K_1K_2 - r_2c_{12}\delta_1K_1K_2 - r_3c_{12}c_{21}K_1K_2)/\Delta, \\ \Delta = & r_1r_2r_3 + r_1\beta\delta_2K_2K_3 + r_2\alpha\delta_1K_1K_2 \\ & -r_3c_{12}c_{21}K_1K_2 - c_{12}\beta\delta_1K_1K_2K_3 - \alpha c_{21}\delta_2K_1K_2K_3, \end{cases}$$

and

$$\begin{cases} J_{11} = r_1\left(1 - \frac{2x^*}{K_1}\right) - c_{12}y^* - \alpha z^*, \\ J_{22} = r_2\left(1 - \frac{2y^*}{K_2}\right) - c_{21}x^* - \beta z^*, \\ J_{33} = r_3\left(1 - \frac{2z^*}{K_3}\right) + \delta_1x^* + \delta_2y^*. \end{cases}$$

In accordance with (2) we deduce

$$\begin{cases} J_{11} = -r_1 \frac{x^*}{K_1}, \\ J_{22} = -r_2 \frac{y^*}{K_2}, \\ J_{33} = -r_3 \frac{z^*}{K_3}. \end{cases}$$

Then

$$J_8 = \begin{bmatrix} -r_1 \frac{x^*}{K_1} & -c_{12}x^* & -\alpha x^* \\ -c_{21}y^* & -r_2 \frac{y^*}{K_2} & -\beta y^* \\ \delta_1 z^* & \delta_2 z^* & -r_3 \frac{z^*}{K_3} \end{bmatrix}.$$

The characteristic polynomial of the variational matrix is

$$P(\lambda) = a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

where

$$\begin{cases} a_3 = 1, \\ a_2 = r_1 \frac{x^*}{K_1} + r_2 \frac{y^*}{K_2} + r_3 \frac{z^*}{K_3}, \\ a_1 = r_1 \frac{x^*}{K_1} r_2 \frac{y^*}{K_2} + r_1 \frac{x^*}{K_1} r_3 \frac{z^*}{K_3} + r_2 \frac{y^*}{K_2} r_3 \frac{z^*}{K_3} + \beta y^* \delta_2 z^* - c_{12}x^* c_{21} y^* + \alpha x^* \delta_1 z^*, \\ a_0 = r_1 \frac{x^*}{K_1} r_2 \frac{y^*}{K_2} r_3 \frac{z^*}{K_3} - c_{12}x^* \beta y^* \delta_1 z^* - \alpha z^* \delta_2 y^* c_{21} x^* \\ \qquad - c_{12}x^* c_{21} y^* r_3 \frac{z^*}{K_3} + \beta y^* \delta_2 z^* r_1 \frac{x^*}{K_1} + \alpha z^* \delta_1 x^* r_2 \frac{y^*}{K_2}. \end{cases}$$

It is easy to prove that a_0, a_1, a_2, and a_3 are positive. Let as add that $a_1a_2 - a_0a_3 > 0$;

$$\begin{aligned} a_1a_2 - a_0a_3 \quad &= \left(r_1 \frac{x^*}{K_1} + r_2 \frac{y^*}{K_2} + r_3 \frac{z^*}{K_3}\right) \left(r_1 \frac{x^*}{K_1} r_2 \frac{y^*}{K_2} + r_1 \frac{x^*}{K_1} r_3 \frac{z^*}{K_3} + r_2 \frac{y^*}{K_2} r_3 \frac{z^*}{K_3} \right. \\ &\left. +\beta y^* \delta_2 z^* - c_{12}x^* c_{21} y^* + \alpha x^* \delta_1 z^*\right) \\ &- \left(r_1 \frac{x^*}{K_1} r_2 \frac{y^*}{K_2} r_3 \frac{z^*}{K_3} - c_{12}x^* \beta y^* \delta_1 z^* - \alpha z^* \delta_2 y^* c_{21} x^* \right. \\ &\left. -c_{12}x^* c_{21} y^* r_3 \frac{z^*}{K_3} + \beta y^* \delta_2 z^* r_1 \frac{x^*}{K_1} + \alpha z^* \delta_1 x^* r_2 \frac{y^*}{K_2}\right) \end{aligned}$$

