Dynamics of a Bertrand Game with Heterogeneous Players and Different Delay Structures

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Abstract: In this paper we consider a duopoly Bertrand game played by two heterogeneous players with different delayed bounded rationality. We suppose that one player adjusts the price strategy according to his own marginal profit with time delay, while another player makes the price strategy as a best response to an expectation that averages the opponent’s actions in the previous periods. The dynamics is built for this decision making process and the local stability of its equilibrium states are studied. Numerical simulations are done to display the influence of the model parameters on the system complexity and system stability. It is shown that the system may lose stability and take complex behaviors, and the stability loss may be due to either period-doubling bifurcation or Neimark-Sacker bifurcation. It is shown that the time delay for the marginal profit has a greater effect on the system stability than the time delay considered for the price variable, and an intermediate level of delay on the marginal profit can greatly improve the system stability.

Keywords: Bertrand game; Time delay; Marginal profit; System stability; System complexity

1 Introduction

Cournot [1] game is a kind of quantity competition game, where each firm takes the quantity of output as its decision variable. It is assumed that two firms produce homogeneous goods for sale in a duopoly market. As market price is dependent on the total supply, each player has to take into account not only the demand function but also the reaction of the other player. Being different from Cournot game, when the firms’ product is perceived as heterogeneous in the perspective of the consumers, any firm may take price instead of quantity as its decision variable. The model based on this idea was first proposed by Bertrand [2].

The traditional oligopoly games were considered as the case with naïve expectation, by which a player supposes that the opponents’ strategies keep as the same level as in the previous period [e.g., 3–7]. However, in a real market it is not always credible that a rational player can remain a constant strategy. As a kind of more sophisticated rationality, the adaptive expectation or adaptive adjustment method was put forward to evaluate the strategy of the opponents [e.g., 8–12].

In recent years, many scholars have paid attention to so-called bounded rationality, with which a player uses his marginal profits to adjust his strategy in the following period. That is, a firm raises its output (in a Cournot game) or its price (in a Bertrand game) if the marginal profit in the last period is positive, otherwise makes a negative adjustment [e.g., 13–18]. These are early models built for the games with homogeneous players.

As players are not all homogeneous in a real market, the models with heterogeneous players are also considered by other authors [e.g., 19–25]. A dynamical Cournot duopoly game which contains a bounded rationality player and a naïve player is studied in [19]. Agiza and Elsadany [20] made work on the duopoly model with a bounded rationality player and an adaptive player. Linear cost function is replaced by a nonlinear one in the duopoly game with heterogeneous players [21]. Quadratic cost functions are considered in [22], where a heterogeneous duopoly game with diseconomies of scale is discussed. Fan et al. [23] analyzed the duopoly game with two heterogeneous players for the case that a naïve player makes its output strategy based on the market price of the previous period, while another bounded rationality player adjusts his strategy by the marginal profit method. Ding et al. [24] analyzed a dynamics of a multi-team Bertrand game played by

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a team consisting of two boundedly rational firms and another team consisting of one naive firm. A master-slave Bertrand game for upstream and downstream monopolies owned by different parties is discussed in [25], in which the upstream monopolist’s quantity is used as the main factor of production by the downstream monopolist who is a small purchaser of the upstream monopolist’s output.

Time delay plays an important role in oligopoly game, which can improve the system stability and delay the occurrence of complex behaviors [e.g., 26–29]. In the duopoly model with linear cost function discussed in [26] and a similar one with nonlinear cost function in [27], each firm is assumed to consider time delay for the output variable and estimate its marginal profit according to the previous-period quantity of all players. A bounded-rationality duopoly game with time delay on opponents’ output was considered in [29], where each firm makes its output strategy by averaging the previous-period quantity of its opponents. Lu [28] considered the case of increasing marginal cost instead of a constant one in [29].

