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A Note on Orthonormal Wavelet Bases Adjusting to a Quasi-Distance

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Abstract: In this note, we construct explicitly the orthonormal wavelets of a homogeneous type space relating to Tricomi operator on $\mathbb{R}^2_+$. The proof is elementary and it follows along the line of Daubechies. The supports of these wavelets are dyadic squares in terms of a quasi-distance. Comparing with classical spaces $\mathbf{V}_j$, the spaces $\mathcal{V}_j$ constructed in this note have a new type "multiscale analysis" of $\mathbb{R}^2_+$. The geometry of these dyadic squares will be useful in calculation as Coifman, Jones and Semmes have done in verifying Shur's criterion.

Keywords: Wavelets; Tricomi operator; Homogeneous type space

1 Introduction

Recall the Haar wavelets on the Euclidean space $\mathbb{R}^2$. Let

$$\phi(x) = \begin{cases} 1 & 0 \le x < 1, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

be the characteristic function on interval $[0, 1)$ and

$$\bar{\Phi}_{j;k,l}(x, y) = 2^{-j}\phi(2^{-j}x - k)\phi(2^{-j}y - l), \quad (j, k, l) \in \mathbb{Z}^3.$$

Then the sequence of spaces $\mathbf{V}_j$ =Span$\{\bar{\Phi}_{j;k,l} : (k, l) \in \mathbb{Z}^2\}$, $j \in \mathbb{Z}$ constitute a "multiscale analysis" of $L^2(\mathbb{R}^2)$ with the following properties:

(1) $\cdots \subset \mathbf{V}_2 \subset \mathbf{V}_1 \subset \mathbf{V}_0 \subset \mathbf{V}_{-1} \subset \mathbf{V}_{-2} \subset \cdots$;

(2) $\bigcap_{j\in\mathbb{Z}} \mathbf{V}_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} \mathbf{V}_j$ dense in $L^2(\mathbb{R}^2)$;

(3) $f \in \mathbf{V}_j \iff f(2^j\cdot, 2^j\cdot) \in \mathbf{V}_0$;

(4) $f \in \mathbf{V}_0 \implies f(\cdot - k, \cdot - l) \in \mathbf{V}_0$ for all $(k, l) \in \mathbb{Z}^2$.

The notion "multiscale analysis" is due to Mallat and Meyer [8, 16, 17]. Let $\mathbf{W}_j$ be the complement space in $\mathbf{V}_{j-1}$ of $\mathbf{V}_j$. It follows that $\mathbf{W}_j$ consists three type orthonormal bases, i.e., the Haar wavelets

$$\bar{\Psi}^{\lambda}_{j;k,l}(x, y) = 2^{-j}\Psi^{\lambda}(2^{-j}x - k, 2^{-j}y - l) : (k, l) \in \mathbb{Z}^2, \lambda = h, v, d$$

with

$$\begin{aligned} \Psi^h(x, y) &= \phi(x)\psi(y), \\ \Psi^v(x, y) &= \psi(x)\phi(y), \\ \Psi^d(x, y) &= \psi(x)\psi(y) \end{aligned} \tag{2}$$

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and

$$\psi(x) = \begin{cases} 1 & 0 \le x < \frac{1}{2}, \\ -1 & \frac{1}{2} \le x < 1, \\ 0 & \text{otherwise,} \end{cases} \tag{3}$$

the Haar function supported on interval $[0, 1]$. The superscripts h, v, d stands for "horizontal", "vertical" and "diagonal" wavelets, see [11]. The supports of these wavelets have the same dyadic length 2^j both in the horizontal and vertical directions.

