Bifurcations of Limit Cycles in A $Z_3$-Equivariant Planar Vector Field of Degree 5

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(Received 8 November 2017, accepted 31 January 2018)

Abstract: In this paper, we study the number and distributions of limit cycles in a $Z_3$-equivariant quintic planar polynomial system. At least 23 limit cycles are found in this system by using the bifurcation methods of double homoclinic loops and Poincare-Bendixson Theorem. The configurations and number of these limit cycles obtained in the above $Z_3$-equivariant planar system are new. The results obtained are useful to the study of weakened Hilbert’s 16th Problem.

Keywords: Double homoclinic loops; Stability; Melnikov function; Bifurcations of limit cycles; Poincaré-Bendixson theorem.

1 Introduction

It is well known that the great Russian mathematician Arnold [1] posed the weakened Hilbert’s 16th Problem in 1977. It is to find the maximal number and relative positions of limit cycles of the planar polynomial vector field:

$$\begin{align*}
\dot{x} &= \frac{\partial H(x, y)}{\partial y} + \varepsilon P_n(x, y), \\
\dot{y} &= -\frac{\partial H(x, y)}{\partial x} + \varepsilon Q_n(x, y),
\end{align*}$$

(1)

where $H(x, y)$ are polynomials of degree $n + 1$, $P_n$ and $Q_n$ are polynomials of degree $n$.

Up to now, many mathematicians have study this problem, but the existence of $H(n)$ is still an open problem, where $H(n)$ stands for the supremum of the number of limit cycles for system (1). Some results of these mathematicians are concluded as follow: $H(2) \geq 4$, $H(3) \geq 13$, $H(4) \geq 20$ (see [2][3][4][18][19][20] for more details).

As to quintic planar polynomial system, there also are some results. In [5], Zhao Liqin obtained at least 23 limit cycles for a $Z_3$ equivariant near-Hamiltonian system of degree 5 which is the perturbation of a $Z_6$ equivariant quintic Hamiltonian system. Li et al.[6] studied a $Z_6$ equivariant perturbed Hamiltonian planar polynomial vector field of degree 5 and found that there exist at least 24 limit cycles. In [7], at least 24 limit cycles were found and two different configurations of them were given in a quintic $Z_3$ equivariant near-Hamiltonian system by Y.Wu. In [8, 9], it was obtained that there were at least 25 limit cycles in $Z_q$ equivariant near-Hamiltonian system of degree 5 where $q = 2, 5$. It was shown in [10] that there were at least 28 limit cycles with four different configurations in a $Z_2$ equivariant quintic planar vector field.

In this paper, the following near Hamiltonian system is considered

$$\begin{align*}
\dot{x} &= \frac{\partial H(x, y)}{\partial y} + \varepsilon P_5(x, y), \\
\dot{y} &= -\frac{\partial H(x, y)}{\partial x} + \varepsilon Q_5(x, y),
\end{align*}$$

(2)

where $\varepsilon > 0$ and small,

$$H(x, y) = -\frac{3x^2}{2} - \frac{113x^3}{250} - \frac{3y^2}{2} - \frac{2x^6}{3} + \frac{330xy^2}{250} + 8x^2y^2 - 5x^4y^2 + 4x^4 + 4y^4 - y^6,$$

(3)

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ISSN 2018.02.15/994
the quintic polynomials $P_5(x, y)$ $Q_5(x, y)$ are given in the following forms

\[
\begin{align*}
P_5(x, y) &= a_1 x + a_2 (x^3 + xy^2) + a_3 (x^5 + 2x^3y^2 + xy^4) + a_4 (x^4 - 6x^2y^2 + y^4) \\
&\quad + a_6 (x^4 - y^4) + a_7 (10x^2y^3 - 5x^4y - y^5) + b_4 (xy^3 - 4x^3 y) \\
&\quad + b_6 (2xy^3 + 2x^3y);
\end{align*}
\]

\[
\begin{align*}
Q_5(x, y) &= a_1 y + a_2 (x^2y + y^3) + a_3 (x^4y + 2x^2y^3 + y^5) + a_4 (4x^3y - 4xy^3) \\
&\quad - a_6 (2xy^3 + 2x^3y) + a_7 (2x^5 - 10x^3y^2 + 5xy^4) + b_4 (x^4 - 6x^2y^2 + y^4) \\
&\quad + b_6 (x^4 - y^4).
\end{align*}
\]

Here we consider the real coefficients $a_i$, $b_j$, $i = 1, 2, 3, 4, 6, 7$, $j = 4, 6$ as parameters.

From [11], we know that the vector field defined by $(P_5(x, y), Q_5(x, y))$ is invariant under $2\pi/3$ rotation with respect to the origin $O$, so it is easy to check that system (2) is $Z_3$ equivariant. Furthermore, we know that system (2) is a Hamiltonian system if and only if $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $4a_4 + a_6 = 0$, $b_4 - b_6 = 0$. It is well known that a Hamiltonian system does not have any limit cycles. Hence, in this paper we assume that

\[
a_1^2 + a_2^2 + a_3^2 + (4a_4 + a_6)^2 + (4b_4 - b_6)^2 \neq 0. \tag{4}
\]

This paper is arranged as follows. In section 2, we will give the phase portraits of the unperturbed system $(2)|_{c=0}$ and make some descriptions of the unperturbed system. Section 3 gives some preliminary lemmas about the existence conditions of double homoclinic loops and the small homoclinic loops of system (2). We give the main results and their proof by using the method of changing the stabilities of double homoclinic loops in section 4. In section 5, we give the conclusion.

## 2 Phase portraits of unperturbed system

The unperturbed system of system (2) is a Hamiltonian system which has the form

\[
\begin{align*}
\dot{x} &= -3y + \frac{339xy}{125} + 16x^2y - 10x^4y + 16y^3 - 6y^5, \\
\dot{y} &= 3x + \frac{339x^2}{250} - 16x^3 + 4x^5 - \frac{339y^2}{250} - 16xy^2 + 20x^3 y^2. \tag{5}
\end{align*}
\]

