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The Structure of Fractal Interpolation Curve in Plane

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Abstract: Fractal interpolation provides an efficient way to describe data that have an irregular or self-similar structure. Fractal interpolation literature focuses mainly on function which are linearly ordered on data points with their abscissa. In practice, however, it is often useful to make curves by fractal interpolation technique. The curve is as important as function. After reviewing existing methods, this paper introduce a new method for curve fitting using fractal interpolation. The continuous fractal interpolation curve is constructed by recursion fractal interpolation method. For a column of discrete points, which are distributed on a plane, are divided into K point sets of type X or Y. Then a class of recursive iterative function system is presented. It is proved that an attractor of the iterative system of the recursive function is the continuous curve of a definite discrete point column. Then, an example and the image of recursion fractal interpolation curve are presented.

Keywords: Recursive iterative function system; Recursive fractal interpolation curve; Curve algorithm

1 Introduction

Recursive fractal interpolation has been developed as an alternative interpolation technique suitable for capturing data with inherent fractal structure, i.e. details at different scales or some degree of self-similarity. In contrast to traditional interpolation, which is built on elementary functions such as polynomials, recursive fractal interpolation is based on the theory of iterated function systems producing interpolants that are convenient for fitting physical or experimental data.

In 1981, Hutchison [1] established the basic framework of fractal geometry. Falconer [2, 3], Edgar [4] studied fractal interpolation function, which is the basis of fractal geometry. In 1989, on the basis of the original fractal interpolation, Barnsley [5] firstly have introduced the recursive fractal interpolation function and given the formula of calculating box dimension of the image of recursion fractal interpolation function on the plane. The recursive fractal interpolation function is more flexible then traditional interpolation function. And it is superior for developing a kind of fractal interpolation function. It is more complex and uncertainty. Barnsley [6–8] systematically studied the iterative function system (IFS). These studies have supported the scoring geometry in theory and these have a wider application space in life. Fractal interpolation literature focuses on function whose data points are linearly ordered with respect to their abscissa. And the interpolation function is a function of (usually) non-integral dimension. In practice, however, there are many cases which the data are suitable for fractal interpolation but define a curve rather than a function, e.g. when models coastlines or cloud edge. Fractal interpolation curve is a fractal geometry firstly proposed in image compression, the smooth curve and curved surface fitting and other fields of research shows the unique advantages. So, it is useful to extend fractal interpolation to include curves as well as functions. This issue have been studied by some professor, but not fully addressed so far. Methods based on generalizations to higher dimensions were introduced in [9–11]. The use of index coordinates was suggested in [12]. Non-affine fractal interpolation has employed in [13]. Various combinations of IFS models and free form curves were proposed in [14] and [15]. A method of data fitting by means of fractal interpolation functions has proposed in [16]. A family of tension trigonometric curves analogous to those of cubic Bzier curves was presented in [17]. Some properties of the proposed curves were discussed. We provide an efficient interpolating method based on the tension trigonometric splines with various propertiessuch as partition of unity; geometric invariance and convex hull property; etc. The new interpolating method is applied to construct curves and surfaces. Manousopoulos use a reversible

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transformation to the data points. Then a Fractal Interpolation Function (FIF) was constructed as usual and its attractor is transformed back to the original coordinates in order to obtain a curve that interpolates the original points in [18].

In this paper we review existing approaches in this area and introduce a new method for curve fitting by recursive fractal interpolation. The new method is more simple and economical than the existing ones. It is more suitable for practical applications. Then an example is given to construct the recursive fractal interpolation curve by this method.

2 Recursive iterative function system

This section mainly introduces the construction method of the iterative function system. First of all, we will construct the compression mapping in four cases, and then we obtain the corresponding iterative function system by solving the equations.

We let point set $A = \{(x_i, y_i) : i = 0, 1, 2, ..., N\}$ be an ordered set of points on the plane. The ordered point set A is divided into K segments(K subsets) in order to apply recursive fractal interpolation method to construct fractal curve on plane which go through each point of A.

$$A_k = \{(x_i, y_i) : i = N_{k-1}, N_{k-1}+1, N_{k-1}+2, ..., N_k\}, k = 1, 2, ..., K,$$

in which $N_0 = 0, N_K = N$. They meet the conditions (1) or (2) in each subset A_k:
(1) The horizontal coordinates are strictly monotonous, $(x_{i+1} - x_i)(x_i - x_{i-1}) > 0$, $i = N_{k-1}+1, N_{k-1}+2, ..., N_{k-1}$, The point set A_k is called a subset of type X;
(2) The ordinate is strictly monotonous, $(y_{i+1} - y_i)(y_i - y_{i-1}) > 0$, $i = N_{k-1}+1, N_{k-1}+2, ..., N_{k-1}$, The point set A_k is called a subset of type Y.