In the above models with time delay, a potential assumption is that the opponents’ output history is a kind of common information for any player. But it may not be true in a real market when opponents’ behaviors are not easily observed. However, each player easily knows his own output and profit information in every period. So Ding et al. [30] considered time delay for the marginal profit and studied a case that each player makes his strategy adjustment by a smoothed marginal profit method, which averages his own previous marginal profits with different weights. On the other hand, the market price is also easily to be observed by each player. Peng et al. [31] studied a repeated Bertrand game with delayed bounded rationality, where all players maximize their profit according to the local information of the market price.

In a real market even with limited information, each player is able to get the information about his own marginal profit and the information about the price as well. In a Cournot market the market price is a kind of common information for all players, and in a Bertrand market any player’s price strategy is also open although it is an individual behavior. So a player may adjust his behavior by considering the history of either the marginal profit or the price variable. In this paper, we consider a Bertrand game played by players with heterogeneous and delayed rationality. One player considers time delay for the marginal profit so that adjusts his price strategy according to the averaged marginal profit in the previous periods. Another player considers time delay for the opponent’s price variable by making an expectation that the opponent adopts a smoothed price strategy which is also averaged from the previous periods. The main purpose is to investigate the different effects caused by the two different time delay structures on the system stability.

2 The model

We consider a Bertrand game focusing on price competition between two firms, which produce similar goods and choose suitable price to sell. Let $p_i$ denote firm $i$’s price strategy and $q_i$ represent the market demand for its product. The two firms’ price $p_i$ and $p_j$ together determine firm $i$’s market demand $q_i$ through a function $Q_i(p_i, p_j)$:

$$ q_i = Q_i(p_i, p_j). \quad (1) $$

To give a full supply $q_i$, firm $i$ has a cost that is described by a function $C_i(q_i)$. We suppose that the two firms are symmetric, and hence both of their demand functions and cost functions are identical; we also suppose that the two functions take linear forms as in [15, 16, 18, 25], i.e.

$$ Q_i(p_i, p_j) = a - p_i + bp_j, \quad (2) $$

$$ C_i(q_i) = cq_i, i = 1, 2, \quad (3) $$

where $a > 0$, $b > 0$ and $c > 0$. Then the profit of firm $i$ is given by

$$ \pi_i = p_iQ_i(p_i, p_j) - C_i(q_i) = (p_i - c)(a - p_i + bp_j), i = 1, 2. \quad (4) $$

Many authors have paid attention to the Cournot game (with incomplete information), where a player may not get accurate information of the opponent’s quantity and hence updates its output strategy in period $t + 1$ according to its marginal profit in period $t$ [e.g., 13, 17, 19–23]. Some other authors also studied the Bertrand game by a similar marginal-profit method, that is, a firm will raise its price strategy if its marginal profit is positive, otherwise reduces the price [e.g., 14–16, 18, 24, 25]. In these models mentioned, they only consider the marginal profit in period $t$, and do not take into account the marginal profit information in the previous periods. In a recent work Ding et al. [30] considered delayed information of the marginal profit in a Cournot game. It may be true that in a Cournot game the opponent’s quantity
strategy can not be observed; but in a Bertrand game, the price information of each firm is open on the market and hence a player may adjust its behavior according to the price history of the opponent. In this work about a Bertrand game, we assume that the two firms are two heterogeneous players and both consider time delay. Firm 1 uses a similar method discussed in [30] so that makes its strategy according to the marginal profit with one step time delay, and firm 2 adjusts its price strategy according to firm 1’s price information in the last two periods.

That is to say, firm 1 updates its price strategy at time \( t + 1 \) according to an average of its marginal profits \( \frac{\partial \pi_1(p_1, p_2)}{\partial p_1} |_{t} \),

\[
p_1(t + 1) = p_1(t) + v p_1(t)[w \frac{\partial \pi_1}{\partial p_1} |_{t} + (1 - w) \frac{\partial \pi_1}{\partial p_1} |_{t - 1}],
\]

where \( 0 \leq w \leq 1 \) is a weight coefficient, \( v \) is a positive parameter which represents a relative adjustment rate (for a Cournot game, see e.g. [30]).