$$\begin{aligned} &= r_1 \frac{x^*}{K_1} r_2 \frac{y^*}{K_2} r_3 \frac{z^*}{K_3} + c_{12}x^* \beta y^* \delta_1 z^* + r_1 \frac{x^*}{K_1} r_2 \frac{y^*}{K_2} r_3 \frac{z^*}{K_3} - \alpha z^* \delta_2 y^* c_{21} x^* \\ &+ r_1^2 \frac{x^*}{K_1^2} r_2 \frac{y^*}{K_2} - r_1 \frac{x^*}{K_1} c_{12} c_{21} y^* + r_2^2 \frac{y^*}{K_2^2} r_3 \frac{z^*}{K_3} + r_2 \frac{y^*}{K_2} \beta \delta_2 z^* \\ &+ r_1^2 \frac{y^*}{K_1^2} r_3 \frac{z^*}{K_3} + r_1 \frac{x^*}{K_1} \alpha \delta_1 z^* + r_3^2 \frac{z^*}{K_3^2} r_1 \frac{x^*}{K_1} + r_3 \frac{z^*}{K_3} \alpha \delta_1 x^* \\ &+ r_2^2 \frac{y^*}{K_2^2} r_1 \frac{x^*}{K_1} - r_2 \frac{y^*}{K_2} c_{12} c_{21} x^* + r_3^2 \frac{z^*}{K_3^2} r_2 \frac{y^*}{K_2} + r_3 \frac{z^*}{K_3} \beta \delta_2 y^*. \end{aligned}$$

So we deduce that $a_1a_2 - a_0a_3 > 0$. Then using Routh-Hurwitz rule it can be concluded that the point $P_8(x^*, y^*, z^*)$ is locally asymptotically stable. ■

4 Numerical simulation

In this Section, we will carry out some numerical simulations to verify the stability and instability of the eight equilibrium points of the biological model, so we will consider the values shown in the table below for the biological parameters of our model which ensure the coexistence of the three fish populations.

For the above values of the parameters, it is found that all the equilibrium points of system (2) are unstable except the last equilibrium point $P_8(484, 308, 416)$ is stable. The following figures confirm these results.

According to the parameter values citing in Table 1, Fig.2 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value x(0)=700, y(0)=0.01, z(0)=0.01. By Fig.2 we find that the steady state $P_2(700, 0, 0)$ is unstable and tend to the equilibrium point $P_8(485, 309, 416)$.

For the parameter values citing in Table 1, Fig.1 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value x(0)=0.01, y(0)=0.01, z(0)=0.01. By Fig.1 we find that the steady state $P_1(0, 0, 0)$ is unstable and tend to the equilibrium point $P_8(485, 309, 416)$.

Following the parameter values citing in Table 1, Fig.3 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value x(0)=0.01, y(0)=500, z(0)=0.01. By Fig.3, one can find that the steady state $P_3(0, 500, 0)$ is unstable and tend to the equilibrium point $P_8(485, 309, 416)$.

Table 1: Parameters and their descriptions/values in the biological model

Parameters	Descriptions	Values
r_1	Growth rate of the first fish population	2
r_2	Growth rate of the second fish population	1
r_3	Growth rate of the third fish population	3
K_1	Environmental carrying capacity of the first fish population	700
K_2	Environmental carrying capacity of the second fish population	500
K_3	Environmental carrying capacity of the tird fish population	400
c_{12}	Coefficient of competition between the first and the second fish populations	0.0009
c_{21}	Cefficient of competition between the second and the first fish populations	0.0007
α	Predation rate coefficient of the third fish population	0.0008
β	Predation rate coefficient of the third fish population	0.0001
δ_1	Maximum predator conversion rates	0.0002
δ_2	Maximum predator conversion rates	0.0001

Using the parameter values citing in Table 1, Fig.4 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value x(0)=0.01, y(0)=0.01, z(0)=400. By Fig.4, one find that the steady state $P_4(0, 0, 400)$ is unstable and tend to the equilibrium point $P_8(485, 309, 416)$.