We let G be the set of all the continuous curves f in the plane that go through the points $(x_i, y_i), i = 1, 2, 3, ..., N$. If A_k is a subset of type X, a section f_k of curve f, which corresponds to the point seton A_k, is a continuous function (recorded as $f_k(x)$) defined on the interval $I_k = \left[x_{N_{K-1}}, x_{N_K}\right]$ or $I_k = \left[x_{N_K}, x_{N_{K-1}}\right]$, $f_k = \{(x, f_k(x))|x \in I_k\}$. If A_k is a subset of type Y, a section f_k of curve f, which corresponds to the point set A_k, is a continuous function (recorded as $f_k(y)$) defined on the interval $I_k = \left[y_{N_{K-1}}, y_{N_K}\right]$ or $I_k = \left[y_{N_K}, y_{N_{K-1}}\right]$, $f_k = \{(f_k(y), y)|y \in I_k\}$, then $\cup_{k=1}^{K} f_k$. For any two adjacent points (x_{i-1}, y_{i-1}) and (x_i, y_i) that belong to a point set A_{k_1}, and any two points $(x_{m(i)}, y_{m(i)})$ and $(x_{M(i)}, y_{M(i)})$ that belong to A_{k_2}. Let $f_{mM(i)}$ be a segment of continuous curve f between $(x_{m(i)}, y_{m(i)})$ and $(x_{M(i)}, y_{M(i)})$. Then, we construct the linear mapping ω_i of curve $f_{mM(i)}$ in four cases, and make

$$\omega_i\left(x_{m(i)}, y_{m(i)}\right) = (x_{i-1}, y_{i-1}), \ \ \omega_i\left(x_{M(i)}, y_{M(i)}\right) = (x_i, y_i). \tag{1}$$

1. A_{k_1} and A_{k_2} are all subsets of type X . We define $\omega_i : D_i \times R \to I_i \times R$ mapping, among them

$$D_i = [x_{m(i)}, x_{M(i)}] \text{ or } D_i = [x_{M(i)}, x_{m(i)}], \ \ I_i = [x_{i-1}, x_i] \text{ or } I_i = [x_i, x_{i-1}].$$

$$\omega_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(x) \\ F_i(x,y) \end{pmatrix} = \begin{pmatrix} a_i & 0 \\ b_i & s_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

2. A_{k_1} is a subset of type X. A_{k_2} is a subset of type Y. We define $\omega_i : D_i \times R \to I_i \times R$ mapping. Among them

$$D_i = [y_{m(i)}, y_{M(i)}] \text{ or } D_i = [y_{M(i)}, y_{m(i)}], \ \ I_i = [x_{i-1}, x_i] \text{ or } I_i = [x_i, x_{i-1}].$$

$$\omega_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} L_i(y) \\ F_i(x,y) \end{pmatrix} = \begin{pmatrix} 0 & a_i \\ s_i & b_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} e_i \\ f_i \end{pmatrix}.$$

3. A_{k_1} is a subset of type Y. A_{k_2} is a subset of type X. We define $\omega_i : D_i \times R \to I_i \times R$ mapping. Among them

$$D_i = [x_{m(i)}, x_{M(i)}] \text{ or } D_i = [x_{M(i)}, x_{m(i)}], \ \ I_i = [y_{i-1}, y_i] \text{ or } I_i = [y_i, y_{i-1}].$$

$$\omega_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_i(x,y) \\ L_i(y) \end{pmatrix} = \begin{pmatrix} b_i & s_i \\ a_i & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_i \\ e_i \end{pmatrix}.$$

4. A_{k_1} and A_{k_2} are all subsets of type Y. We define $\omega_i : D_i \times R \to I_i \times R$ mapping. Among them

$$D_i = [y_{m(i)}, y_{M(i)}] \text{ or } D_i = [y_{M(i)}, y_{m(i)}], \quad I_i = [y_{i-1}, y_i] \text{ or } I_i = [y_i, y_{i-1}].$$

$$\omega_i \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} F_i(x,y) \\ L_i(y) \end{pmatrix} = \begin{pmatrix} s_i & b_i \\ 0 & a_i \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} f_i \\ e_i \end{pmatrix}.$$

Among them, $s_i, i = 1, 2, ..., N$ is a set of given real numbers, a_i, b_i, e_i, f_i can be determined by condition (1).