The above dyadic square is measured by the classical Euclidean distance. A space of homogeneous type (X, ρ, μ) is a set X together with a quasi-metric ρ and a nonnegative measure μ on X such that $0 < \mu(B(x, r)) < \infty$ for all $x \in X$ and $r > 0$, and so that there exists $C < \infty$ such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \le C\mu(B(x, r)).$$

Here $B(x, r) = \{y \in X : \rho(x, y) < r\}$. Similar to the case of classical Euclidean spaces, Christ [4] apply the stopping time arguments to construct a family of open sets of diameter roughly dyadic size for arbitrary homogeneous type space and prove a variant of the Tb-theorem which is applicable to a question about analytic capacity of subsets of the complex plane, see also David [11] for another construction. Coifman, Jones and Semmes [5] use a pseudo-orthogonal wavelet to show the L^2 boundedness of Cauchy integrals on Lipschitz curves. Using Carleson's theorem on Carleson measure, David [11] give a proof of the Tb-theorem on $\mathbb{R}^n$ with b replaced by para-accretive functions. In [5] and [11], the dyadic intervals and dyadic cubes are measured by the classical Euclidean metric. For study of homogeneous type spaces, see, for example, [10, 12, 18, 23].

In this note, we construct explicitly the orthonormal wavelets of a homogeneous type space relating to Tricomi operator on $\mathbb{R}^2_+$. Our proof is elementary, it follow along the line of Daubechies [8, 9]. The supports of these wavelets are dyadic squares in terms of a quasi-distance. We show that spaces supported on these dyadic squares have a new type "multiscale analysis" of $\mathbb{R}^2_+$. The geometry of these dyadic squares will be useful in calculation as Coifman, Jones and Semmes [5] have done in verifying Shur's criterion.

2 Wavelets on a homogeneous type space

Consider the Tricomi operator

$$T = y\partial_{xx} + \partial_{yy} \tag{4}$$

on the upper plane $\mathbb{R}^2_+ = \{(x, y) \in \mathbb{R}^2, y > 0\}$. The operator degenerates on the line $y = 0$, is invariant under the translation of the variable x and has a natural dilation $\delta(x, y) = (\delta x, \delta^{3/2}y)$ such that $T(u(\delta(x, y))) = \delta^3(Tu)(\delta(x, y))$. Tricomi type operator is well studied, such as [1, 13–15, 20–22]. Tricomi operator can be written as the sum of squares of vector fields, i.e., $T = X_1^2 + X_2^2$ with $X_1 = y^{1/2}\partial_x$ and $X_2 = \partial_y$. Observe that X_1 is only Hölder continuous in $\mathbb{R}^2_+$. There is a natural distance associating vector fields, see [19] for smooth vector fields and [13] for Hölder continuous ones. For our purposes it is useful to introduce the following quasi-distance ρ relating to T

$$\rho((x, y), (x_1, y_1)) = \min\{y^{-1/2}|x - x_1|, |x - x_1|^{2/3}\} + |y - y_1| \tag{5}$$

for $(x, y), (x_1, y_1) \in \mathbb{R}^2_+$, see [13, 14] for definition and the properties and [15] for Grushin operator of the form $y^2\partial_{xx} + \partial_{yy}$. The triple $(\mathbb{R}^2_+, \rho, \mathfrak{L})$ with the quasi-distance ρ and the Lebesgue measure $\mathfrak{L}$ on $\mathbb{R}^2_+$ is a homogeneous type space in the sense of Coifman and Weiss [6, 7].

With the notation of dyadic intervals in classical Euclidean metric

$$I_{j;k} = [2^j k, 2^j(k + 1)),$$

we define the "dyadic square" corresponding to ρ to be the rectangles

$$R_{j;,k,l} = I_{j+m;k} \times I_{j;l}, \quad (k, l) \in \mathbb{Z} \times \mathbb{Z}_+,$$

where

$$m = m(j, l) \tag{6}$$

is the integer such that

$$2^j(l+1) \in (2^{2(m-1)}, 2^{2m}].$$

Here $\mathbb{Z}_+$ represents the set of nonnegative integers. In the following we usually use m instead of $m(j, l)$ for simplicity of notation.