Firm 2 is supposed to have a different rationality and consider a time delay structure built for the price strategy of its opponent. In the period \( t + 1 \) firm 2 is not able to know what price strategy firm 1 will take, but it may make some expectation about the behavior of firm 1. We suppose that firm 2 takes a naïve expectation that firm 1 will adopt a smoothed level of price strategy \( p_1^E(t + 1) \), which averages firm 1’s price in the previous two periods with different weights, i.e.

\[
p_1^E(t + 1) = \alpha p_1(t) + (1 - \alpha)p_1(t - 1),
\]

where \( 0 \leq \alpha \leq 1 \). Then for its price strategy \( p_2(t + 1) \) at time \( t + 1 \), firm 2 has an expected profit \( \pi_2^E(t + 1) \) as follows:

\[
\pi_2^E(t + 1) = [p_2(t + 1) - c_2][a - p_2(t + 1) + bp_2^E(t + 1)].
\]

To maximize its expected profit, firm 2 will take an action as a best reply to (7). That is to say, firm 2 will make a price strategy \( p_2(t + 1) \) according to the following formula:

\[
p_2(t + 1) = \arg \max_{p_2(t + 1)} \pi_2^E(t + 1).
\]

So far, for the Bertrand game we have described two different time delay structures by (5) and (8), which are together written as a system:

\[
\begin{align*}
\begin{cases}
p_1(t + 1) = p_1(t) + v p_1(t)[w \frac{\partial \pi_1}{\partial p_1} |_{t} + (1 - w) \frac{\partial \pi_1}{\partial p_1} |_{t - 1}], \\
p_2(t + 1) = \arg \max_{p_2(t + 1)} \pi_2^E(t + 1).
\end{cases}
\end{align*}
\]

From (4), we get the marginal profit of firm 1 as

\[
\frac{\partial \pi_1}{\partial p_1} |_{t} = a - 2p_1(t) + bp_2(t) + c_1.
\]

And from (6) and (7), we obtain firm 2’s expected profit \( \pi_2^E(t + 1) \) as

\[
\pi_2^E(t + 1) = [p_2(t + 1) - c_2][a - p_2(t + 1) + b(\alpha p_1(t) + (1 - \alpha)p_1(t - 1))],
\]

which has a maximization solution

\[
p_2(t + 1) = \arg \max_{p_2(t + 1)} \pi_2^E(t + 1) = \frac{a + c_2 + b(\alpha p_1(t) + (1 - \alpha)p_1(t - 1))}{2}.
\]

All the equations (9), (10) and (12) build a discrete dynamic system with one step delay:

\[
\begin{align*}
\begin{cases}
p_1(t + 1) = p_1(t) + v p_1(t)[w(a - 2p_1(t) + bp_2(t) + c_1) \\
p_2(t + 1) = a + c_2 + b(\alpha p_1(t) + (1 - \alpha)p_1(t - 1))]/2.
\end{cases}
\end{align*}
\]

By setting \( p_3(t + 1) = p_1(t) \) and \( p_4(t + 1) = p_2(t) \), we rewrite system (13) as a four-dimensional system as follows:

\[
\begin{align*}
\begin{cases}
p_1(t + 1) = p_1(t) + v p_1(t)[w(a - 2p_1(t) + bp_2(t) + c_1) \\
p_2(t + 1) = a + c_2 + b(\alpha p_1(t) + (1 - \alpha)p_1(t))/2, \\
p_3(t + 1) = p_1(t), \\
p_4(t + 1) = p_2(t).
\end{cases}
\end{align*}
\]

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And let \( p_i(t + 1) = p_i(t)(i = 1, 2, 3, 4) \) in system (14), then we obtain a boundary equilibrium \( E^0 \) and a nonnegative interior equilibrium \( E^* \):

\[
E^0 = (0, \frac{a + c_2}{2}, 0, \frac{a + c_2}{2}),
E^* = (p_1^*, p_2^*, p_3^*, p_4^*),
\]

where

\[
p_1^* = p_3^* = \left(\frac{2a + ab + 2c_1 + bc_2}{4 - b^2}\right),
\]
\[
p_2^* = p_4^* = \left(\frac{2a + ab + 2c_2 + bc_2}{4 - b^2}\right).
\]

Since we only need to consider a nonnegative state, we suppose \( 0 < b < 2 \) in our model.