It is easy to prove, for any $f \in G$, we have $\omega(f) = \cup_{i=1}^{N} \omega_i \left(f_{mM(i)}\right) \in G$. So, $\{G; \omega_i, i = 1, 2, ..., N\}$ is a recursive iterative function system defined on G.

3 Recursive iterative function series invariant set

This section has given and proven that the attractor of the iterative function system in the first chapter is the continuous curve. In other words, the recursion fractal interpolation curve of interpolation points can be obtained by this method.

Theorem 1 *For any $f \in G$, we make $\|f\| = \max\limits_{1\le k\le K} \max\limits_{t\in I_k} |f_k(t)|$. Then, $\|.\|$ is a norm of G, and $(G; \|.\|)$ is complete metric space.*

Proof.

1. First, let's prove that $\|.\|$ is a norm G

   (a) Qualitative
   Because $|f_k(t)| \ge 0$, so $\|f\| = \max\limits_{1\le k\le K} \max\limits_{t\in I_k} |f_k(t)| \ge 0$. If and only if $|f_k(t)| = 0$,
   $$\|f\| = \max_{1\le k\le K} \max_{t\in I_k} |f_k(t)| = 0.$$

   (b) Homogeneity
   $$\|cf\| = \max_{1\le k\le K} \max_{t\in I_k} |cf_k(t)| = \max_{1\le k\le K} \max_{t\in I_k} |f_k(t)||c| = |c| \max_{1\le k\le K} \max_{t\in I_k} |f_k(t)| = |c|\|f\|.$$

   (c) Triangle inequality
   $$\|f+g\| = \max_{1\le k\le K} \max_{t\in I_k} |f_k(t)+g_k(t)| \le \max_{1\le k\le K} \max_{t\in I_k} |f_k(t)| + \max_{1\le k\le K} \max_{t\in I_k} |g_k(t)| = \|f\| + \|g\|.$$

2. Then, let's prove that $(G; \|.\|)$ is a complete metric space.
   From (1) we know that $(G; \|.\|)$ is the norm space. So it's a metric space. We demonstrate the completeness of the metric space nest. Let $f_n \in G, n = 1, 2, 3...$ be a cauchy sequence converging to f. So for $\forall \varepsilon > 0, \exists N \in 1, 2, 3...$, when $n > N$, constant has $|f_n - f| < \varepsilon$, $|f_n - f| \to 0 (n \to \infty)$.

3. So
   $$\|f_n - f\| = \max_{1\le k\le K} \max_{t\in I_k} |f_{n(k)} - f_k(t)| \to 0(n \to \infty), (G; \|.\|)$$
   is a complete metric space. ■

Theorem 2 *When $s = \max\{|s_i|; i = 1, 2, ..., N\} < 1$, there is a curve element f in space G. So f is the invariant set of the recursive iterative function $\{G; \omega_i, i = 1, 2, ..., N\}$, $f = \omega(f) = \cup_{i=1}^{N} \omega_i(f_{mM(i)})$. And continuous curve f is interpolated by the interpolation points $A = \{(x_i, y_i) : i = 0, 1, 2, ..., N\}$. We call f the fractal interpolation curve which is generated by the recursive iterative function $\{G; \omega_i, i = 1, 2, ..., N\}$.*

Proof. For every $f, g \in G$, we let two adjacent points (x_{i-1}, y_{i-1}) and (x_i, y_i) both belong to A_{k_1}, while $(x_{m(i)}, y_{m(i)})$ and $(x_{M(i)}, y_{M(i)})$ both belong to A_{k_2}. If A_{k_1} is a subset of type Y and A_{k_2} is a subset of type X. According to the mapping (1), the curve $\omega(f)$ and $\omega(g)$ between (x_{i-1}, y_{i-1}) and (x_i, y_i) are function images:

$$F(y) = s_i f(L_i^{-1}(y)) + b_i L_i^{-1}(y) + f_i,$$

and

$$G(x) = s_i f(L_i^{-1}(y)) + b_i f(L_i^{-1}(y)) + f_i.$$

They are graphs on the interval $I_i = [y_{i-1}, y_i]$ or $I_i = [y_i, y_{i-1}]$, Then

$$\max_{y \in I_i} |F(y) - G(y)| = |s_i| \max_{y \in I_i} |f(L_i^{-1}(y)) - f(L_i^{-1}(y))| \leq s\|f - g\|.$$