According to the parameter values citing in Table 1, Fig.5 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value x(0)=609, y(0)=286, z(0)=0.01. By Fig.5, one find that the steady state $P_5(609, 286, 0)$ is unstable and tend to the equilibrium point $P_8(485, 309, 416)$.

Following the parameter values citing in Table 1, Fig.6 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value x(0)=583, y(0)=0.01, z(0)=415. By Fig.5, one find that the steady state $P_6(583, 0, 415)$ is unstable and tend to the equilibrium point $P_8(485, 309, 416)$.

According to the parameter values citing in Table 1, Fig.7 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value x(0)=0.01, y(0)=479, z(0)=406. By Fig.5, one find that the steady state $P_7(0, 479, 406)$ is unstable and tend to the equilibrium point $P_8(485, 309, 416)$.

Following the parameter values citing in Table 1, Fig.8 show the dynamical behaviors and phase portrait of the three fish populations against time, beginning with the initial value x(0)=484, y(0)=308, z(0)=416. By Fig.8, one find that the steady state $P_8(484, 308, 416)$ is stable and according to the above phase-space trajectories we can confirm that this equilibrium point is a global attractor.

More precisely, the following graphs show that the equilibrium point $P_8(485, 309, 416)$ is globally asymptotically stable since all the points with different initial values converge tend to this point.

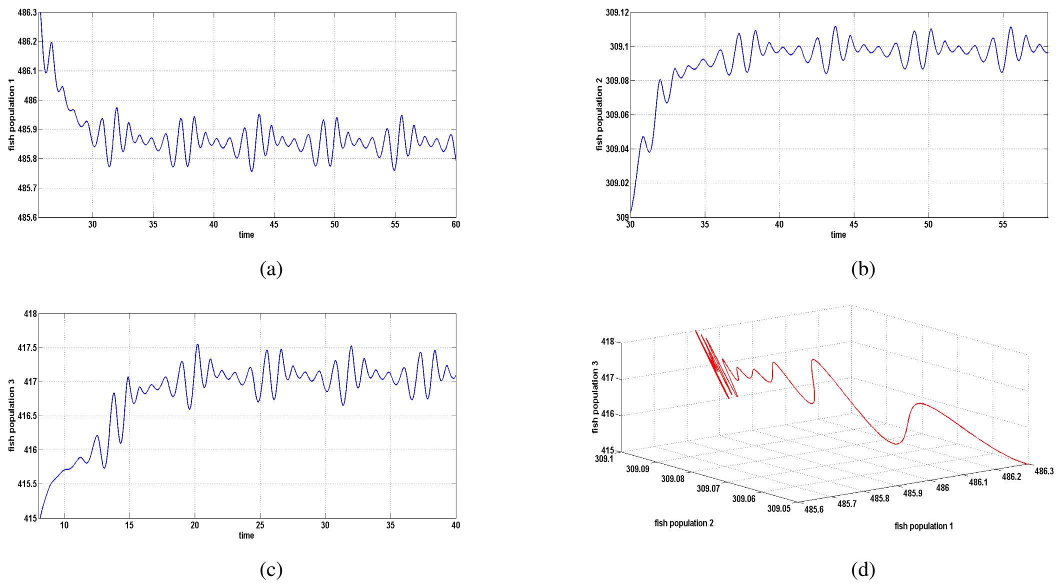

Figure 1: Behaviors and phase portrait of system (2) at $P_1(0, 0, 0)$.

