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Fractional Reduced Differential Transform Method and Its Applications

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Abstract: Here, the fractional Reduced differential (FRDTM) transform method is used to develop a scheme-to study numerical solution of time fractional nonlinear evolution equations under initial conditions. Two models of special interest in physics with fractional-time derivative of order α are considered. The behaviour of RDTM and the effects of different values of α are shown graphically. Numerical examples are tested to illustrate the pertinent feature of the proposed algorithm. The numerical results demonstrate the significant features, efficiency and reliability of the proposed method and the effects of different values are shown graphically. The paper shows that the results obtained from the fractional analysis appear to be general.

Keywords: Fractional Reduced differential transform method; Fractional-time differential equations; Approximate and exact solutions; Adomian decomposition method

1 Introduction

One of the generalizations of classical ordinary calculus can be considered as fractional calculus in the field of the mathematical analysis with applications of derivatives and integrals with an arbitrary order. The history of fractional calculus went back to a few centuries ago. Fractional differential equations are generalizations of classical differential equations of integer order.

Fractional calculus (that is the theory of integrals and derivatives of arbitrary real or complex orders) were planted over 300 years ago. Science than, the subject of factional calculus [1–4] is a rapidly growing field of research, at the interface between chaos, probability, differential equations, and mathematical physics.

The fractional differential equations and have gained much attention as they are widely used to describe various complex phenomena in many fields such as the fluid flow, signal processing, theory, systems identification, biology, and other areas. These equations appear in a wide great array discretization span, Jumarie's defined the fractional derivative in a variety of contexts, such as physics, biology, engineering, signal processing, systems identification, control theory, finance, and fractional dynamics. The exact solutions of FDEs play an important role in many physical phenomena, which can be modeled by these equations. In recent years, a large number of studies have been done concerning the analytical and numerical solutions of the nonlinear FDEs [5–24] and so on.

The rest of this paper is arranged as follows. In section 2, we describe the fractional reduced differential transform method [11, 12]. In section 3, in order to illustrate the method two models of fractional nonlinear differential equation of special interest physically are chosen, namely, the coupled MKdV equations and coupled system of nonlinear physical equations. Also, with our examples it will be shown that the RDTM is very useful in the discussion of fractal equations. Finally, conclusion and discussion are given in section 4.

2 Methodlogy

In what the follows some definitions and important properties of fractional calculus [3]. Let us consider two variables $u(x,t)$ and suppose that it can be represented as a product of two single-variable functions, i.e., $u(x,t) = f(x)g(t)$. Based

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Table 1: Reduced Differential Transformation.

$Functional$	$Transformed$
$u(x,t)$	$U_k = \frac{1}{\Gamma(k\alpha+1)}[\frac{\partial^{k\alpha}u(x,t)}{\partial t^{k\alpha}}]_{t=0}$
$w(x,t) = u(x,t) \pm v(x,t)$	$W_k(x) = U_k(x) \pm V_k(x)$
$w(x,t) = \alpha u(x,t)$	$W_k(x) = \alpha U_k(x)$,α $isaconstant$
$w(x,t) = x^m t^n$	$W_k(x) = x^m \delta(k\alpha - n)$
$w(x,t) = x^m t^n u(x,t)$	$W_k(x) = x^m U(k\alpha - n)$
$w(x,t) = u(x,t)v(x,t)$	$W_k(x) = \sum_{r=0}^{k} U_r(x)V_{k-r}(x)$
$w(x,t) = \frac{\partial^{N\alpha}}{\partial t^{N\alpha}}u(x,t)$	$W_k(x) = \frac{(\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}(x)$
$w(x,t) = \frac{\partial^m}{\partial x^m}u(x,t)$	$W_k(x) = \frac{\partial^m}{\partial x^m}U_k(x)$

on the properties of differential transform, function $u(x,t)$ can be represented as

$$u(x,t) = \sum_{k=0}^{\infty} U_k(x)t^{k\alpha}, \tag{1}$$

where α is a parameter describing the order of the time fractional derivative in Caputo and sense and $U_k(x)$ is called t-dimensional spectrum function of $u(x,t)$.