Similarly, when A_{k_1} and A_{k_2} are subsets of other types, similar inequalities can be founded. Thus $\|\omega(f) - \omega(g)\| \leq s\|f - g\|$. That, ω is a compact mapping on the complete metric space G. So we have element f in space G, and f is the invariant set of the recursive iterative function system $\{G; \omega_i, i = 1, 2, ..., N\}$. $f = \omega(f) = \cup_{i=1}^{N} \omega_i(f_{mM(i)})$. ■

From the theorem, the invariant set of the recursive iterative function system, which is constructed earlier, is the continuous curve through the point set. $A = \{(x_i, y_i) : i = 0, 1, 2, ..., N\}$. We call it the fractal interpolation curve.

4 The algorithm of fractal interpolation curve

According to the above theory, for a given interpolation point $A = \{(x_i, y_i) : i = 0, 1, 2, ..., N\}$ and iteration relationship, the iterative function system $\{\omega_i, i = 1, 2, ..., N\}$ can be determined. Let f be the fractal interpolation curve generated by the recursive iterative function $\{G; \omega_i, i = 1, 2, ..., N\}$. So the interpolation points $A \subset f$, and $f = \omega(f)$. So $\omega(A) \subset f$. Similarly $\omega^{(n)}(A) \subset f$. So, when $M(i) - m(i) \geq 2, i = 1, 2, ..., N$, the number of iterations increases. The insertion points on the curve increase. So when the number of iterations is sufficiently large, the precision demand curve can be obtained.

Example: As shown in figure 1, data points are

$$A_0(0, 10),\ A_1(2, 8),\ A_2(4, 4),\ A_3(6, 2),\ A_4(10, 4),\ A_5(6, 6),\ A_6(4, 4),\ A_7(2, 0).$$

Among them, type X subsets are: $A_1\{(0, 10), (2, 8), (4, 4), (6, 2)\}$, $A_3\{(6, 6), (4, 4), (2, 0)\}$.
Type Y subset is: $A_2\{(6, 2), (10, 4), (6, 6)\}$.

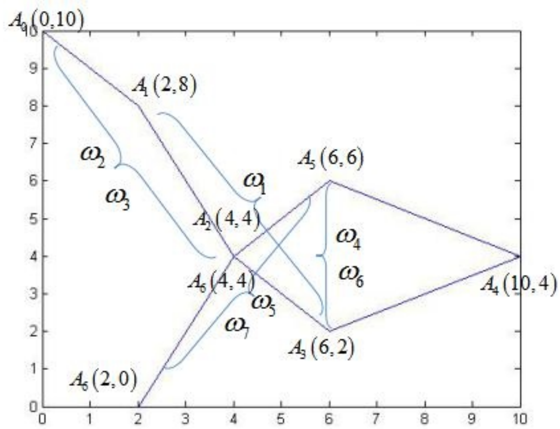
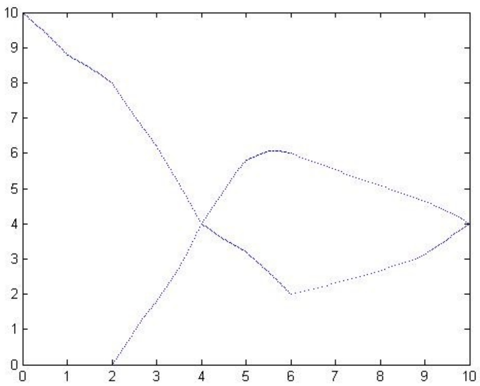


Figure 1: The interpolation points.

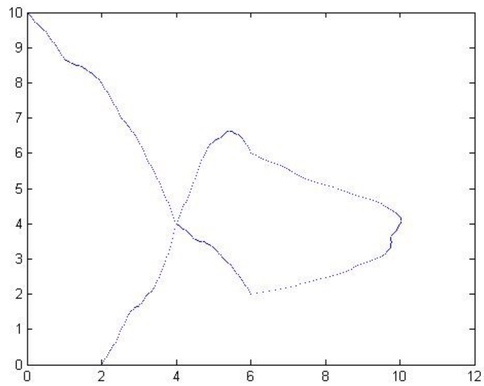
The data points and interpolation points corresponding to the iteration are follows:

Table1 Interpolation point iteration relationship.