It is easy to check that unperturbed system (5) has 19 singular points by solving polynomial equations, which are 9 centers $A_i (i = 1, 2, \cdots, 9)$ and 9 saddle points $S_j (j = 1, 2, \cdots, 9)$ and the origin $O$. The coordinates of these singular points are listed as follows:

\[
\begin{align*}
A_1(x_{A_1}, y_{A_1}), &\quad A_7(-x_{A_7}, -y_{A_7}), \\
A_2(-x_{A_2}, y_{A_2}), &\quad A_8(-x_{A_8}, y_{A_8}), \\
A_3(-x_{A_3}, 0), &\quad A_9(x_{A_9}, 0), \\
A_4(-x_{A_4}, y_{A_4}), &\quad A_4(-x_{A_4}, y_{A_4}), \\
A_5(x_{A_5}, -y_{A_5}), &\quad A_5(-x_{A_5}, y_{A_5}), \\
A_6(x_{A_6}, 0), &\quad A_6(x_{A_6}, 0), \\
S_1(x_{S_1}, y_{S_1}), &\quad S_1(x_{S_1}, -y_{S_1}), \\
S_2(x_{S_2}, y_{S_2}), &\quad S_2(x_{S_2}, -y_{S_2}), \\
S_3(x_{S_3}, y_{S_3}), &\quad S_3(x_{S_3}, -y_{S_3}), \\
S_4(x_{S_4}, y_{S_4}), &\quad S_4(x_{S_4}, -y_{S_4}), \\
S_5(x_{S_5}, y_{S_5}), &\quad S_5(x_{S_5}, -y_{S_5}), \\
S_6(x_{S_6}, y_{S_6}), &\quad S_6(x_{S_6}, -y_{S_6}), \\
S_7(x_{S_7}, y_{S_7}), &\quad S_7(x_{S_7}, -y_{S_7}), \\
S_8(x_{S_8}, y_{S_8}), &\quad S_8(x_{S_8}, -y_{S_8}), \\
S_9(x_{S_9}, y_{S_9}), &\quad S_9(x_{S_9}, -y_{S_9}), \\
\end{align*}
\]

where $x_{A_1} \approx 0.997688$, $y_{A_1} \approx 1.72805$, $x_{A_2} \approx 0.950548$, $y_{A_2} \approx 1.6464$, $x_{A_3} \approx 1.99538$, $x_{A_6} \approx 1.9011$, $x_{A_7} \approx 0.24714$, $y_{A_7} \approx 0.428659$, $x_{A_8} \approx 0.49428$, $x_{A_9} \approx 1.40307$, $y_{A_9} \approx 0.70365$, $x_{S_1} \approx 0.092154$, $y_{S_1} \approx 1.56692$, $x_{S_2} \approx 1.31091$, $y_{S_2} \approx 0.863267$, $y_{S_3} \approx 0.94641$.

**Remark 1** From [11], it is easy to check that the unperturbed system (5) is $Z_3$-equivariant.

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To plot the phase portraits of system (5), we list the representative orbits of system (5) in the following.

System (5) is a Hamiltonian system, and it has the first integral of the form $H(x, y) = h$, where function $H(x, y)$ is defined in (3). Then we have $H(0) = 0$, $H(S_i) = h_3$, $H(S_j) = h_2$, $H(A_m) = h_4$, $H(A_n) = h_5$, $H(A_3) = h_1$, $i = 1, 2, \ldots, 6$, $j = 7, 8, 9$, $m = 2, 4, 6$, $n = 1, 3, 5$, where $h_1 \geq -1.92019$, $h_2 \geq -0.111403$, $h_3 \geq 5.47628$, $h_4 \geq 12.2493$, $h_5 \geq 18.9507$.

The level curve defined by $H(x, y) = h_3$ consists of the six saddle points $S_i$ and six heteroclinic loops denoted by $\Gamma_{i,j}^{h_3} \cup \Gamma_{j,i}^{h_3}$ ($j = i + 1$ as $i = 1, 2, \ldots, 6$ and $j = 1$ as $i = 6$), where $\Gamma_{i,j}^{h_3}$ represents the saddle connection between $S_i$ and $S_j$ with the direction from point $S_i$ to point $S_j$. Every heteroclinic loops $\Gamma_{i,j}^{h_3} \cup \Gamma_{j,i}^{h_3}$ embraces the focus $A_i(i = 1, 2, \ldots, 6)$). Similarly, the level curve which is defined by $H(x, y) = h_2$ consists of the three saddle points $S_i$ and five heteroclinic loops which are denoted by $\Gamma_{i,j}^{h_2} \cup \Gamma_{j,i}^{h_2}$ ($j = i + 1$ as $i = 7, 8$, and $j = 7$ as $i = 9$) and $\Gamma_{5,8}^{h_2} \cup \Gamma_{8,9}^{h_2} \cup \Gamma_{9,7}^{h_2}$, and $\Gamma_{9,7}^{h_2} \cup \Gamma_{8,9}^{h_2} \cup \Gamma_{7,8}^{h_2}$. Every heteroclinic loops $\Gamma_{i,j}^{h_2} \cup \Gamma_{j,i}^{h_2}$ embraces the focus $A_i(i = 7, 8, 9)$.

Then we denote the close orbit of the unperturbed system (5) by $\Gamma: H(x, y) = h$, and these close orbits have the following properties.

(i) If $h < h_3$, the close orbit $\Gamma: H(x, y) = h$ shrinks as $h$ increases, and surrounds all the singular points of the unperturbed system (5).

(ii) If $h_2 < h < h_3$, the close orbit $\Gamma: H(x, y) = h$ expands as $h$ increases, and surrounds $O, A_1, S_3(i = 7, 8, 9)$.

(iii) If $h_2 < h < 0$, the close orbit $\Gamma: H(x, y) = h$ shrinks as $h$ increases, and only surrounds the origin $O$.

(iv) If $h_1 < h < h_2$, the close orbit $\Gamma: H(x, y) = h$ expands as $h$ increases, and only surrounds the single singular point $A_i(i = 7, 8, 9)$ respectively.

(v) If $h_3 < h < h_5$, the close orbit $\Gamma: H(x, y) = h$ shrinks as $h$ increases, and only surrounds the single singular point $A_i(i = 1, 2, \ldots, 6)$ respectively.

From the above analysis, we obtain the phase portraits of unperturbed system (5) in Figure 1.

![Figure 1: The phase portraits of unperturbed system (5)](image)

3 Preliminary lemmas

As $0 < \varepsilon \ll 1$, system (2) can be regarded as perturbation of system (5). After perturbation, it is easy to find that the number of singular points is preserved, so system (2) still has 19 singular points. Denote the singular points of system (2) by $A_i(\varepsilon), S_i(\varepsilon)$ which near $A_i, S_i, i = 1, 2, \ldots, 9$. Generally speaking, the saddle connections $\Gamma_{i,i+1}, \Gamma_{i+1,i}$, $(i = 1, 2, \ldots, 9)$ of system (5) will break after perturbation. Denote $\Gamma_{S_i(\varepsilon)}^s, \Gamma_{S_i(\varepsilon)}^u$ the stable and unstable manifold of saddle point $S_i(\varepsilon)$. From [7], we know that the distance between $\Gamma_{S_i(\varepsilon)}^u$, $\Gamma_{S_i(\varepsilon)}^u$ is $d(\varepsilon, \Gamma_{i,j}) = \varepsilon N_{i,j} \cdot M(\Gamma_{i,j}) + O(\varepsilon^2)(N_{i,j} > 0, 0 < \varepsilon \ll 1, j = i+1)$, and $M(\Gamma_{i,j})$ is called Melnikov function of the saddle connections $\Gamma_{i,j}$, which is defined by the following function

$$
M(\Gamma_{i,j}) = \int_{\Gamma_{i,j}} Q_5(x, y)dx - \int_{\Gamma_{i,j}} P_5(x, y)dy.
$$

Then we obtain the following results.