As in the classical case we use the tensor product of two one-dimension characteristic function to construct wavelets. Let ϕ and ψ be defined in the introduction. Write

$$\Phi(x, y) = \phi(x)\phi(y), \quad \Psi^t(x, y) = 2^{-1/2}\psi(2^{-1}x)\phi(y). \tag{7}$$

The superscript t stands for "transition" wavelets. Write, for $(j, k, l) \in \mathbb{Z}^2 \times \mathbb{Z}_+$,

$$\Phi_{j;k,l} = T_{j;k,l}(\Phi), \tag{8}$$

$$\Psi^\lambda_{j;k,l} = T_{j;k,l}(\Psi^\lambda) \quad \text{for } \lambda = h, v, d, \quad \Psi^t_{j;k} = T_{j;k,0}(\Psi^t), \tag{9}$$

with

$$T_{j;k,l}(u(x)v(y)) = 2^{-(2j+m)/2}u(2^{-j-m}x - k)v(2^{-j}y - l).$$

Remark 1 *Set $\mathcal{V}_j$ =Span$\{\Phi_{j;k,l} :\ (k, l) \in \mathbb{Z} \times \mathbb{Z}_+\}$, $j \in \mathbb{Z}$. We see that the sequence of spaces constitute a "multiscale analysis" of $L^2(\mathbb{R}^2_+)$ having the following properties:*

(1) $\cdots \subset \mathcal{V}_2 \subset \mathcal{V}_1 \subset \mathcal{V}_0 \subset \mathcal{V}_{-1} \subset \mathcal{V}_{-2} \subset \cdots$;

(2) $\bigcap_{j\in\mathbb{Z}} \mathcal{V}_j = \{0\}$, $\bigcup_{j\in\mathbb{Z}} \mathcal{V}_j$ *dense in* $L^2(\mathbb{R}^2_+)$;

(3) $f \in \mathcal{V}_j \iff f(2^{j+m}\cdot, 2^j\cdot) \in \mathcal{V}_0$;

(4) $f \in \mathcal{V}_0 \implies f(\cdot - k, \cdot) \in \mathcal{V}_0$ *for all* $k \in \mathbb{Z}$.

Comparing with classical "multiscale analysis" in $\mathbb{R}^2$, one finds that: (i) there is an additional factor m in (3); (ii) the property (4) here is only translate invariant in the first variable x.

We state some orthonormal and merge properties of these functions.

Lemma 2 *(i) The following wavelets are orthonormal,*

$$\Psi^\lambda_{j;k,l},\ \Psi^t_{j;k},\ \lambda = h, v, d,\ j, k \in \mathbb{Z},\ l \in \mathbb{Z}_+. \tag{10}$$

(ii) For $(j, k, l) \in \mathbb{Z}^2 \times \mathbb{Z}_+$ satisfying $j = 2j' + 1$ or $l > 0$, and $c_{s,t} \in \mathbb{R}$, we have

$$2^{-1}\sum_{s,t=0}^{1} c_{s,t}\Phi_{j;2k+s,2l+t} = c\Phi_{j+1;k,l} + \sum_{\lambda=h,v,d} c_\lambda \Psi^\lambda_{j+1;k,l} \tag{11}$$

with

$$\begin{aligned}
c &= (c_{0,0} + c_{1,0} + c_{0,1} + c_{1,1})/4,\\
c_h &= (-c_{0,0} - c_{1,0} + c_{0,1} + c_{1,1})/4,\\
c_v &= (-c_{0,0} + c_{1,0} - c_{0,1} + c_{1,1})/4,\\
c_d &= (c_{0,0} - c_{1,0} - c_{0,1} + c_{1,1})/4.
\end{aligned}$$

(iii) For $(j,k) \in \mathbb{Z}^2$ and $c_s, c_{s,1} \in \mathbb{R}$, we have

$$2^{-3/2}\sum_{s=0}^{3} c_s \Phi_{2j;4k+s,0} + 2^{-1}\sum_{s=0}^{1} c_{s,1}\Phi_{2j;2k+s,1}$$
$$= c\Phi_{2j+1;k,0} + \sum_{\lambda=h,v,d} c_\lambda \Psi^{\lambda}_{2j+1;k,0} + \sum_{s=0}^{1} d_s \Psi^{t}_{2j;k+s} \tag{12}$$

with

$$d_0 = (c_0 - c_1)/2, \qquad d_1 = (c_2 - c_3)/2$$

and c, c_h, c_v, c_d as defined in (ii) with

$$c_{0,0} = (c_0 + c_1)/2, \qquad c_{1,0} = (c_2 + c_3)/2.$$

Proof. (i). By direct calculation, orthonormality is easy to establish.