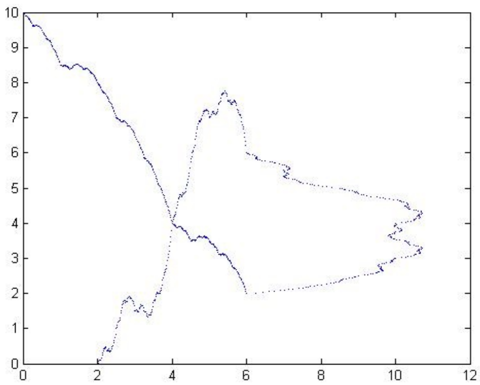
$A_{i-1}(x_{i-1}, y_{i-1}),\ \ A_i(x_i, y_i)$	$A_{l(i-1)}(x_{l(i-1)}, y_{l(i-1)}),\ \ A_{r(i)}(x_{r(i)}, y_{r(i)})$
$A_1(0,10),\ \ A_2(2,8)$	$A_0(2,8),\ \ A_2(6,2)$
$A_1(2,8),\ \ A_2(4,4)$	$A_0(0,10),\ \ A_2(4,4)$
$A_2(4,4),\ \ A_3(6,2)$	$A_0(0,10),\ \ A_2(4,4)$
$A_3(6,2),\ \ A_4(10,4)$	$A_3(6,2),\ \ A_5(6,6)$
$A_4(10,4),\ \ A_5(6,6)$	$A_5(6,6),\ \ A_7(2,0)$
$A_5(6,6),\ \ A_6(4,4)$	$A_3(6,2),\ \ A_5(6,6)$
$A_5(4,4),\ \ A_6(2,0)$	$A_3(6,6),\ \ A_5(2,0)$



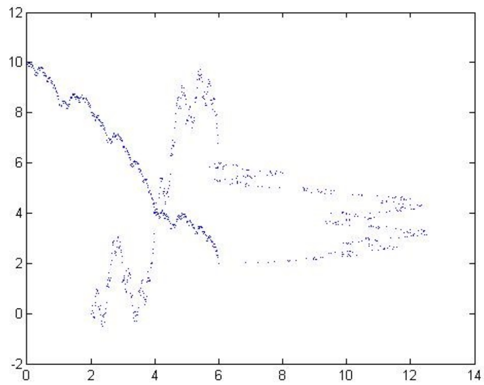
(a) $s_i = 1/5$.



(b) $s_i = 1/3$.



(c) $s_i = 1/2$.



(d) $s_i = 2/3$.

Figure 2: (a) In the case of $s_i = 1/5$, the image is obtained through 6 iterations. (b) In the case of $s_i = 1/3$, the image is obtained through 6 iterations. (c) In the case of $s_i = 1/2$, the image is obtained through 6 iterations. (d) In the case of $s_i = 2/3$, the image is obtained through 6 iterations.

We can get

$$\omega_1 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ (-1+3s_1)/2 & s_1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} -1 \\ 11-11s_1 \end{pmatrix}.$$
$$\omega_2 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ -1+3s_2/2 & s_2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 2 \\ 8-10s_2 \end{pmatrix}.$$
$$\omega_3 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ (1+3s_3)/2 & s_3 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4 \\ 4-10s_3 \end{pmatrix}.$$
$$\omega_4 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} s_4 & 1 \\ 0 & 0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4-6s_4 \\ 1 \end{pmatrix}.$$
$$\omega_5 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 1-1.5s_5 & s_5 \\ -0.5 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 4+3s_5 \\ 7 \end{pmatrix}.$$
$$\omega_6 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & -0.5 \\ s_6 & -0.5 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 7 \\ 7-6S_6 \end{pmatrix}.$$
$$\omega_7 \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0.5 & 0 \\ 1-3s_7/2 & s_7 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 1 \\ 2+3S_7 \end{pmatrix}.$$

And because $S(1) = \text{diag}\{s_1, s_2, s_3, s_4, s_5, s_6, s_7\}$, and through $Q = CS(1)$, we get the strong connected branch $\langle T^1 \rangle = \{V_1, V_2, V_3\}$, $\langle T^3 \rangle = \{V_4, V_5, V_6, V_7\}$ of $D(Q)$. $Q = CS(1)$ The dimension of the image is going to change with the variation. This is the next question.

5 Conclusion

In this paper, a new method for constructing fractal curves is given. The fractal interpolation curves of given points are constructed by recursive fractal interpolation method. This is a more simple and accurate fractal interpolation curve construction method. This method does not have to transform the interpolation point, but only choose the appropriate type of iteration function in the four iteration functions. Then, these iterative functions are constructed to form an iterative function. And the fractal curve can be obtained by finding the attractor.

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