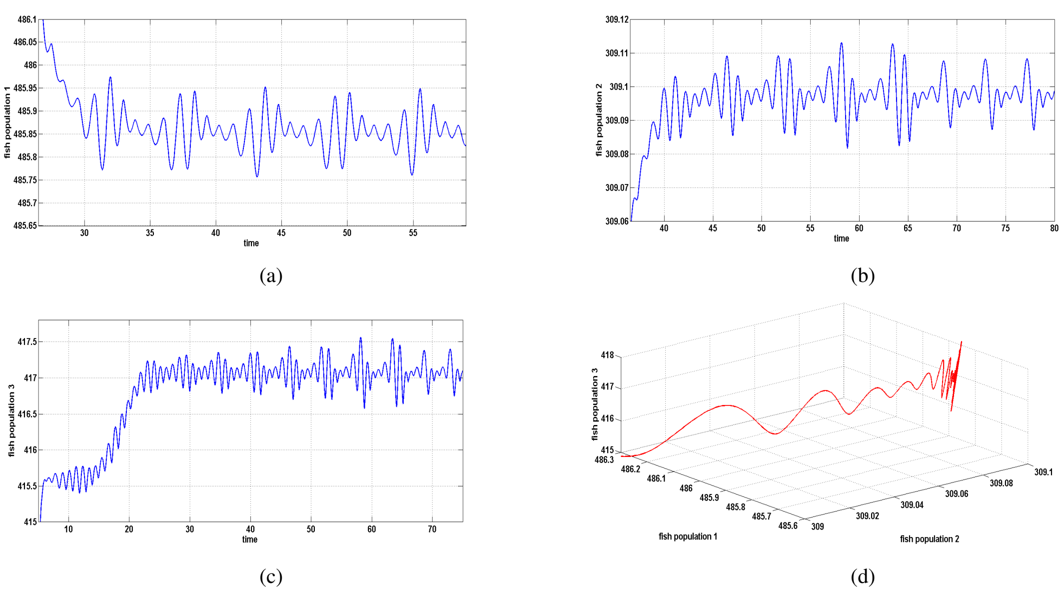

Figure 2: Behaviors and phase portrait of system (2) at $P_2(700, 0, 0)$.

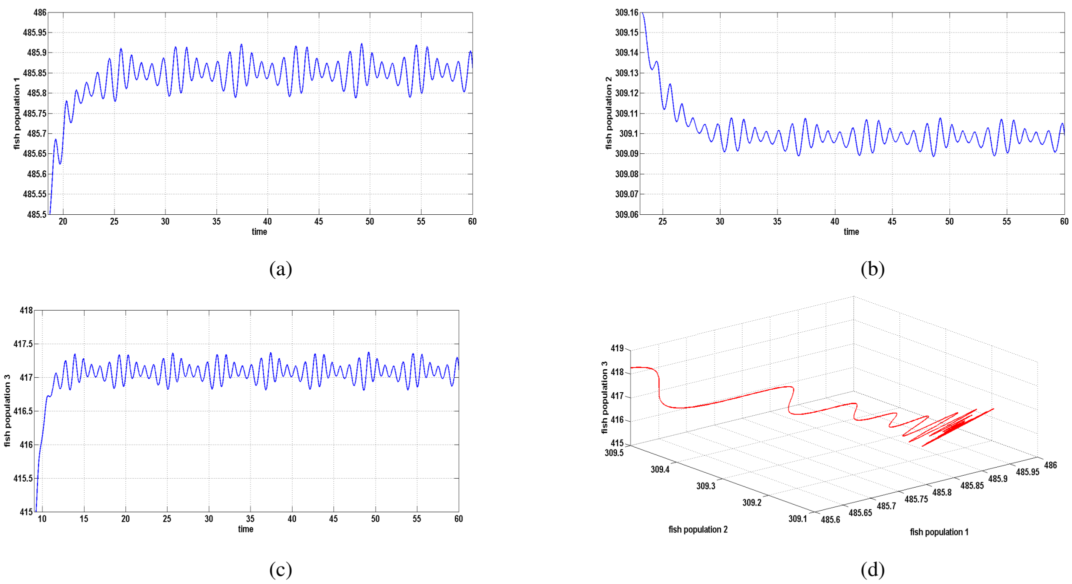

Figure 3: Behaviors and phase portrait of system (2) at $P_3(0, 500, 0)$.