(ii). By assumption, we have $m(j, 2l) = m(j, 2l + 1) = m_0$. For $(s,t) = (0,0), (1,0), (0,1), (1,1)$, $\Phi_{j;2k+s,2l+t}$ support on $I_{j+m_0;2k} \times I_{j;2l}$, $I_{j+m_0;2k+1} \times I_{j;2l}$, $I_{j+m_0;2k} \times I_{j;2l+1}$, $I_{j+m_0;2k+1} \times I_{j;2l+1}$ respectively. See Fig.1. The union of these four rectangles is $I_{j+1+m_0;k} \times I_{j+1;l}$. Then explicit calculations yield (11).

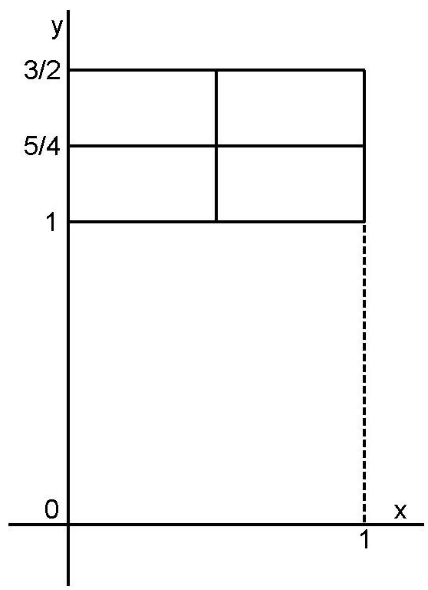

Fig.1. $j = -2,\ k = 0,\ 2l = 4 > 0,\ m = 1.$

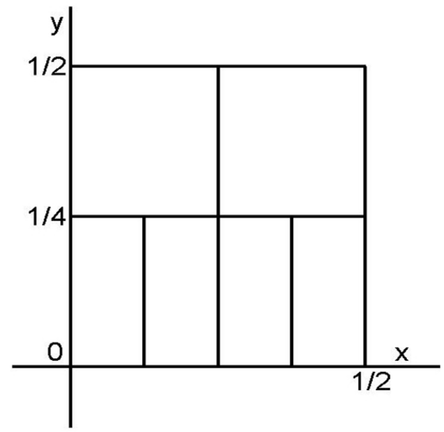

Fig.2. $2j = -2,\ k = 0,\ l = 0,\ m = -1.$

(iii). The definition of Ψ^t yields

$$\sum_{s=0}^{3} c_s \Phi_{2j;4k+s,0} = \sum_{s=0}^{1} c_{s,0}(\Phi_{2j;4k+2s,0} + \Phi_{2j;4k+2s+1,0}) + \sum_{s=0}^{1} d_s \Psi^{t}_{2j;k+s}. \tag{13}$$

We have $m(2j, 0) = j$. For $s = 0, 1$, $\Phi_{2j;4k+2s,0} + \Phi_{2j;4k+2s+1,0}$ support on $(I_{3j;4k} \times I_{2j;0}) \cup (I_{3j;4k+1} \times I_{2j;0}) = I_{3j+1;2k} \times I_{2j;0} = R_1$ and $(I_{3j;4k+2} \times I_{2j;0}) \cup (I_{3j;4k+3} \times I_{2j;0}) = I_{3j+1;2k+1} \times I_{2j;0} = R_2$ respectively. Since $m(2j, 1) = j + 1$, for $s = 0, 1$, $\Phi_{2j;2k+s,1}$ support on $I_{3j+1;2k} \times I_{2j;1} = R_3$, $I_{3j+1;2k+1} \times I_{2j;1} = R_4$ respectively. The union of R_i, $i = 1, 2, 3, 4$ is $I_{3j+2;k} \times I_{2j+1;0}$. See Fig. 2. Then explicit calculations yield (12). ■