Definition 1 *If function $u(x,t)$ is analytic and differentiated continuously with respect to t and x in the domian of interst, then let*

$$U_k(x) = \frac{1}{\Gamma(k\alpha+1)}[[\frac{\partial^{k\alpha}u(x,t)}{\partial t^{k\alpha}}]|_{t=0}, \tag{2}$$

t dimensional spectrum function $u_k(x)$ is the transformed function.

In this paper, the lowercase $u(x,t)$ represent the original function while the uppercase $U_k(x)$ stand for the transformed function. Then combining Eqs.(1) and (2) we write

$$u(x,t) = \sum_{k=0}^{\infty} \frac{1}{\Gamma(k\alpha+1)}[\frac{\partial^{k\alpha}u(x,t)}{\partial t^{k\alpha}}]|_{t=0}t^{k\alpha}, \tag{3}$$

with the aid above definitions, it can be found that the concept of the reduced differential transform is derived from the power series expansion.

For the purpose of illustration of the methodlology to the proposed method,we write nonlinear equation in the s-tandered operator form as follows

$$L(u(x,t)) + R(u(x,t)) + N(u(x,t)) = g(x,t), \tag{4}$$

with initial condition

$$u(x,0) = f(x), \tag{5}$$

where $L = \frac{\partial^\alpha}{\partial t^\alpha}$, R is a linear operator which has partial derivatives, $u(x,t)$ is a nonlinear term and $g(x,t)$ is an inhomogeneous term.

According to the RDTM and Table 1, we can constructing the following iteration formulas

$$\frac{\Gamma(k\alpha+N\alpha+1)}{\Gamma(k\alpha+1)}U_{k+N}(x) = G_k(x) - R(U_k(x)) - N(U_k(x)), \tag{6}$$

where $U_k(x)$, $R(U_k(x))$, $N(U_k(x))$ and $G_k(x)$ are transformation of the functions $Lu(x,t)$, $Ru(x,t)$, $Nu(x,t)$ and $g(x,t)$ respectively. From the initial condition, we write

$$U_0(x) = f(x), \tag{7}$$

substituting Eqs.(7) and (6) and by a straight forward iterative formula, we get the following $U_k(x)$ values. Then the inverse transformation of the set of values $U_k(x)_{k=0}^{n}$ gives approximation solution as

$$u^*(x,t) = \sum_{k=0}^{n} U_k(x) t^{k\alpha}, \tag{8}$$

where n is order of approximation solution. Therefore, the exact solution of problem is given by

$$u(x,t) = lim_{n \longrightarrow \infty} u_n^*(x,t), i = 1, 2, ..., n. \tag{9}$$

3 New applications

To illustrate the effectiveness and the advantages of the proposed method, two models of nonlinear evaluation equations arising in physics are chosen, namely, the coupled MKdV equations and coupled system of nonlinear physical equations.

3.1 Example(1): A coupled system of nonlinear physical equations

Let us first consider a coupled system of nonlinear physical equations reads [4]

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = u(1 - u^2 - v) + u_{xx}, t > 0, \tag{10}$$

$$\frac{\partial^\alpha v(x,t)}{\partial t^\alpha} = v(1 - u - v) + v_{xx}, \tag{11}$$

with initial conditions [4]

$$u(x,0) = \frac{e^{kx}}{[1 + e^{kx}]}, \tag{12}$$

$$v(x,0) = \frac{1 + (3/4)e^{kx}}{[1 + e^{kx}]^2}. \tag{13}$$

The above sysetem for two coupled nonlinear equations of reaction-diffusion type arising in chemical reaction or ecology, and other fields of physics. This model can be used to describe the ecological process of two specimen, where k is constant. For the solution procedure, we first take the differential transform of Eqs.(10-14) by the use of Table 1 and have the following equations

$$\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} U_{k+1}(x) = U_k(x) - A_k(x) - B_k(x) + \frac{\partial^2}{\partial x^2} U_k(x), \tag{14}$$

$$\frac{\Gamma(k\alpha + \alpha + 1)}{\Gamma(k\alpha + 1)} V_{k+1}(x) = V_k(x) - B_k(x) - C_k(x) + \frac{\partial^2}{\partial x^2} V_k(x), \tag{15}$$