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Lemma 1 Melnikov functions of system (2) satisfy the following equations

\[ M(\Gamma_{1,6}) = M(\Gamma_{5,4}) = M(\Gamma_{3,2}), \quad M(\Gamma_{6,1}) = M(\Gamma_{4,5}) = M(\Gamma_{2,3}); \]
\[ M(\Gamma_{3,4}) = M(\Gamma_{5,6}) = M(\Gamma_{1,2}), \quad M(\Gamma_{4,3}) = M(\Gamma_{6,5}) = M(\Gamma_{2,1}); \]
\[ M(\Gamma_{7,8}) = M(\Gamma_{8,9}) = M(\Gamma_{9,7}), \quad M(\Gamma_{8,7}) = M(\Gamma_{9,8}) = M(\Gamma_{7,9}). \]

Proof. Noticing the fact that system (2) is \( Z_3 \)-equivariant, it is easy to prove the lemma. ■

From Lemma 1, we only need to compute the following six Melnikov functions: \( M(\Gamma_{1,6}), M(\Gamma_{6,1}), M(\Gamma_{3,4}), M(\Gamma_{4,3}), M(\Gamma_{7,8}), M(\Gamma_{8,7}) \). With the aid of numeric calculation of Mathematica 8.0, we obtain the following results.

Lemma 2 For \( 0 < \varepsilon \ll 1 \), Melnikov functions \( M(\Gamma_{1,6}), M(\Gamma_{6,1}), M(\Gamma_{3,4}) \) and \( M(\Gamma_{4,3}) \) respectively have the following forms

\[ M(\Gamma_{1,6}) = -0.04316a_1 + 3.8659a_2 + 7.98226a_3 + 1.99840a_4 + 5.18956a_5 + 1.72086a_7; \]
\[ M(\Gamma_{6,1}) = -0.76515a_1 - 13.7964a_2 - 56.3786a_3 - 32.8209a_4 - 12.8951a_5 - 1.72086a_7; \]
\[ M(\Gamma_{3,4}) = -1.22092a_1 - 20.2829a_2 - 92.3631a_3 + 49.0403a_4 + 16.9500a_5 + 1.72086a_7; \]
\[ M(\Gamma_{4,3}) = -0.08185a_1 + 3.8703a_2 + 7.30355a_3 + 0.16772a_4 - 4.64803a_5 - 1.72086a_7. \]

Proof. By using Mathematica 8.0, we obtain the following functions

\[ \Gamma_{1,6}: \quad y = y_{16}(x), \quad x_2 \leq x \leq x_1; \quad x = x_{16}(y), \quad -y_1 \leq y \leq y_1; \]
\[ \Gamma_{4,3}: \quad y = y_{43}(x), \quad x_3 \leq x \leq x_4; \quad x = x_{43}(y), \quad -y_2 \leq y \leq y_2; \]
\[ \Gamma_{6,1}: \quad y = y_{61}(x), \quad x_1 \leq x \leq x_2; \quad y = y_{61}(x), \quad x_0 \leq x \leq x_1; \quad x = x_{61}(y), \quad -y_1 \leq y \leq y_1; \]
\[ \Gamma_{3,4}: \quad y = y_{34}(x), \quad x_3 \leq x \leq x_4; \quad y = y_{34}(x), \quad x_0 \leq x \leq x_3; \quad x = x_{34}(y), \quad -y_2 \leq y \leq y_2. \]

All the above functions are determined by the equation \( H(x, y) = h_3 \), where \( x_1 = x_{S_1}, x_2 = 1.35480, x_3 = -x_{S_1}, x_4 = -1.23340, x_1^* = 1.46390, x_2^* = 2.21126, x_3^* = -1.34914, x_4^* = -2.372014, y_1 = y_{S_1}, y_2 = y_{S_2}. \)

From (6), Melnikov function of the saddle connection \( \Gamma_{1,6} \) is calculated as follows:

\[
\int_{\Gamma_{1,6}} Q_5(x, y)dx = \int_{x_1}^{x_2} Q_5(x, y_{16}(x))dx + \int_{x_2}^{x_1} Q_5(x, -y_{16}(x))dx \\
= a_1k_{1,1} + a_2k_{2,1} + a_3k_{3,1} + a_4k_{4,1} + a_5k_{6,1} + a_7k_{7,1};
\]
\[
\int_{\Gamma_{1,6}} P_5(x, y)dy = \int_{y_1}^{-y_1} P_5(x_{16}(y), y)dy \\
= a_1k_{1,2} + a_2k_{2,2} + a_3k_{3,2} + a_4k_{4,2} + a_5k_{6,2} + a_7k_{7,2}.
\]

Melnikov function of the saddle connection \( \Gamma_{6,1} \) is calculated as follows:

\[
\int_{\Gamma_{6,1}} Q_5(x, y)dx = \int_{x_1}^{x_2} Q_5(x, y_{60}(x))dx + \int_{x_2}^{x_1} Q_5(x, -y_{60}(x))dx + \\
\int_{x_2}^{x_1} Q_5(x, -y_{60}(x))dx + i\int_{x_2}^{x_1} Q_5(x, -y_{60}(x))dx \\
= a_1k_{1,3} + a_2k_{2,3} + a_3k_{3,3} + a_4k_{4,3} + a_5k_{6,3} + a_7k_{7,3};
\]
\[
\int_{\Gamma_{6,1}} P_5(x, y)dy = \int_{y_1}^{-y_1} P_5(x_{61}(y), y)dy \\
= a_1k_{1,4} + a_2k_{2,4} + a_3k_{3,4} + a_4k_{4,4} + a_5k_{6,4} + a_7k_{7,4}.
\]

Melnikov function of the saddle connection \( \Gamma_{3,4} \) is calculated as follows:
From above numeric results and equation (6), the lemma is proved.

W.Zhu and Y. Wu: Bifurcations of Limit Cycles in A Z₂-Equivariant Planar Vector Field of Degree 5

After a lot of numerical calculations, we obtain the following results by using Mathematica 8.0:

\[
\int_{\Gamma_{3,4}} Q_5(x, y) dx = \int_{x_3}^{x_4} Q_5(x, y_{30}(x)) dx + \int_{x_3}^{x_4} Q_5(x, y_{04}(x)) dx + \int_{x_3}^{x_4} Q_5(x, -y_{04}(x)) dx + int_{x_3}^{x_4} Q_5(x, -y_{30}(x)) dx
\]
\[
= a_1 k_{1,5} + a_2 k_{2,5} + a_3 k_{3,5} + a_4 k_{4,5} + a_0 k_{6,5} + \alpha_7 k_{7,5};
\]
\[
\int_{\Gamma_{3,4}} P_5(x, y) dy = \int_{-y_2}^{y_2} P_5(x_{34}(y), y) dy
\]
\[
= a_1 k_{1,6} + a_2 k_{2,6} + a_3 k_{3,6} + a_4 k_{4,6} + a_0 k_{6,6} + \alpha_7 k_{7,6}.
\]