Let J_1 be a positive integer sufficiently large. Define the classical dyadic squares

$$Q_{-J_1;k,l} = I_{-J_1;k} \times I_{-J_1;l}, \quad (k,l) \in \mathbb{Z} \times \mathbb{Z}_+$$

in Euclidean distance of $\mathbb{R}^2_+$. Here $I_{-J_1;k} = [2^{-J_1}k, 2^{-J_1}(k+1))$ are dyadic intervals of length 2^{-J_1}. Let $\chi_{-J_1;k,l}$, $(k,l) \in \mathbb{Z} \times \mathbb{Z}_+$ be characteristic functions on squares $Q_{-J_1;k,l}$.

Lemma 3 *Let $J_2 \geq 4J_1$ be a positive integer and*

$$g = \sum_{|k|,l<2^{J_2},l\geq 0} g_{-J_1;k,l}\chi_{-J_1;k,l} \tag{14}$$

with $g_{-J_1;k,l} \in \mathbb{R}$. Then we have

$$g = \frac{b}{2^{3n/2}}\Phi_{J_2+2n;0,0} + \Psi'_{J_2+2n} + \Psi^t_{J_2+2n} \tag{15}$$

for all $n \in \mathbb{Z}_+$, where b is a real number and Ψ'_{J_2+2n} and $\Psi^t_{J_2+2n}$ are linear combination of wavelets to be determined later.

Proof. (1). We decompose each dyadic square $Q_{-J_1;k,l}$ in Euclidean space into dyadic squares of the form $R_{j;k,l}$ in homogeneous type space $\mathbb{R}^2_+$.

Case 1: $m = m(-J_1, l) \geq 0$, that is, $2^{-J_1}(l+1) > 1/4$. In this time, subdivide dyadically each square $Q_{-J_1;k,l}$ m times in the horizontal direction such that each rectangle is of scale 2^{-J_1-m} in terms of the para-distance ρ,

$$Q_{-J_1;k,l} = \bigcup_{n=0}^{2^m-1} R_{-J_1-m;k,2^ml+n}.$$

See Fig. 3. So

$$\chi_{-J_1;k,l} = \sum_{n=0}^{2^m-1} 2^{J_1+m/2}\Phi_{-J_1-m;k,2^ml+n}. \tag{16}$$

Case 2: $m = m(-J_1, l) \leq -1$, that is, $2^{-J_1}(l+1) \leq \frac{1}{4}$. Subdivide dyadically each square $Q_{-J_1;k,l}$ m times in the vertical direction such that each rectangle is of scale 2^{-J_1} in terms of the para-distance ρ. So

$$Q_{-J_1;k,l} = \bigcup_{s=0}^{2^{-m}-1} R_{-J_1;2^{-m}k+s,l}.$$

For example, $3/64 \to 4/64$ in y direction, see Fig. 4. So

$$\chi_{-J_1;k,l} = \sum_{s=0}^{2^{-m}-1} 2^{J_1-m/2}\Phi_{-J_1;2^{-m}k+s,l}. \tag{17}$$