where $A_k(x)$, $B_k(x)$, and $C_k(x)$ are transformed form of the nonlinear terms. From the initial conditions (12) and (13), we write

$$U_0(x) = \frac{e^{kx}}{[1 + e^{kx}]}, \tag{16}$$

$$V_0(x) = \frac{1 + (3/4)e^{kx}}{[1 + e^{kx}]^2}. \tag{17}$$

The first five nonlinear term reads

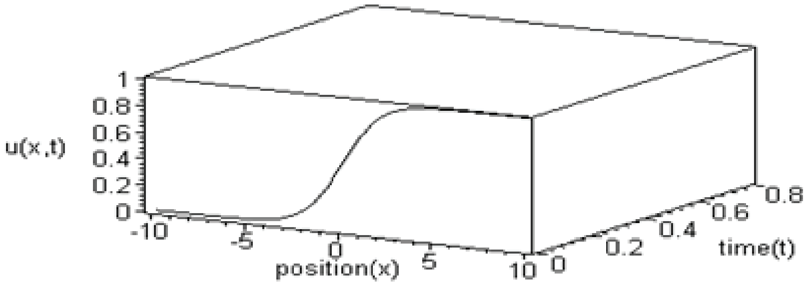

Figure 1a: The numerical solution of u(x,t) obtained by RDTM for the fractional order $\alpha = 3/4$ for a coupled system of diffusion reaction equations.

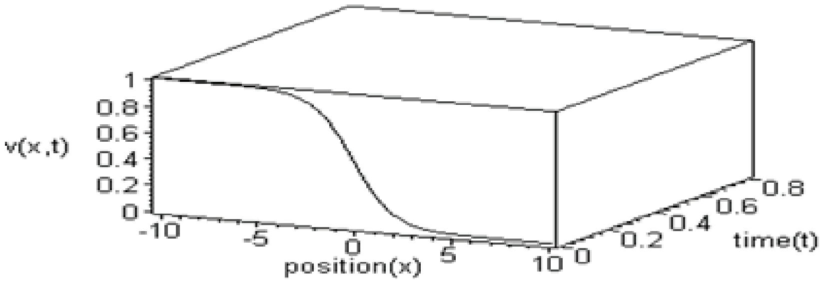

Figure 1b: The numerical solution of v(x,t) obtained by RDTM for the fractional order $\alpha = 3/4$ for a coupled system of diffusion reaction equations.

$$A_0 = U_0^3, A_1 =, A_2 = 3U_0^2U_2 + 3U_0U_1^2, A_3 = U_1^3 + 3U_0^2U_3 + 6U_0U_1U_2,$$

$$A_4 = 3U_0^3U_4 + 6U_0U_1U_3 + 3U_0U_2^2 + 3U_1^2U_2, C_0 = V_0^2, C_1 = 2V_0V_1, C_2 = 2V_0V_2 + V_1^2, C_3 = 2V_0V_3 + 2V_1V_2,$$

$$C_4 = 2V_0V_4 + 2V_1V_3 + V_2^2, B_0 = U_0V_0, B_1 = U_1V_0 + U_0V_1, B_2 = V_0U_2 + U_0V_2 + U_1V_1,$$

$$B_3 = U_0V_3 + V_2U_1 + U_2V_1 + U_3V_0, B_4 = U_0V_4 + V_3U_1 + U_2V_2 + U_0V_4 + U_3V_1. \tag{18}$$

With the aid of Eqs.(16) and (17) into Eqs.(14) and (15), we can directly evaluated the values of $U_k(x)$ and $V_k(x)$. Explicit form for $U_k(x)$ and $V_k(x)$ are obtained for n order of approximation. For simplicity should be omitte here.