Melnikov function of the saddle connection \( \Gamma_{4,3} \) is calculated as follows:

\[
\int_{\Gamma_{4,3}} Q_5(x, y) dx = \int_{x_3}^{x_4} Q_5(x, y_{43}(x)) dx + \int_{x_3}^{x_4} Q_5(x, -y_{43}(x)) dx
\]
\[
= a_1 k_{1,7} + a_2 k_{2,7} + a_3 k_{3,7} + a_4 k_{4,7} + a_0 k_{6,7} + \alpha_7 k_{7,7};
\]
\[
\int_{\Gamma_{4,3}} P_5(x, y) dy = \int_{-y_2}^{y_2} P_5(x_{43}(y), y) dy
\]
\[
= a_1 k_{1,8} + a_2 k_{2,8} + a_3 k_{3,8} + a_4 k_{4,8} + a_0 k_{6,8} + \alpha_7 k_{7,8}.
\]

After a lot of numerical calculations, we obtain the following results by using Mathematica 8.0:

\[
k_{1,1} \doteq -0.04316137, k_{2,1} \doteq -0.09482117, k_{3,1} \doteq -0.2095090, k_{4,1} \doteq -0.39051156, k_{5,1} \doteq 0.26267075, k_{7,1} \doteq 0.562356668;
\]
\[
k_{1,2} \doteq 0, k_{2,2} \doteq -3.9608195, k_{3,2} \doteq -8.19177, k_{4,2} \doteq 2.38891, k_{6,2} \doteq -4.9268956, k_{7,2} \doteq -1.158506660;
\]
\[
k_{1,3} \doteq -0.76515175, k_{2,3} \doteq -2.54847278, k_{3,3} \doteq -8.78324015, k_{4,3} \doteq -15.0207547, k_{5,3} \doteq 9.0094163, k_{7,3} \doteq 31.1831989;
\]
\[
k_{1,4} \doteq 0, k_{2,4} \doteq 11.247930, k_{3,4} \doteq 47.595420, k_{4,4} \doteq 17.800185, k_{5,4} \doteq 21.9046152, k_{7,4} \doteq 32.9046222;
\]
\[
k_{1,5} \doteq -1.22092592, k_{2,5} \doteq -4.31925507, k_{3,5} \doteq -16.0690833, k_{4,5} \doteq 24.0859745, k_{6,5} \doteq -15.546579, k_{7,5} \doteq 50.3377220;
\]
\[
k_{1,6} \doteq 0, k_{2,6} \doteq 15.963738, k_{3,6} \doteq 76.29409, k_{4,6} \doteq -24.954327, k_{6,6} \doteq -32.496619, k_{7,6} \doteq 48.6168589;
\]
\[
k_{1,7} \doteq -0.08185760, k_{2,7} \doteq -0.16738669, k_{3,7} \doteq -0.34742379, k_{4,7} \doteq -0.51804027, k_{6,7} \doteq -0.4296418, k_{7,7} \doteq 0.569370864;
\]
\[
k_{1,8} \doteq 0, k_{2,8} \doteq -4.0377181, k_{3,8} \doteq -7.6509821, k_{4,8} \doteq 0.350312, k_{6,8} \doteq 4.218390, k_{7,8} \doteq 2.29023383.
\]

From above numeric results and equation (6), the lemma is proved.

Lemma 3 For \( 0 < \epsilon \ll 1 \), Melnikov functions \( M(\Gamma_{7,8}) \), \( M(\Gamma_{8,7}) \) respectively have the following forms

\[
M(\Gamma_{7,8}) \doteq 0.348711099a_1 + 0.20254578a_2 + 0.06900199a_3 + 0.07863389a_4 + 0.01965847a_6;
\]
\[
M(\Gamma_{8,7}) \doteq -0.147538383a_1 - 0.02527178a_2 - 0.00294147a_3 + 0.00449010a_4 + 0.00112252a_6.
\]

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**Proof.** By using Mathematica 8.0, we obtain the following functions

\[ \Gamma_{7,8} : y = y_{70}(x), x_5 \leq x \leq x_6; \quad y = y_{80}(x), x_6 \leq x \leq x_7; \]

\[ x = x_{70}(y), y_3 \leq y \leq y_4; \quad x = x_{80}(y), y_4 \leq y \leq y_5; \]

\[ \Gamma_{8,7} : y = y_{87}(x), x_5 \leq x \leq x_7; \quad x = x_{87}(y), y_3 \leq y \leq y_5. \]

All the above functions are determined by the equation \( H(x, y) = h_2 \), where \( x_5 = -0.4, x_6 = -0.481108, x_7 = 0.2, y_3 = 0, y_4 = 0.59688, y_5 = y_5^a. \)

From (6), Melnikov function of the saddle connection \( \Gamma_{7,8} \) is calculated as follows:

\[
\int_{\Gamma_{7,8}} Q_5(x, y)dx = \int_{x_5}^{x_6} Q_5(x, y_{70}(x))dx + \int_{x_6}^{x_7} Q_5(x, y_{80}(x))dx \\
= a_1k_{1,9} + a_2k_{2,9} + a_3k_{3,9} + a_4k_{4,9} + a_6k_{6,9} + a_7k_{7,9} + b_4k_{8,9} + b_6k_{9,9};
\]

\[
\int_{\Gamma_{7,8}} P_5(x, y)dy = \int_{y_3}^{y_4} P_5(x_{70}(y), y)dy + \int_{y_4}^{y_5} P_5(x_{80}(y), y)dy \\
= a_1k_{1,10} + a_2k_{2,10} + a_3k_{3,10} + a_4k_{4,10} + a_6k_{6,10} + a_7k_{7,10} + b_4k_{8,10} + b_6k_{9,10}.
\]

Melnikov function of the saddle connection \( \Gamma_{8,7} \) is calculated as follows:

\[
\int_{\Gamma_{8,7}} Q_5(x, y)dx = \int_{x_5}^{x_6} Q_5(x, y_{87}(x))dx \\
= a_1k_{1,11} + a_2k_{2,11} + a_3k_{3,11} + a_4k_{4,11} + a_6k_{6,11} + a_7k_{7,11} + b_4k_{8,11} + b_6k_{9,11};
\]

\[
\int_{\Gamma_{8,7}} P_5(x, y)dy = \int_{y_3}^{y_5} P_5(x_{87}(y), y)dy \\
= a_1k_{1,12} + a_2k_{2,12} + a_3k_{3,12} + a_4k_{4,12} + a_6k_{6,12} + a_7k_{7,12} + b_4k_{8,12} + b_6k_{9,12}.
\]