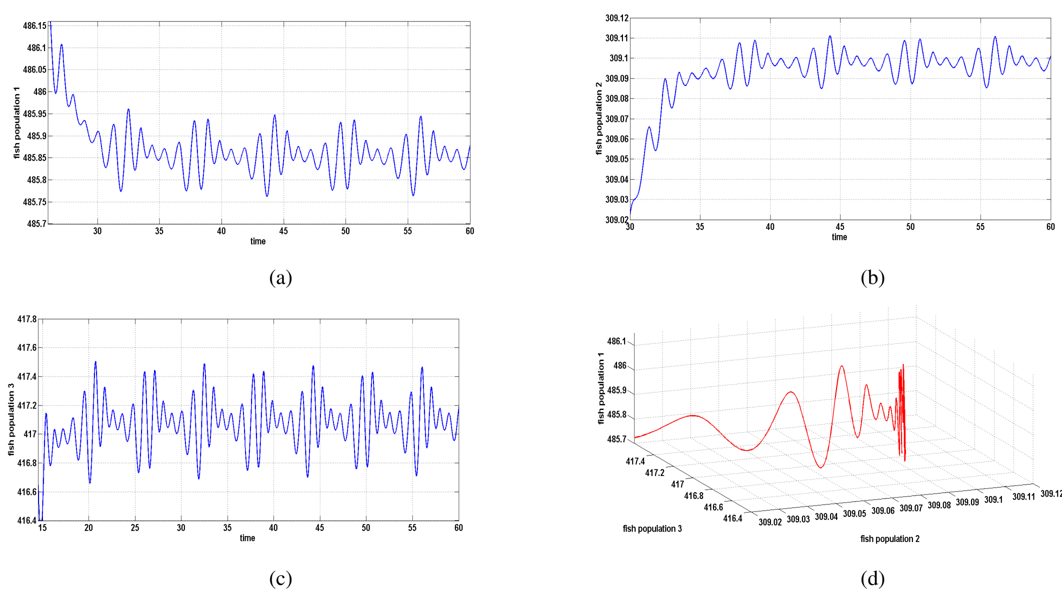

Figure 4: Behaviors and phase portrait of system (2) at $P_4(0, 0, 400)$.

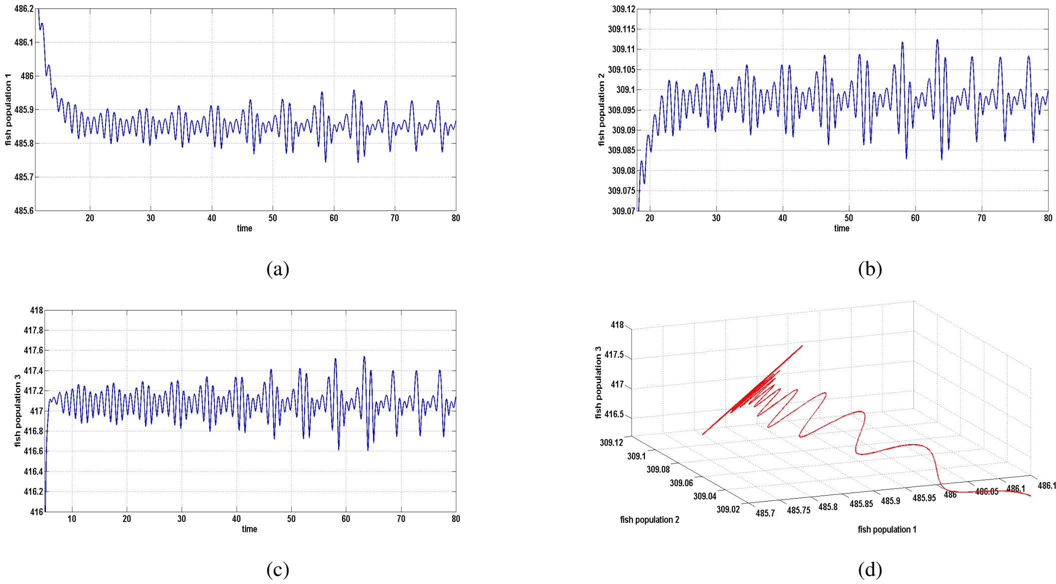

Figure 5: Behaviors and phase portrait of system (2) at $P_5(609, 286, 0)$.