The differential inverse transform of $U_k(x)$ and $V_k(x)$ gives

$$u(x,t) = \sum_{k=0}^{n} U_k(x)t^k, \tag{19}$$

$$v(x,t) = \sum_{k=0}^{n} V_k(x)t^k. \tag{20}$$

The numerical behavior of the approximate solutions of $u(x,t)$ and $v(x,t)$ obtained by RDTM with different values of fractional time derivative order α are shown graphically see Figs.(1.(a-d)) and the exact solutions are shown graphically in Figs.(1.(e,f)) for a different values of time t with a fixed values of $k = 0.1$, $\lambda = 1.5$ and $b_1 = 0.1$, which proves the two solutions are quite good. It is to be noted that the exact solutions of $u(x,t)$ and $v(x,t)$ [4].

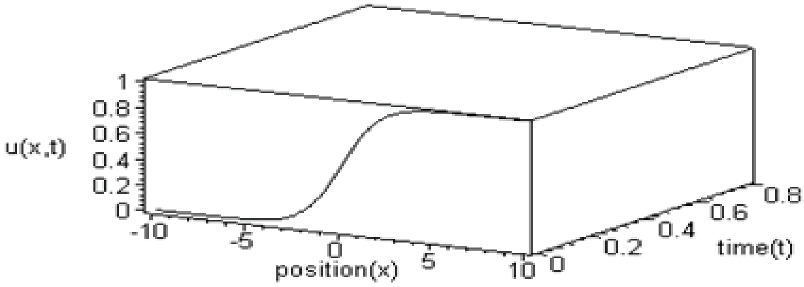

Figure 1c: The numerical solution of u(x,t) obtained by RDTM for the fractional order $\alpha = 1$ for a coupled system of diffusion reaction equations.

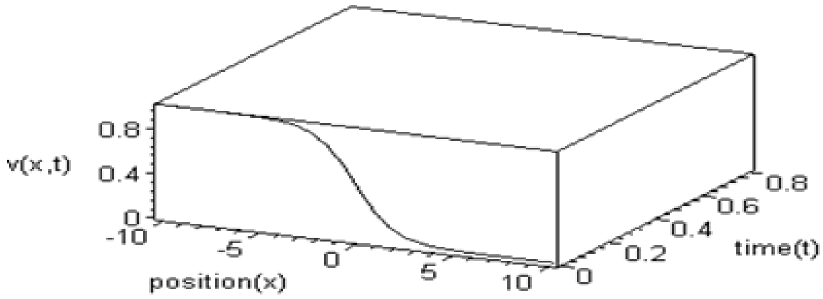

Figure 1d: The numerical solution of v(x,t) obtained by RDTM for the fractional order $\alpha = 1$ for a coupled system of diffusion reaction equations.

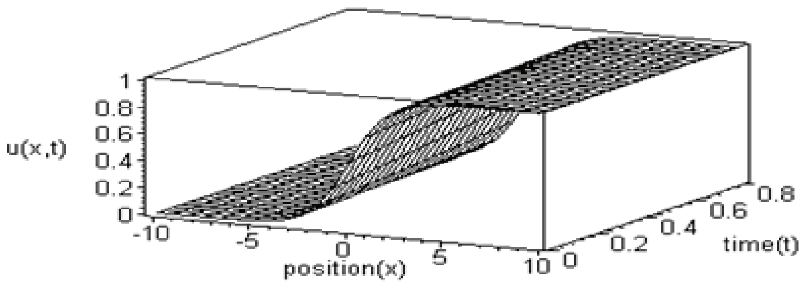

Figure 1e: The exact solution of u(x,t) of Eq.(21) for $\alpha = 1$ for a coupled system of diffusion reaction equations.

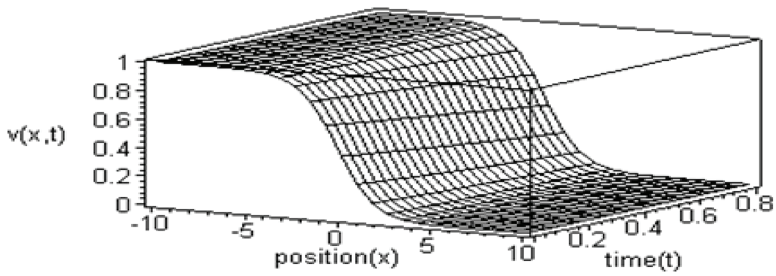

Figure 1f: The exact solution of v(x,t) of Eq.(22) for $\alpha = 1$ for a coupled system of diffusion reaction equation.