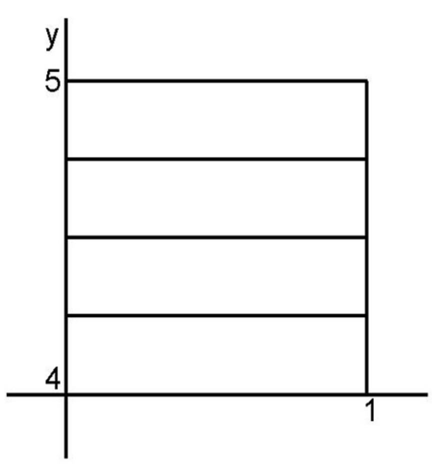

Fig.3. $J_1 = 0,\ k = 0,\ l = 4,\ m = 2.$

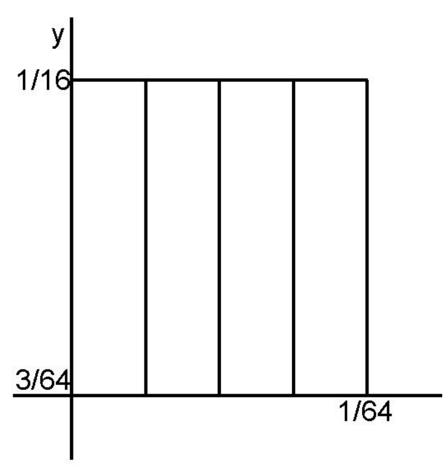

Fig.4. $J_1 = 6,\ k = 0,\ l = 3,\ m = -2.$

(2). For $l \in \mathbb{Z}_+$ such that $m = m(-J_1, l) \geq 1$, (16) implies

$$\begin{aligned}\sum_{|k|<2^{J_2}} g_{-J_1;k,l}\chi_{-J_1;k,l} &= \sum_{|k|<2^{J_2}} \sum_{n=0}^{2^m-1} g'_{-J_1;k,l}\Phi_{-J_1-m;k,2^m l+n} \\ &= \sum_{|k|<2^{J_2-1}} \sum_{n=0}^{2^{m-1}-1} \sum_{s,t=0}^{1} g'_{-J_1;k,l}\Phi_{-J_1-m;2k+s,2(2^{m-1}l+n)+t}\end{aligned}$$

for real numbers $g'_{k,l}$. Applying the results of the case (ii) of Lemma 2, the right hand side of the last equation becomes

$$\sum_{|k|<2^{J_2-1}} \sum_{n=0}^{2^{m-1}-1} \left(g^n_{-J_1;k,l}\Phi_{-J_1-m+1;k,2^{m-1}l+n} + \sum_{\lambda=h,v,d} g^{n,\lambda,1}_{-J_1;k,l}\Psi^\lambda_{-J_1-m+1;k,2^{m-1}l+n} \right)$$

Repeating such process, it follows that

$$\sum_{|k|<2^{J_2}} g_{-J_1;k,l}\chi_{-J_1;k,l} = \sum_{|k|<2^{J_2-m}} g^0_{-J_1;k,l}\Phi_{-J_1;k,l} + \Psi'_{-J_1;l} \tag{18}$$

with

$$\Psi'_{-J_1;l} = \sum_{i=1}^{m} \sum_{|k|<2^{J_2-i}} \sum_{n=0}^{2^{m-i}-1} \sum_{\lambda=h,v,d} g^{n,\lambda,i}_{-J_1;k,l}\Psi^\lambda_{-J_1-m+i;k,2^{m-i}l+n}. \tag{19}$$

Note that $J_2 \geq m = m(-J_1, l)$ for $l \leq 2^{J_2}$ by the assumption $J_2 \geq 4J_1$ and J_1 is sufficiently large.

Combining (17) and (18) together, one finds

$$g = \sum_{0\leq l<2^{J_2}} \sum_{|k|<2^{J_2-m}} a_{-J_1;k,l}\Phi_{-J_1;k,l} + \sum_{0\leq l<2^{J_2}} \Psi'_{-J_1;l}. \tag{20}$$

Using the method to get (18), one gets, for $-J_1 \leq p \leq J_2 - J_1$, $p \in \mathbb{Z}$,

$$\sum_{2^{J_1+p}\leq l<2^{J_1+p+1}} \sum_{|k|<2^{J_2-m}} a_{-J_1;k,l}\Phi_{-J_1;k,l} = \sum_{|k|<2^{J_2-m-p-J_1}} b_{p;k}\Phi_{p;k,1} + \Psi'_p$$

with Ψ'_p is a linear combination of wavelets of the form (19). By (20)

$$g = \sum_{|k|<2^{J_2-m}} a_{-J_1;k,0}\Phi_{-J_1;k,0} + \sum_{-J_1\leq p\leq J_2-J_1} \sum_{|k|<2^{J_2-m-p-J_1}} b_{p;k}\Phi_{p;k,1} + \Psi' \tag{21}$$

with Ψ' in the form (19).