After a lot of numerical calculations, we obtain the following results by using Mathematica 8.0:

\[
k_{1,9} \approx 0.348711, k_{2,9} \approx 0.12041, k_{3,9} \approx 0.043109, k_{4,9} \approx 0.03843, k_{6,9} \approx 0.041111 \\
k_{7,9} \approx 0.0077271, k_{8,9} \approx 0.001536, k_{9,9} \approx -0.0541830; \\
k_{1,10} \approx 0, k_{2,10} \approx -0.08213, k_{3,10} \approx -0.02589, k_{4,10} \approx -0.040203, k_{6,10} \approx 0.02145 \\
k_{7,10} \approx 0.0077271, k_{8,10} \approx 0.001536, k_{9,10} \approx -0.0541830; \\
k_{1,11} \approx -0.1475, k_{2,11} \approx -0.0161, k_{3,11} \approx -0.0018, k_{4,11} \approx 0.0031, k_{6,11} \approx -0.0004 \\
k_{7,11} \approx 0.0003180, k_{8,11} \approx -0.001536, k_{9,11} \approx 0.0018087; \\
k_{1,12} \approx 0, k_{2,12} \approx 0.00915, k_{3,12} \approx 0.001111, k_{4,12} \approx -0.001358, k_{6,12} \approx -0.001605 \\
k_{7,12} \approx 0.0003180, k_{8,12} \approx -0.001536, k_{9,12} \approx 0.0018087.
\]

From above numeric results and equation (6), the lemma is proved. ■

Next, from [12], we give the existence conditions of double homoclinic loops and the small homoclinic loops of system (2).

**Lemma 4** As \( 0 < \varepsilon \ll 1 \), there exist functions

\[
d(\varepsilon, \Gamma_{6,1}, \Gamma_{1,6}) = \varepsilon N_1(M(\Gamma_{6,1}) + M(\Gamma_{1,6})) + O(\varepsilon^2), \quad N_1 > 0 \\
d(\varepsilon, \Gamma_{3,4}, \Gamma_{4,3}) = \varepsilon N_2(M(\Gamma_{3,4}) + M(\Gamma_{4,3})) + O(\varepsilon^2), \quad N_2 > 0 \\
d(\varepsilon, \Gamma_{7,8}, \Gamma_{8,7}) = \varepsilon N_3(M(\Gamma_{7,8}) + M(\Gamma_{8,7})) + O(\varepsilon^2), \quad N_3 > 0
\]

such that
Then we will give the criteria to determine the stabilities of the double homoclinic loops and the small homoclinic loops near $\Gamma_{6,1} \cup \Gamma_{1,6}$, which passes through the saddle point $S_1(\varepsilon)$ (resp., $S_0(\varepsilon)$) if $d(\varepsilon, \Gamma_{6,1}) < 0$ (resp., $d(\varepsilon, \Gamma_{6,1}) > 0$); (ii) when $d(\varepsilon, \Gamma_{3,4}, \Gamma_{4,3}) = 0$, then system (2) has a homoclinic loop denoted by $\Gamma_{3,4}(S_2(\varepsilon))$ (resp., $\Gamma_{4,3}(S_2(\varepsilon))$) near $\Gamma_{3,4} \cup \Gamma_{4,3}$, which passes through the saddle point $S_2(\varepsilon)$ (resp., $S_1(\varepsilon)$) if $d(\varepsilon, \Gamma_{3,4}) > 0$ (resp., $d(\varepsilon, \Gamma_{3,4}) < 0$); (iii) when $d(\varepsilon, \Gamma_{7,8}, \Gamma_{8,7}) = 0$, then system (2) has a homoclinic loop denoted by $\Gamma_{7,8}(S_7(\varepsilon))$ (resp., $\Gamma_{7,8}(S_8(\varepsilon))$) near $\Gamma_{7,8} \cup \Gamma_{8,7}$, which passes through the saddle point $S_7(\varepsilon)$ (resp., $S_8(\varepsilon)$) if $d(\varepsilon, \Gamma_{7,8}) < 0$ (resp., $d(\varepsilon, \Gamma_{7,8}) > 0$).

From Lemma 4, we got the following result.

**Lemma 5** There exist functions $\varphi_1, \varphi_2, \varphi_3, \phi_1$

\[
\varphi_1(a_2, a_3, a_4, a_6, \varepsilon) = -12.2853439a_2 - 59.873330a_3 - 38.131921a_4 - 9.5329803a_6 + O(\varepsilon);
\]

\[
\varphi_2(a_3, a_4, a_6, \varepsilon) = -17.318557a_3 + 242.653402a_4 + 60.663349a_6 + O(\varepsilon);
\]

\[
\varphi_3(a_4, a_6, \varepsilon) = 20.3320277a_4 + 5.083007a_6 + O(\varepsilon);
\]

\[
\phi_1(a_3, a_4, a_6) = 38.10311a_3 - 622.0189a_4 - 158.23007a_6;
\]

such that the following conclusions hold.

(i) If $a_1 = \varphi_1, a_2 = \varphi_2, a_3 > \phi_1$, then the system (2) has three double homoclinic loops $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$, $\Gamma_{2,3}(S_1(\varepsilon)) \cup \Gamma_{4,3}(S_1(\varepsilon))$, $\Gamma_{4,5}(S_5(\varepsilon)) \cup \Gamma_{5,6}(S_5(\varepsilon))$, where $\Gamma_{i,j}(S_1(\varepsilon))$ is the homoclinic loop passing through saddle point $S_1(\varepsilon)$ and tending to $\Gamma_{i,j}$, $i, j = 1, 3, 5$.

(ii) Assume the condition (1) is hold, further if $a_3 = \varphi_3$, then the system (2) has three more small homoclinic loops $\Gamma_{7,8}(S_7(\varepsilon))$, $\Gamma_{8,9}(S_8(\varepsilon))$, $\Gamma_{9,7}(S_9(\varepsilon))$, where $\Gamma_{i,j}(S_1(\varepsilon))$ is the homoclinic loop passing through saddle point $S_1(\varepsilon)$ and tending to $\Gamma_{i,j}$, $i, j = 7, 8, 9$.

**Proof.** From Lemma 4, we can study the conditions that the system (2) have three double homoclinic loops and three small homoclinic loops. Let $d(\varepsilon, \Gamma_{6,1}, \Gamma_{1,6}) = d(\varepsilon, \Gamma_{3,4}, \Gamma_{4,3}) = 0$ and $d(\varepsilon, \Gamma_{7,8}, \Gamma_{8,7}) = 0$. Then from the implicit function theorem, we know that there exist functions $\varphi_1, \varphi_2$, to let the following equation hold

\[
d(\varepsilon, \Gamma_{6,1}, \Gamma_{1,6}) = d(\varepsilon, \Gamma_{3,4}, \Gamma_{4,3}) = 0 \Leftrightarrow a_1 = \varphi_1, a_2 = \varphi_2.
\]

When equations $a_1 = \varphi_1, a_2 = \varphi_2$ hold, we have

\[
M(\Gamma_{6,1}) = 65.57024a_1 + 1070.409a_4 - 272.9232a_6 - 1.720863a_7,
\]

\[
M(\Gamma_{3,4}) = 72.24030a_3 - 1186.4623a_4 - 291.9256a_6 + 1.720863a_7.
\]

Let $M(\Gamma_{6,1}) = 0$, we get $a_7 = \phi_1(a_2, a_4, a_6)$, where $\phi_1$ is given in (7). Therefore, if $a_2 > \phi_1$, then $d(\varepsilon, \Gamma_{6,1}) < 0$, $d(\varepsilon, \Gamma_{1,6}) > 0$, and $d(\varepsilon, \Gamma_{7,8}) > 0$. Hence, from $Z_3$-equivariance of system (2) and Lemma 4, we prove the first part of the lemma.