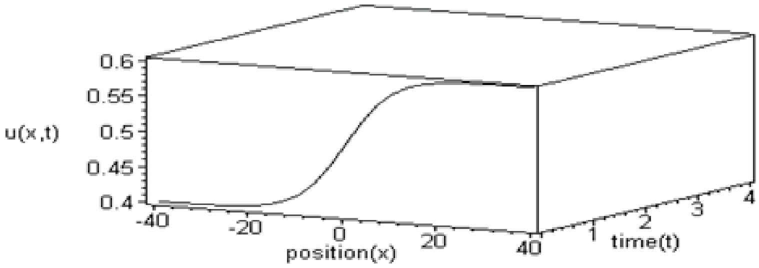

Figure 2a: The numerical solution of u(x,t) obtained by RDTM for the fractional order $\alpha = 1/2$ for fractional time of a coupled MKdV equations with a fixed values of x and t.

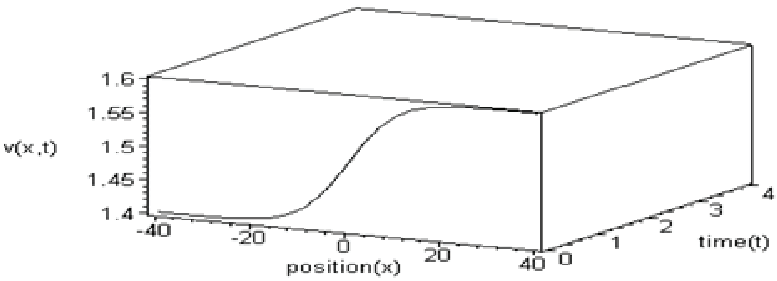

Figure 2b: The numerical solution of v(x,t) obtained by RDTM for the fractional order $\alpha = 1/2$ for fractional time of a coupled MKdV equations with a fixed values of x and t.

$$u(x,t) = \frac{e^{k(x+ct)}}{(1+e^{k(x+ct)})}, \tag{21}$$

$$v(x,t) = \frac{1+(3/4)e^{k(x+ct)}}{(1+e^{k(x+ct)})^2}, \tag{22}$$

where k and c are constants. It is worth noting that the accuracy of the RDTM is significantly enhanced by calculation as many more terms as we like. This may be made but only at the expense of a considerable increase in the complexity of analysis.

3.2 Example(2): A coupled MKdV equation

A second instructive model is the coupled MKdV equations

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = \frac{1}{2}u_{xxx} - 3u^2u_x + \frac{3}{2}v_{xx} + 3(uv)_x - 3\lambda u_x,$$

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha} = -v_{xxx} - 3vv_x - 3u_xv_x + 3u^2v_x + 3\lambda v_x, \tag{23}$$

with initial conditions

$$u(x,0) = \frac{b_1}{2k} + ktanh[kx],$$

$$v(x,0) = \frac{\lambda}{2}(1+\frac{b_1}{2}) + b_1tanh[kx], \tag{24}$$

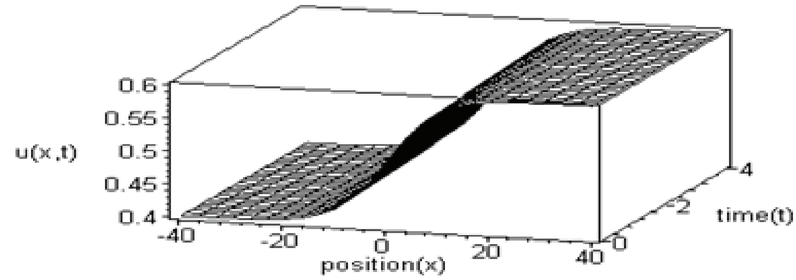

Figure 2c: The numerical solution of u(x,t) obtained by RDTM for $\alpha = 1$ for fractional time of a coupled MKdV equations with a fixed values of x and t.