(3). Without loss of generality, we assume J_1 is an odd integer, then $m(-J_1, 0) = m(-J_1, 1) = m$. Applying the results of the case (ii) of Lemma 2, we have

$$\sum_{|k|<2^{J_2-m}} (a_{-J_1;k,0}\Phi_{-J_1;k,0} + b_{-J_1;k}\Phi_{-J_1;k,1}) = \sum_{|k|<2^{J_2-m-1}} b_{-J_1+1;k,0}\Phi_{-J_1+1;k,0} + \Psi_{-J_1+1}$$

with Ψ_{-J_1+1} in the form (19). Now $-J_1 + 1$ is an even integer. Let $m = m(-J_1, 0)$, then $m(-J_1, 0) = m + 1$. Using the results of the case (iii) of Lemma 2, we get

$$\begin{aligned}&\sum_{|k|<2^{J_2-m-1}} (b_{-J_1+1;k,0}\Phi_{-J_1+1;k,0} + b_{-J_1+1;k}\Phi_{-J_1+1;k,1}) \\ &= \Psi_{-J_1+2} + \sum_{|k|<2^{J_2-m-2}} b_{-J_1+2;k,0}\Phi_{-J_1+2;k,0} + \sum_{|k|<2^{J_2-m-2}} \sum_{s=0}^{1} d_{-J_1+2;k,s}\Psi^t_{-J_1+2;k+s}\end{aligned}$$

with Ψ_{-J_1+2} in the form (19).

Repeating such process, (21) implies, for $-J_1 \le 2q \le J_2 - J_1$,

$$g = b_{2q;k,0}\Phi_{2q;k,0} + \sum_{2q \le p \le J_2 - J_1} \sum_{|k| < 2^{J_2 - m - 2q - J_1}} b_{p;k}\Phi_{p;k,1} + \Psi'_{2q+1} + \Psi^{t\prime}_{2q+1} \tag{22}$$

with Ψ'_{2q+1} in the form (19) and $\Psi^{t\prime}_{2q+1}$ in the same form but with Ψ^λ replaced by Ψ^t. We see that $m(2q, 0) = q$ and $m(2q, 1) = q + 1$. Then for all q sufficiently large, the sum for k in (22) has only one term with $k = 0$. Taking $q = J_1$, the assumption $J_2 \ge 4J_1$ yields

$$g = b_{2J_1;k,0}\Phi_{2J_1;k,0} + \sum_{2J_1 \le p \le J_2 - J_1} b_{p;k}\Phi_{p;k,1} + \Psi'_{2J_1+1} + \Psi^{t\prime}_{2J_1+1}.$$

Without loss of generality, we assume J_2 is an odd integer. Arguing in the same way, we have

$$g = b\Phi_{J_2;0,0} + \Psi'_{J_2} + \Psi^{t\prime}_{J_2}.$$

Here $b = b_{J_2;0,0}$.

In the following calculation, we take $b_{p;1} = b_{p+1;0} = b_{p+1;1} = 0$ for $p > J_2$. Repeating such process, we have

$$g = \frac{b}{2^{3/2}}\Phi_{J_2+2;0,0} + \Psi'_{J_2+2} + \Psi^{t\prime}_{J_2+2}$$

with the help (11) and (12). By induction

$$g = \frac{b}{2^{3n/2}}\Phi_{J_2+2n;0,0} + \Psi'_{J_2+2n} + \Psi^{t\prime}_{J_2+2n}$$

for all $n \in \mathbb{Z}_+$. Thus we get the desired results. ∎

3 Main theorem

We come to the main results of this note.