From the implicit function theorem, we can also know that there exist function $\varphi_3$ to let the following equation hold

\[
d(\varepsilon, \Gamma_{7,8}, \Gamma_{8,7}) = 0 \Leftrightarrow a_3 = \varphi_3.
\]

Then in the same way, we prove that the second part of the lemma is true. ■

In the above context, we have already got the existence conditions of double homoclinic loops and the small homoclinic loops of system (2). Next, the stabilities of the double homoclinic loops and the small homoclinic loops of system (2) need to be studied. Denote the divergence quantity of equilibrium point $P$ of system (2) by $\text{div}(P) = (\frac{\partial P_x}{\partial x} + \frac{\partial P_y}{\partial y})(P)$. Then we will give the criteria to determine the stabilities of the double homoclinic loops and the small homoclinic loops in the following lemma.

**Lemma 6** Suppose that the parameters of system (2) satisfy the following conditions $a_1 = \varphi_1, a_2 = \varphi_2, a_3 = \varphi_3$ and $a_2 > \phi_1$. Then we have

(i) if $a_4 = \varphi_4 < 0$ (resp., $a_4 = \varphi_4 > 0$), then the double homoclinic loop $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$ of system (2) is inner and outer stable(unstable), where $\varphi_4$ is given in (7).
orbits

The equation

From [15], we can know that

prove that the second part of the lemma is true.

and outer stable(unstable).

homoclinic loop
respectively, so it easy to prove the first and last part of the lemma is true.

For

We denote the Melnikov function of the close orbit

From the above analysis, the proof is completed.

Proof. By direct calculation, we have the following result:

From [13, 14, 17], we know that the stabilities of the double homoclinic loop \( \Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon)) \) and the small homoclinic loop \( \Gamma_{7,8}(S_7(\varepsilon)) \) are determined by the sign of divergence quantities of the saddle point \( S_1(\varepsilon) \) and \( S_7(\varepsilon) \) respectively, so it easy to prove the first and last part of the lemma is true.

Let \( \text{div}(S_1(\varepsilon)) = 0 \), then we get \( a_4 = \varphi_4 \) (where \( \varphi_4 \) is given in (7)). From [13, 14, 17], we also know if \( a_4 = \varphi_4 \), the double homoclinic loop \( \Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon)) \) of system (2) is degenerated, and its stability is determined by the sign of the integrals of the following divergence quantities

\[
\begin{align*}
\sigma_1 &= \varepsilon \int_{L_{1,2}(S_1(\varepsilon))} \left( \frac{\partial P_5}{\partial x} + \frac{\partial Q_5}{\partial y} \right) dt; \\
\sigma_2 &= \varepsilon \int_{L_{6,1}(S_1(\varepsilon))} \left( \frac{\partial P_5}{\partial x} + \frac{\partial Q_5}{\partial y} \right) dt.
\end{align*}
\]

From [15], we can know that \( \sigma_1, \sigma_2 \) and \( \text{div}(S_2(\varepsilon)) \) have the same sign as \( \text{div}(S_2(\varepsilon)) \neq 0 \). So from [13–15, 17], we prove that the second part of the lemma is true.

From the above analysis, the proof is completed. ■

We denote the Melnikov function of the close orbit \( \Gamma : H(x, y) = h \) by \( M(\Gamma) = \int_Q (Q_5(x, y) dx - P_5(x, y) dy) \). The equation \( H(x, y) = -0.09 \) determines the close orbits \( \Gamma_s, \Gamma_m, \Gamma_l \), where \( \Gamma_s \) only surrounds the origin \( \Gamma_m \) surrounds \( O, A_i, S_i (i = 7, 8, 9) \) and \( \Gamma_l \) surrounds all the singular points of the unperturbed system (5). As \( 0 < \varepsilon \ll 1 \), we study the breaking way of the close orbits \( \Gamma_s, \Gamma_m, \Gamma_l \). From [16], we know that the breaking ways of the close orbits \( \Gamma_s, \Gamma_m, \Gamma_l \) are closely related with the sign of \( M(\Gamma_s), M(\Gamma_m) \) and \( M(\Gamma_l) \). So we compute the Melnikov functions of these three close orbits in the following lemma.

Lemma 7 For \( \varepsilon > 0 \) and small, then Melnikov function \( M(\Gamma_s), M(\Gamma_m) \) and \( M(\Gamma_l) \) respectively have the following expressions

\[
\begin{align*}
M(\Gamma_s) &= -0.2386675a_1 - 0.0365510a_2 - 0.0028328a_3 + 0.0028260a_4 + 0.0007194a_6; \\
M(\Gamma_m) &= 1.0469147a_1 + 0.7281667a_2 + 0.2627759a_3 + 0.2377991a_4 + 0.0594497a_6; \\
M(\Gamma_l) &= -14.105015a_1 - 130.05528a_2 - 615.35708a_3 + 41.97422a_4 + 10.493556a_6.
\end{align*}
\]

Proof. By using Mathematica 8.0, we get the following functions:

\[
\begin{align*}
\Gamma_s : y &= y_0(x), x_8 \leq x \leq x_9; \ y = y_0(x), x_8 \leq x \leq x_9; \\
&= x_9(x), y_0 \leq y \leq y_9; \ x = x_9(x), y_0 \leq y \leq y_9; \\
\Gamma_m : y &= y_0(x), x_{10} \leq x \leq x_{11}; \ y = y_0(x), x_{10} \leq x \leq x_{12}; \\
&= y_{12}(x), x_{10} \leq x \leq x_{12}; \ y = y_{12}(x), x_{10} \leq x \leq x_{11}; \\
&= x_0(x), y_8 \leq y \leq y_9; \ x = x_0(x), y_8 \leq y \leq y_9; \\
&= x_{12}(y), y_{10} \leq y \leq y_{11}; \ x = x_{12}(y), y_{10} \leq y \leq y_{11}; \\
\Gamma_l : y &= y_0(x), x_8 \leq x \leq x_9; \ y = y_0(x), x_8 \leq x \leq x_9.
\end{align*}
\]

All the above functions are determined by the equation \( H(x, y) = -0.09 \), where \( x_8 = -0.29753, x_9 = 0.25786, x_{10} = -0.50423, x_{11} = -0.48259, x_{12} = 0.6529, y_0 = 0.27302, y_9 = -0.27302, y_{10} = -0.61442, y_{11} = 0.61442, y_{10} = 0.27929, y_{11} = -0.27929. \)

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By direct calculation, Melnikov function of the close orbit $\Gamma_s$ is calculated as follows:

$$M(\Gamma_s) = \int_{x_0}^{x_9} Q_5(x, y_{s0}(x)) \, dx + \int_{x_9}^{x_n} Q_5(x, y_{0s}(x)) \, dx$$

$$- \int_{y_9}^{y_0} P_3(x_{s0}(y), y) \, dy - \int_{y_0}^{y_9} P_3(x_{0s}(y), y) \, dy.$$  