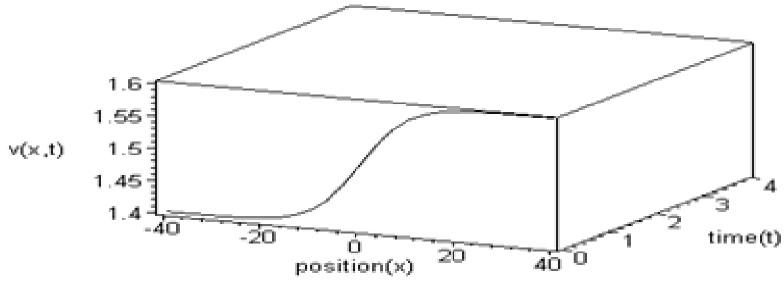

Figure 2d: The numerical solution of v (x,t) obtained by RDTM for $\alpha = 1$ for fractional time of a coupled MKdV equations with a fixed values of x and t.

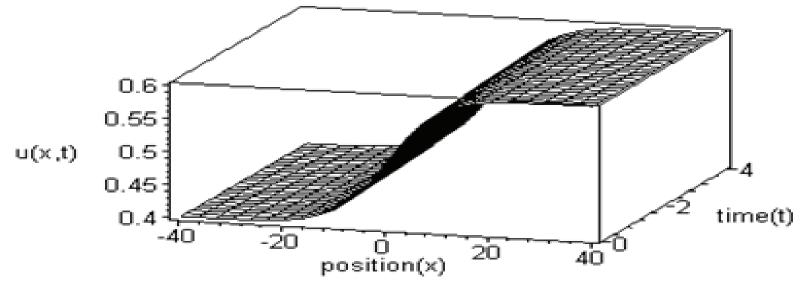

Figure 2e: The exact solution of u (x,t) of Eq.(32) for fractional time of a coupled MKdV equations with a fixed values of x and t.

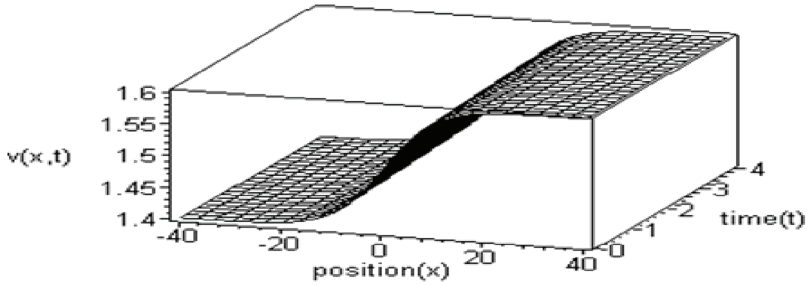

Figure 2f: The exact solution of v (x,t) of Eq.(32) for fractional time of a coupled MKdV equations with a fixed values of x and t.

where k,$b_1 \neq 0$, and λ are arbitrary constants. This equation describes interactions of two long waves with different dispersion relations.

In view RDTM Table 1, Eq.(23) reduces to

$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}U_{k+1}(x)=\frac{1}{2}\frac{\partial^3}{\partial x^3}U_k(x)-3A_k(x)+3B_k(x)+\frac{3}{2}\frac{\partial^2}{\partial x^2}V_k(x)-3\lambda\frac{\partial}{\partial x}U_k(x),$$
$$\frac{\Gamma(k\alpha+\alpha+1)}{\Gamma(k\alpha+1)}V_{k+1}(x)=-\frac{1}{2}\frac{\partial^3}{\partial x^3}V_k(x)-3C_k(x)-3D_k(x)+E_k(x)+3\lambda\frac{\partial}{\partial x}V_k(x), \tag{25}$$

where $A_k(x)$,$B_k(x)$,$C_k(x)$ and $D_k(x)$ are transformed form of the nonlinear terms. From the initial conditions (24), we write

$$U_0(x)=\frac{b_1}{2k}+ktanh[kx],$$
$$V_0(x)=\frac{\lambda}{2}(1+\frac{b_1}{2})+b_1tanh[kx]. \tag{26}$$