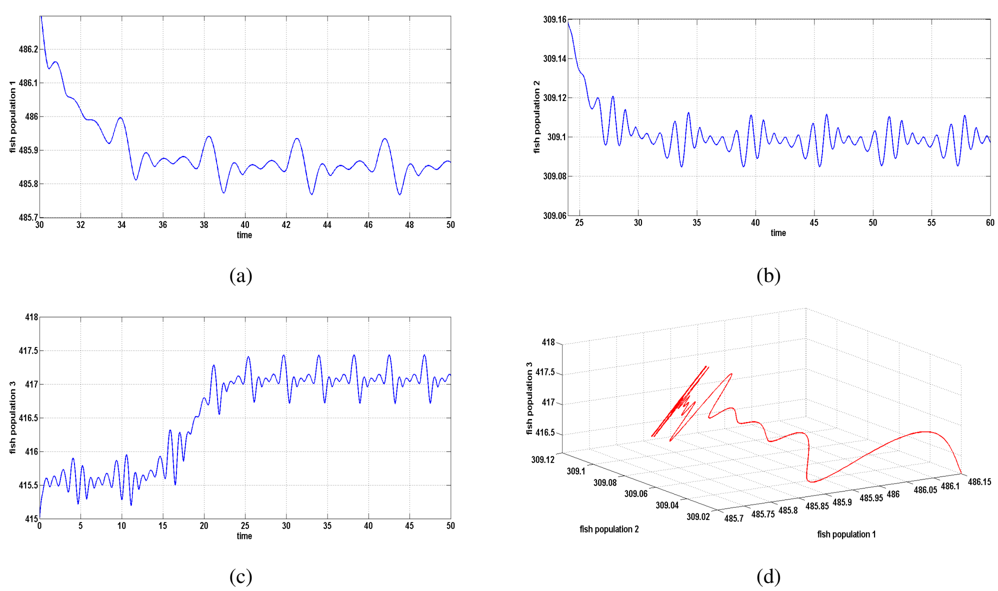

Figure 6: Behaviors and phase portrait of system (2) at $P_6(583, 0, 415)$.

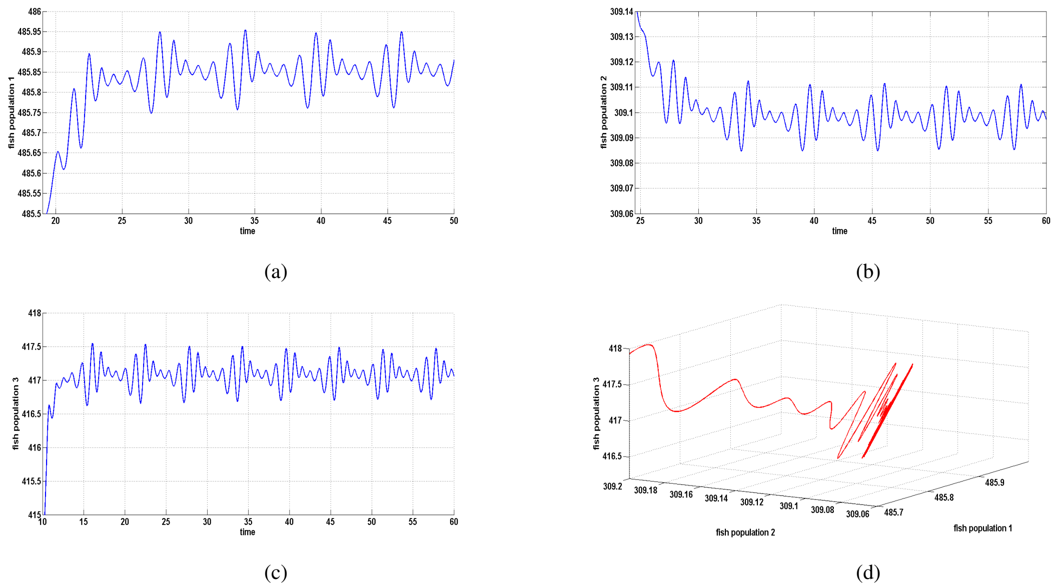

Figure 7: Behaviors and phase portrait of system (2) at $P_7(0, 479, 406)$.