Melnikov function of the close orbit $\Gamma_m$ is calculated as follows:

$$M(\Gamma_m) = \int_{x_1}^{x_{10}} Q_5(x, y_{m0}(x)) \, dx + \int_{x_{10}}^{x_{12}} Q_5(x, y_{01}(x)) \, dx$$

$$+ \int_{x_{12}}^{x_{10}} Q_5(x, y_{12}(x)) \, dx + \int_{x_{10}}^{x_{112}} Q_5(x, y_{2m}(x)) \, dx$$

$$- \int_{y_{10}}^{y_9} P_3(x_{m0}(y), y) \, dy - \int_{y_9}^{y_{10}} P_3(x_{01}(y), y) \, dy$$

$$- \int_{y_{11}}^{y_{10}} P_3(x_{12}(y), y) \, dy - \int_{y_{10}}^{y_{11}} P_3(x_{2m}(y), y) \, dy.$$  

Hence, using numeric computation, we get the expression of $M(\Gamma_s)$ and $M(\Gamma_m)$. Then using the same way, we can compute the expression of $M(\Gamma_l)$. ■

It is well known that as $0 < \varepsilon \ll 1$, the stabilities of singular points $A_i(\varepsilon)$, $i = 1, 2, \cdots, 9$ of system (2) are closely related with the sign of divergence quantity of the points and the first order focus quantity of the points. Denote $V_3(P)$ the first order focus quantity of the singular point $P$ of system (2). Then from focus quantity formulae given in [16] and computing, we get the following lemma.

**Lemma 8** For $\varepsilon > 0$ and small, we get the following formula

$$\text{div}(A_1(\varepsilon)) \doteq (a_1 + 15.92614a_2 + 95.11514a_3 - 63.55727a_4 - 15.88952a_6)\varepsilon + O(\varepsilon^2);$$

$$\text{div}(A_6(\varepsilon)) \doteq (a_1 + 14.45662a_2 + 78.37387a_3 + 54.96742a_4 + 13.74183a_6)\varepsilon + O(\varepsilon^2);$$

$$\text{div}(A_7(\varepsilon)) \doteq (a_1 + 0.977251a_2 + 0.358132a_3 + 0.966071a_4 + 0.241518a_6)\varepsilon + O(\varepsilon^2);$$

$$\text{div}(S_7(\varepsilon)) \doteq (a_1 + 0.64006a_2 + 0.153623a_3 - 0.5127346a_4 - 0.12864a_6)\varepsilon + O(\varepsilon^2);$$

$$\text{div}(S_2(\varepsilon)) \doteq (a_1 + 9.854a_2 + 36.419a_3 + 5.424a_4 + 1.356a_6 + 30.457a_4 - 7.614a_6)\varepsilon + O(\varepsilon^2);$$

$$V_3(A_1(\varepsilon)) \doteq (-34.486a_2 - 328.64a_3 + 260.48a_4 + 65.12a_6)\varepsilon + O(\varepsilon^2), \text{ when } \text{div}(A_4(\varepsilon)) = 0.$$  

### 4 Main results and their proof

Now by using the method of changing the the stabilities of homoclinic loops and the Poincare-Bendixson Theorem, we give the main results and their proof of this paper.

Our main results are stated as follows.

**Theorem 1** There exist functions

$$\varphi_1(a_2, a_3, a_4, a_6, \varepsilon) \doteq -12.2853a_2 - 59.8733a_3 - 38.1319a_4 - 9.5329a_6 + O(\varepsilon);$$

$$\varphi_2(a_3, a_4, a_6, \varepsilon) \doteq -17.31855a_3 + 242.653402a_4 + 60.663349a_6 + O(\varepsilon);$$

$$\varphi_3(a_4, a_6, \varepsilon) \doteq 20.332027a_4 + 5.083007a_6 + O(\varepsilon);$$

$$\varphi_4(a_6, b_4, b_6, \varepsilon) \doteq -0.25a_6 - 0.125076b_4 + 0.031269b_6 + O(\varepsilon);$$

$$\varphi_5(a_3, a_4, a_6) \doteq 38.10311a_3 - 622.0189a_4 - 158.23007a_6,$$

which is calculated as follows:

$$\int_{x_s}^{x_0} Q_5(x, y_{s0}(x)) \, dx + \int_{x_0}^{x_n} Q_5(x, y_{0s}(x)) \, dx$$

$$- \int_{y_0}^{y_9} P_3(x_{s0}(y), y) \, dy - \int_{y_9}^{y_0} P_3(x_{0s}(y), y) \, dy.$$  

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such that for fixed $4b_4 - b_6 > 0$, $a_7 > \phi_1$ and $\varepsilon > 0$ and small, the following two conclusions hold.

(i) If $0 < a_1 - \varphi_1 \ll a_2 - \varphi_2 \ll \varphi_3 - a_3 \ll \varphi_4 - a_4 \ll \varepsilon^2$, then the system (2) at least has 19 limit cycles with the configuration given in Figure 2(a).

(ii) If $0 < a_1 - \varphi_1 \ll \varphi_2 - a_2 \ll \varphi_3 - a_3 \ll \varphi_4 - a_4 \ll \varepsilon^2$, then the system (2) at least has 19 limit cycles with the configuration given in Figure 2(b).

**Proof.** First, we suppose that the parameters of system (2) satisfy the following conditions: $4b_4 - b_6 > 0$, $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_3 = \varphi_3$, $a_4 = \varphi_4$. From Lemma 5 and Lemma 6, we can know that for $0 < \varepsilon < 1$, the system (2) has 3 double homoclinic loops and 3 small homoclinic loops. The 3 double homoclinic loops are degenerated and their stabilities are determined by the sign of $\text{div}(S_2(\varepsilon))$, while the 3 small homoclinic loops’s stabilities are determined by the sign of $\text{div}(S_7(\varepsilon))$.

Noticing the above assumptions and Lemma 8, we know $\text{div}(S_2(\varepsilon)) > 0$, $\text{div}(S_7(\varepsilon)) < 0$. That means the double homoclinic loop $\Gamma_{1.2}(S_1(\varepsilon)) \cup \Gamma_{6.1}(S_1(\varepsilon))$ is inner and outer unstable and the small homoclinic loop $\Gamma_{7.8}(S_7(\varepsilon))$ is inner and outer stable. From Lemma 8, we also know $\text{div}(A_1(\varepsilon)) < 0$, $\text{div}(A_6(\varepsilon)) < 0$ and $\text{div}(A_7(\varepsilon)) > 0$, it mean that $A_1(\varepsilon)$ and $A_6(\varepsilon)$ are stable, and $A_7(\varepsilon)$ is unstable. So by applying Poincaré Bendixson theorem, we are not certain that system (2) has any limit cycle surrounding $A_i(\varepsilon)$, $i = 1, 2, \cdots, 9$. From Lemma 2 and Lemma 7, we also get $M(\Gamma_{1.6}) + M(\Gamma_{4.3}) > 0$ and $M(\Gamma_{3.9}) < 0$. By applying Poincaré Bendixson theorem again, we get that system (2) has one limit cycles surrounding $S_1(\varepsilon)$, $A_i(\varepsilon)$, $O$, $i = 7, 8, \cdots, 9$. From the above analysis, we conclude that system (2) only has one limit cycle under the above assumptions.