The first three component of the nonlinear term reads

$$A_0=U_0^2U_{0x}, A_1=2U_0U_1U_{0x}+U_0^2U_{1x}, A_2=2U_0U_2U_{0x}+U_1^2U_{0x}+2U_0U_1U_{1x}+U_{2x}U_0^2,$$

$$B_0=(U_0V_0)_x, B_1=(V_0V_1+U_0V_1)_x, B_2=(V_0U_2+U_0V_2+U_1V_1)_x,$$

$$C_0=V_0V_{0x}, C_1=(V_0V_{1x}+V_1V_0)_x, C_2=V_2V_{0x}+V_1V_{1x}+V_0V_{2x},$$

$$D_0=U_{0x}V_{0x}, D_1=U_{1x}V_{0x}, D_2=U_{2x}V_{0x}+U_{1x}V_{1x}+U_{0x}V_{2x},$$

$$E_0=U_0^2V_{0x}, E_1=2U_0V_{0x}U_1+V_0^2V_{1x}, E_2=2U_0V_{0x}U_2+U_1^2V_{0x}+2U_0U_1V_{1x}+V_{2x}U_0^2. \tag{27}$$

Knowing the above recursive relationship Eq.(25) with Eq.(26), we can directly evaluated the values of $U_k(x)$ and $V_k(x)$.Explicit form for $U_k(x)$ and $V_k(x)$ are obtained for n order of approximation. For simplicity should be omitte here.

The inverse transform of of the set of values $U_k(x)$ and $V_k(x)$ gives n term approximate solutions as

$$u^*(x,t)=\sum_{k=0}^{n}U_k(x)t^k, \tag{28}$$
$$v^*(x,t)=\sum_{k=0}^{n}V_k(x)t^k. \tag{29}$$

The approximate solutions of $u(x,t)$ and $v(x,t)$ are readily found to be

$$u(x,t)=lim_{n\longrightarrow\infty}u_n^*(x,t), i=1,2,....,n \tag{30}$$

$$v(x,t)=lim_{n\longrightarrow\infty}v_n^*(x,t), i=1,2,....,n \tag{31}$$

The numerical behavior of the approximate solutions of $u(x,t)$ and $v(x,t)$ by RDTM with different values of fractional time derivative order α are shown graphically see Figs.(2.(a-d)), with the exact solutions are shown graphically in Figs.(2.(e,f)) for a different values of time t with a fixed values of $k=0.1$,$\lambda=1.5$ and $b_1=0.1$, which proofs the two solutions are quite good.It is to be noted that the exact solutions of $u(x,t)$ and $v(x,t)$.

$$u(x,t)=\frac{b_1}{2k}+ktanh[k\xi],$$
$$v(x,t)=\frac{\lambda}{2}(1+\frac{b_1}{2})+b_1tanh[k\xi], \tag{32}$$

with $\xi=x+\frac{1}{4}[-4k^2-6\lambda+\frac{6k\lambda}{b_1}+\frac{3b_1^2}{k^2}]t$, where k, $b_1\neq 0$, and λ is arbitrary constants.

4 Conclusions

In this work, the fractional RDTM method have been used for finding approximate and numerical solutions of two coupled systems of fractional nonlinear differential equations, namely, the coupled MKdV equations and coupled system of nonlinear physical equations with initial conditions. The approximate solutions resulting from the RDTM and the effects of time fractional order α are shown graphically. The results obtained from the fractional analysis appear to be general since the obtained solutions go back to the classical one when $\alpha = 1$.

The proposed method was clearly very efficient and powerful technique in finding the solutions of the proposed time fractional nonlinear differential equations. It is clear that this method avoids liberalization and biologically unrealistic assumptions, and provides an efficient numerical solution.

A clear conclusion can be drawn from the numerical results that the RDTM provides highly accurate numerical soultions without spatial discretization for nonlinear differential equations and the advantage of the decomposition methodology displays a fast convergence of the solutions.

Finally, it is worth noting that the RDTM is used to solve nonlinear fractional time derivative equations. This leads one to ask whether the approach can be extended to deal with nonlinear equations with fractional space-derivative. The answer of these questions is our task in future work.

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