### 3 Stability of the equilibriums

The local stability of an equilibrium point \((p_1, p_2, p_3, p_4)\) is determined by the eigenvalues of the Jacobian matrix \( J \), which takes the form as:

\[
J(p_1, p_2, p_3, p_4) = \begin{pmatrix}
an_{11} & bwp_1 & -2v(1 - w)p_1 & bv(1 - w)p_1 \\
bwp_1 & 0 & \frac{b(1 - \alpha)}{2} & 0 \\
0 & \frac{b(1 - \alpha)}{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

where \( an_{11} = 1 + v[w(a - 4p_1 + bp_2 + c_1) + (1 - w)(a - 2p_3 + bp_3 + c_1)] \).

If all the absolute values of the eigenvalues of the Jacobian matrix \( J(p_1, p_2, p_3, p_4) \) are less than 1, then the equilibrium \((p_1, p_2, p_3, p_4)\) will be locally asymptotically stable. On the contrary, the equilibrium \((p_1, p_2, p_3, p_4)\) will be unstable if there is an eigenvalue such that its absolute value is greater than 1.

At the boundary equilibrium point \( E^0 \), the Jacobian matrix (15) takes a form as:

\[
J(E^0) = \begin{pmatrix}
a_{11} & bwp_1 & -2v(1 - w)p_1 & bv(1 - w)p_1 \\
bwp_1 & 0 & \frac{b(1 - \alpha)}{2} & 0 \\
0 & \frac{b(1 - \alpha)}{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

where \( a_{11} = 1 + v[w(a - 4p_1 + bp_2 + c_1) + (1 - w)(a - 2p_3 + bp_3 + c_1)] \).

It is obvious that \( J(E^0) \) has an eigenvalue \( \lambda = 1 + \frac{v(2a + ab + 2c_1)}{2} > 1 \), so we conclude that the boundary equilibrium point \( E^0 \) is unstable.

Now we consider the asymptotic stability of \( E^* \). At the interior equilibrium \( E^* \), the condition \( w(a - 2p_1^* + bp_2^* + c_1) + (1 - w)(a - 2p_3^* + bp_3^* + c_1) = 0 \) must holds. Then the Jacobian matrix at \( E^* \) is given by:

\[
J(E^*) = \begin{pmatrix}
1 - 2vwp_1 & bwp_1 & -2v(1 - w)p_1 & bv(1 - w)p_1 \\
bwp_1 & 0 & \frac{b(1 - \alpha)}{2} & 0 \\
0 & \frac{b(1 - \alpha)}{2} & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]

Let \( Y(\lambda) \) denotes the characteristic polynomial of \( J(E^*) \), then it has a form as:

\[
Y(\lambda) = \lambda^4 + Y_1\lambda^3 + Y_2\lambda^2 + Y_3\lambda + Y_4,
\]

where

\[
Y_1 = -1 + 2vwp_1^*,
Y_2 = 2v(1 - w)p_1^* + \frac{b^2vwp_1^*}{2},
Y_3 = \frac{b^2vwp_1^*(2w(1 - w) - \alpha)}{2},
Y_4 = \frac{b^2vwp_1^*(w - 1)(1 - \alpha)}{2}.
\]
According to the Schur-Cohn Criterion (see e.g. [32]), all the roots of the polynomial \( Y(\lambda) \) (i.e., the eigenvalues of the Jacobian matrix \( J(E^*) \)) will lie inside the unit disk if the following inequalities hold:

\[
(i) \quad Y(1) = 1 + Y_1 + Y_2 + Y_3 + Y_4 > 0; \\
(ii) \quad (-1)^4 Y(-1) = 1 - Y_1 + Y_2 - Y_3 + Y_4 > 0; \\
(iii) \quad \text{The determinants of the } 1 \times 1 \text{ matrices } B_1 \text{ and the } 3 \times 3 \text{ matrices } B_3 \text{ are all positive, where}
\]

\[
B_1^+ = (1 + Y_4), \\
B_1^- = (1 - Y_4), \\
B_3^\pm = \begin{pmatrix} 1 & 0 & 0 \\ Y_1 & 1 & 0 \\ Y_2 & Y_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & Y_4 \\ 0 & Y_3 & Y_4 \\ Y_4 & Y_3 & Y_2 \end{pmatrix}.
\]

It is obvious that the condition \( 1 - Y_4 > 0 \) always hold in our model, and \( 1 + Y_1 + Y_2 + Y_3 + Y_4 = vp_1(2 - \frac{b^2}{2}) \) is also positive since \( 0 < b < 2 \) is supposed in our model. Thus, we conclude that the interior equilibrium point \( E^* \) of system (14) is locally asymptotically stable if it satisfies the following conditions:

\[
1 - Y_1 + Y_2 - Y_3 + Y_4 > 0, 1 + Y_4 > 0, \\
\text{Det}(B_3^+) > 0, \text{Det}(B_3^-) > 0,
\]

which can be reduced to a group of inequalities as:

\[
(a) \quad 1 + Y_4 > 0, \\
(b) \quad 1 - Y_1 + Y_2 - Y_3 + Y_4 > 0, \\
(c) \quad 1 - Y_1 Y_3 Y_4 + Y_4^2 + Y_2 Y_4^2 + Y_3^2 < Y_2 + Y_4 - Y_1 Y_3 + Y_2^2 Y_3 Y_4 - Y_4^3 < 1 + Y_1 Y_3 Y_4 + 2 Y_2 Y_4 - Y_2^2 Y_4^2 - Y_3^2.
\]

The interior equilibrium \( E^* \) will be locally stable if the coefficients of characteristic polynomial satisfy the conditions (a), (b) and (c). In the following section, we will provide numerical evidences for the complex dynamical behaviors of

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system when the stability conditions are broken, and use these stability conditions to analyze the stability region in the parameter space by numerical simulations.

4 Numerical simulations

In this section, some detailed numerical results including bifurcation diagrams, phase portraits and stable regions are used to show the complicated dynamic behavior of the system and the different effect on the system stability of the two different delay structures. The numerical work is done for the main model parameters such as the delay weights \( w \) and \( \alpha \) and the adjustment speed \( v \). In all the numerical simulations, we set the other parameters \( a, b, c_1 \) and \( c_2 \) as: 

\[
\begin{align*}
  a &= 3, \\
  b &= 0.6, \\
  c_1 &= 1, \text{ and } c_2 &= 1.5.
\end{align*}
\]

Fig.1 and Fig.2 show the bifurcation diagrams with respect to the adjustment rate \( v \) when the system loses stability. To show the influence of the delay weight \( \alpha \) on the system stability loss, Fig.1 is done for three cases that \( \alpha \) takes three different values (\( \alpha = 0.2, 0.6 \) and 1) while \( w \) is fixed as \( w = 0.2 \). Comparing the three bifurcation diagrams in Fig.1, we observe that there is a little difference in the values of \( v \) for the system to begin stability loss. That is to say, the delay weight \( w \) has little effect on the system stability. Similarly, Fig.2 is plotted for three cases with different values of \( w (w = 0.2, 0.6, 1) \) and a fixed \( \alpha (\alpha = 0.2) \). Comparing the three diagrams in Fig.2, it is obvious that there is a big difference in the values of \( v \) for the system to lose stability and an intermediate level of \( w \) can make the system keep stability in a large region. So we can say that the delay weight \( w \) has a stronger effect on the system stability. From both Fig.1 and Fig.2, we can also see that the system will be stable for small values of the adjustment speed, but the system becomes unstable as \( v \) increases.