In the following, by using the disturbing skill, we prove that system (2) has 18 more limit cycles. In first step, fix the value of $4b_4 - b_6 > 0$, slightly change the value of $a_4$ to satisfy that $0 < \varphi_2 - a_4 \ll \varepsilon^2$. At the same time, let $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_3 = \varphi_3$ and $a_7 > \varphi_1$. So the 3 double homoclinic loops of system (2) still existing but $\text{div}(S_1(\varepsilon)) < 0$. Hence the double homoclinic loop $\Gamma_{1.2}(S_1(\varepsilon)) \cup \Gamma_{6.1}(S_1(\varepsilon))$ have changed from a unstable one to the stable one. By applying Poincaré Bendixson theorem, we get 3 limit cycles near $A_i(\varepsilon)$, $O$ and $S_7(\varepsilon)$, $i = 1, 2, \cdots, 3$.

In second step, fix the value of $a_4$ and continue to let $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_3 > \varphi_1$ and slightly change $a_3$ to satisfy $0 < \varphi_3 - a_3 \ll \varphi_4 - a_4$. Then the small homoclinic loop $\Gamma_{7.8}(S_7(\varepsilon))$ is broken, and a stable limit cycle appears.

The last step, fix the value of $a_4$, $a_7$, and keep $a_1 = \varphi_1$, $a_7 > \varphi_1$ change $a_2$ slightly to satisfy that $|a_2 - \varphi_2| \ll \varphi_3 - a_3$. Then there are two cases (see [13] more details):

Case i. $0 < a_1 - \varphi_1 \ll a_2 - \varphi_2$. Then there are 2 limit cycles;

Case ii. $0 < a_1 - \varphi_1 \ll \varphi_2 - a_2$. Then there are 2 limit cycles.

Noticing the fact that system (2) is $Z_3$-equivariant, we can conclude that we get 18 more limit cycles by using disturbing skill. From the above analysis, we prove the system (2) totally at least has 19 limit cycles whose configurations are given in Figure 2.

The proof is completed.

**Theorem 2** There exist functions $\varphi_1$, $\varphi_2$, $\phi_1$ which are given in (7) and

\[
\begin{align*}
\varphi_5(a_4, a_6, b_4, b_6, \varepsilon) & = 33.39732a_4 + 8.34933a_6 + 1.63416b_4 - 0.40853b_6 + O(\varepsilon); \\
\varphi_6(a_4, b_4, b_6, \varepsilon) & = -0.25a_6 - 0.30905b_4 + 0.07726b_6 + O(\varepsilon),
\end{align*}
\]

such that for fixed $4b_4 - b_6 > 0$, $a_7 > \phi_1$ and $\varepsilon > 0$ and small, the following two conclusions hold.

(i) If $0 < a_1 - \varphi_1 \ll a_2 - \varphi_2 \ll \varphi_5 - a_3 \ll a_4 - \varphi_6 \ll \varepsilon^2$, then the system (2) at least has 23 limit cycles with the configuration given in Figure 3(a).

(ii) If $0 < a_1 - \varphi_1 \ll a_2 - \varphi_2 \ll \varphi_5 - a_3 \ll a_4 - \varphi_6 \ll \varepsilon^2$, then the system (2) at least has 23 limit cycles with the configuration given in Figure 3(b).

**Proof.** First, we suppose that the parameters of system (2) satisfy the following conditions: $4b_4 - b_6 > 0$, $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_7 > \varphi_1$. It mean that the system (2) only has 3 double homoclinic loops. Let $\text{div}(S_1(\varepsilon)) = \text{div}(A_1(\varepsilon)) = 0$, then we get $a_3 = \varphi_5(a_4, a_6, b_4, b_6, \varepsilon)$, $a_4 = \varphi_6(a_4, b_4, b_6, \varepsilon)$, where $\varphi_5$, $\varphi_6$ are given in (8). From Lemma 6, we know the system (2) has 3 degenerated double homoclinic loops and their stabilities are determined by the sign of $\text{div}(S_2(\varepsilon))$.

Suppose all the above assumptions are hold, from Lemma 8, we know $\text{div}(S_2(\varepsilon)) > 0$, it mean that the double homoclinic loop $\Gamma_{1.2}(S_1(\varepsilon)) \cup \Gamma_{6.1}(S_1(\varepsilon))$ is inner and outer unstable. From Lemma 8, we also can have $V_3(A_1(\varepsilon)) > 0$, it mean that $A_1(\varepsilon)$ is a unstable fine focus. So by applying Poincaré Bendixson theorem and noticing the $Z_3$-equivariance of system (2), we get that system (2) has 3 limit cycles repectively surrounding $A_i(\varepsilon)$, $i = 1, 3, 5$. Under the assumptions,
we also get $M(\Gamma_3) > 0$ and $M(\Gamma_9, 1) + M(\Gamma_{3,4}) < 0$, $M(\Gamma_{1,6}) + M(\Gamma_{4,3}) > 0$ and $M(\Gamma_{m}) < 0$. By applying Poincaré Bendixson theorem again, we get that system (2) has two limit cycles, the first one of them is surrounding all the singular points, the second one is surrounding $S_{i}(\varepsilon), A_{i}(\varepsilon), O, i = 7, 8, \cdots, 9$. From the above analysis, we conclude that system (2) has 5 limit cycles under the above given assumptions.

Next, by using same disturbing skill just as the one given in the proof of theorem 1, we can prove that system (2) has 3 more limit cycles which respectively surround $A_{i}(\varepsilon), i = 1, 3, 5$ and 15 more limit cycles near the double homoclinic loops of system (2) with two different configurations. From the above analysis, we prove that the system (2) totally at least has 23 limit cycles whose configurations are given in Figure 3.

The proof is completed.

Remark 2 The configurations of these limit cycles are new and different from the configurations obtained by Zhao liqin [5], where the unperturbed systems is a $Z_6$ equivariant quintic Hamiltonian system.

5 Conclusion

In this paper, the qualitative method of differential equation is used to study the number and distribution of limit cycles of a perturbed quintic Hamiltonian system. The existence and stability theory of heteroclinic loop and homoclinic loop are applied to study the heteroclinic loop and homoclinic loop bifurcation of such system under $Z_3$-equivariant quintic perturbation. By using the method of changing the the stabilities of homoclinic loops and the Poincare-Bendixson Theorem, we find the perturbed system has at least 23 limit cycles with two different configurations.

Acknowledgments

The project was supported by National Natural Science Foundation of China (NSFC 11101189).

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References