Theorem 4 *The following functions*

$$\Psi^\lambda_{j;k,l} : (j, k, l) \in \mathbb{Z}^2 \times \mathbb{Z}_+,\ \lambda = h, v, d, \quad \Psi^t_{2j;k} : (j, k) \in \mathbb{Z}^2 \tag{23}$$

are orthonormal basis for $L^2(\mathbb{R}^2_+)$. More precisely, for any $f \in L^2(\mathbb{R}^2_+)$, we have

$$f = \sum_{(j,k,l) \in \mathbb{Z}^2 \times \mathbb{Z}_+} \langle f, \Psi^\lambda_{j;k,l}\rangle \Psi^\lambda_{j;k,l} + \sum_{(j,k) \in \mathbb{Z}^2} \langle f, \Psi^t_{2j;k}\rangle \Psi^t_{2j;k} \tag{24}$$

and

$$\|f\|^2_{L^2} = \sum_{(j,k,l) \in \mathbb{Z}^2 \times \mathbb{Z}_+} |\langle f, \Psi^\lambda_{j;k,l}\rangle|^2 + \sum_{(j,k) \in \mathbb{Z}^2} |\langle f, \Psi^t_{2j;k}\rangle|^2. \tag{25}$$

Here $\langle \cdot, \cdot \rangle$ is the inner product of $L^2(\mathbb{R}^2_+)$.

Proof. Clearly functions in (23) are orthonormal. To prove (24) and (25), we need only to prove that any functions in $L^2(\mathbb{R}^2_+)$ can be approximated, up to any arbitrarily small precision, by a finite linear combination of functions in the form (23). That is, given any $f \in L^2(\mathbb{R}^2_+)$, for any $\varepsilon > 0$, there exists

$$h = \sum_{-J \le j \le J} \sum_{|k|, l \le 2^J,\, l \ge 0} c^\lambda_{j,k,l}\Psi^\lambda_{j;k,l} + \sum_{-J \le j \le J} \sum_{|k| \le 2^J} c^t_{j,k}\Psi^t_{j;k}$$

for some sufficiently large positive integer J, depending on ε, such that

$$\|f-h\|_{L^2}<\varepsilon. \tag{26}$$

By the fact that characteristic functions on Euclidean squares $Q_{j;k,l}$ are dense in $L^2(\mathbb{R}^2_+)$, there are, for any $\varepsilon>0$, sufficient large positive integers J_1 and J_2 such that

$$\|f-g\|_{L^2}<\frac{\varepsilon}{2}$$

with g in the form (14). By Lemma 3,

$$\|g-\Psi'_{J_2+2n}-\Psi^{t\prime}_{J_2+2n}\|_{L^2}=\frac{b}{2^{3n/2}}\|\Phi_{J_2+2n;0,0}\|_{L^2}<\frac{\varepsilon}{2}$$

if we choose n sufficiently large. The last two inequalities yield (26). ■

Remark 5 *Let $\mathcal{W}_j$ be the complement space in $\mathcal{V}_{j-1}$ of $\mathcal{V}_j$. It follows that $\mathcal{W}_{2j+1}$ consists three type of wavelets*

$$\Psi^{\lambda}_{2j+1;k,l},\ (j,k,l)\in\mathbb{Z}^2\times\mathbb{Z}_+,\ \lambda=h,v,d,$$

while $\mathcal{W}_{2j}$ consists four type of wavelets

$$\Psi^{\lambda}_{2j;k,l},\ (j,k,l)\in\mathbb{Z}^2\times\mathbb{Z}_+,\ \lambda=h,v,d,\quad \textit{and}\quad \Psi^{t}_{2j;k},\ (j,k)\in\mathbb{Z}^2.$$

Comparing with classical wavelets in the plane, there are additional wavelets $\Psi^t_{2j;k},\ (j,k)\in\mathbb{Z}^2$.

4 Conclusions

In conclusion, we construct explicitly the orthonormal wavelets of a homogeneous type space relating to Tricomi operator on $\mathbb{R}^2_+$. Our proof is elementary. The supports of these wavelets are dyadic squares in terms of a quasi-distance. These dyadic squares have a new type "multiscale analysis" on $\mathbb{R}^2_+$. The geometry of these dyadic squares will be useful in later calculation.

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