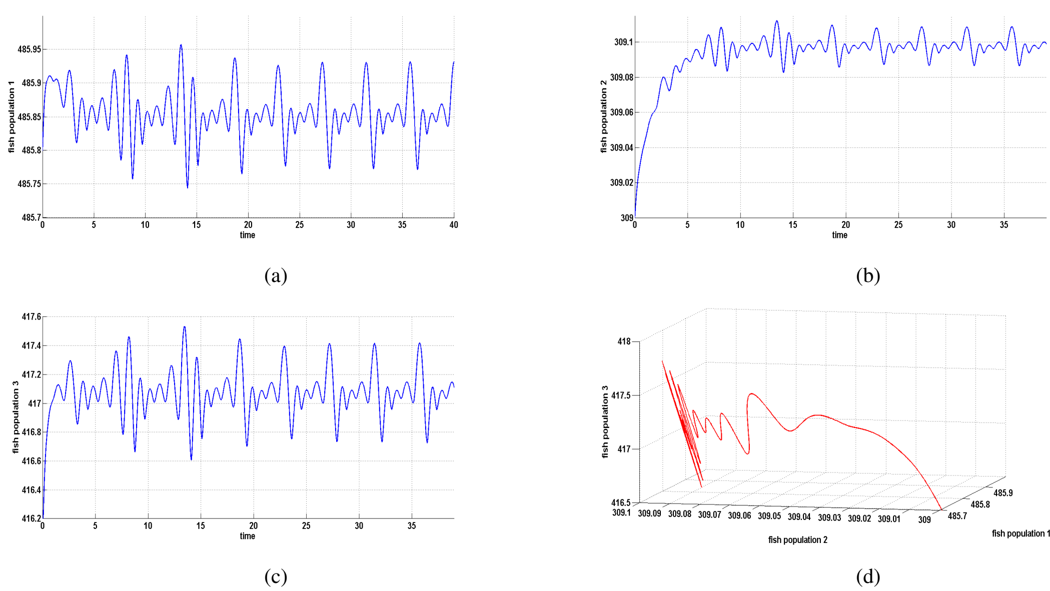

Figure 8: Behaviors and phase portrait of system (2) at $P_8(484, 308, 416)$.

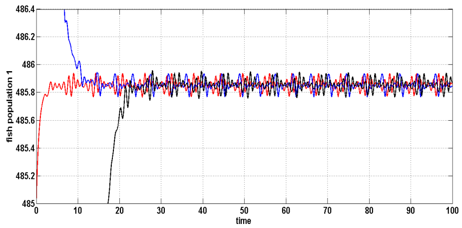

Figure 9: Dynamic behaviours of the first fish population with reference to different initial values.

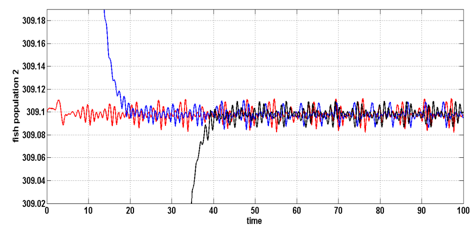

Figure 10: Dynamic behaviours of the second fish population with reference to different initial values.

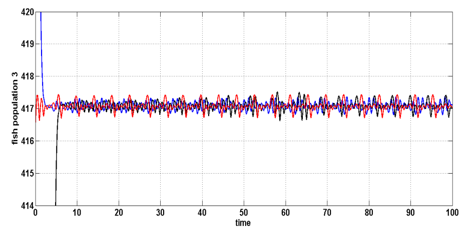

Figure 11: Dynamic behaviours of the third fish population with reference to different initial values.

5 Conclusion

In this paper we have proposed a biological model for three fish populations. This model combines a model of competition and a model of prey-predator. We have supposed that the first fish population grows according to a logistic equation and competes with second one for space or food and it is a prey of the third one; the second fish population grows according to a logistic equation and competes with first one for space or food and it is a prey of the third one; we have also supposed that the third fish population grows according to a logistic equation and it is the predator of the first and the second one. In this work we have calculated the steady states of the biological model and we have studied their stability using eigenvalue analysis and the Routh-Hurwitz Criterion. Finally, some numerical simulations are given to illustrate the results obtained using Matlab. As perspectives we intend to develop the bioeconomic model of this three fish populations and to maximize the income of the fishermen who exploit them using the theory of linear complementarity problems.

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