Furthermore, the stability conditions (a), (b) and (c) obtained in Section 3 can also be used to show the different influence of the delay weights \( w \) and \( \alpha \) on the system stability. Based on the numerical solution of the three the stability conditions, Fig.3 and Fig.4 plot some two-dimensional parameter regions, from which every couple of parameters together with other fixed ones satisfy the conditions (a), (b) and (c) and hence make the system stable in its evolution. Fig.3 shows the \((v, w)\)-stability regions for four different values of \( \alpha (\alpha = 0.2, 0.6, 0.75, 1) \), and Fig.4 plots the stability regions.
Figure 3: Stability region in $(v, w)$-plane for different levels of $\alpha$

Figure 4: Stability region in $(v, \alpha)$-plane for different levels of $w$
in the $(v, \alpha)$-plane with four different values of $w$ ($w = 0.2, 0.6, 0.75, 1$). From Fig.3 we can find that there is nearly no difference among the four stability regions that are plotted for different values of $\alpha$. However, form Fig.4 that is associated with different values of $w$ we can see that the four stability regions are much more distinguishable. And Fig.4 shows that the stability region for the delay weight $w = 0.75$ is much larger than the one for a non-delay case ($w = 1$) and the one for an excessive delay case ($w = 0.2$). From these numerical simulations for stability region, we also get a conclusion that the delay weight $w$ for the marginal profit has a stronger effect on the system stability while the delay weight $\alpha$ for the price has little effect, and an appropriate level of marginal-profit delay weight $w$ can improve the system stability.

If the stability conditions (a), (b) and (c) are not satisfied, the system will lose stability and displays some complicated dynamical behaviors, as shown in Fig.1 and Fig.2. From Fig.1 and Fig.2, we can see that the system can lose stability either through period-doubling bifurcations or Neimark-Sacker bifurcations. It is obvious that in Fig.1(a)(b)(c) and Fig.2(a) the system loses stability through Neimark-Sacker bifurcations, while in Fig.2(c) the system stability loss is caused by a period doubling bifurcation. In order to specify the two different ways of the stability loss, we can consider the phase portraits for each case. For instance, we plot the two-dimensional phase portraits associated with Fig.1(a) and Fig.2(c) in Fig.5. The first line of phase portraits are done for Fig.1(a) with different values of $v$, which obviously shows a Neimark-Sacker bifurcation because closed invariant curves take place when the system loses stability, and which also show there are period windows in the process of Neimark-Sacker bifurcation. The second line in Fig.5 plots the phase portraits that are associated with Fig.2(c); it is evidently related to a period-doubling process, which leads the system from doubling periods to chaos in the end.

5 Conclusions

In a Bertrand market even with limited information, each player is able to get the information not only about his own marginal profit but also the opponents price strategy that is open in the market. When considering time delay, there may be players paying attention to either the marginal profit history or the open price memory. So we consider a duopoly Bertrand game played by heterogeneous players with two different time-delay rationality. We suppose that one firm
adjusts its price strategy according to a kind of smoothed information that average its marginal profit in the previous period and the one in a delayed period, while another firm makes its price strategy as a best response to an expectation that averages the opponents actions in the previous periods with also a one-step delay. Built for this game model, the dynamic system has a boundary equilibrium and an interior equilibrium. The instability of the boundary equilibrium is proved, and the stability conditions of the interior equilibrium are obtained. Numerical simulations including bifurcation diagrams, stability regions and phase portraits have been done to show the influence of the main model parameters on system evolution. We have shown that when the stability conditions are broken the system may take complex behaviors, and the loss of the system stability may be caused by Neimark-Sacker bifurcation or period doubling bifurcation; the time delay for the marginal profit has a stronger effect on the system stability than the time delay for the price variable; an intermediate level of time delay on the marginal profit can greatly improve the system stability.

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