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Bifurcations of Limit Cycles in A Z_3-Equivariant Planar Vector Field of Degree 5

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Abstract: In this paper, we study the number and distributions of limit cycles in a Z_3-equivariant quintic planar polynomial system. At least 23 limit cycles are found in this system by using the bifurcation methods of double homoclinic loops and Poincare-Bendixson Theorem. The configurations and number of these limit cycles obtained in the above Z_3-equivariant planar system are new. The results obtained are useful to the study of weakened Hilbert's 16th Problem.

Keywords: Double homoclinic loops; Stability; Melnikov function; Bifurcations of limit cycles; Poincaré-Bendixson theorem.

1 Introduction

It is well known that the great Russian mathematician Arnold [1] posed the weakened Hilbert's 16th Problem in 1977. It is to find the maximal number and relative positions of limit cycles of the planar polynomial vector field:

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,y)}{\partial y} + \varepsilon P_n(x,y), \\ \dot{y} &= -\frac{\partial H(x,y)}{\partial x} + \varepsilon Q_n(x,y), \end{aligned} \tag{1}$$

where $H(x,y)$ are polynomials of degree $n+1$, P_n and Q_n are polynomials of degree n.

Up to now, many mathematicians have study this problem, but the existence of $H(n)$ is still an open problem, where $H(n)$ stands for the supremum of the number of limit cycles for system (1). Some results of these mathematicians are concluded as follow: $H(2) \geq 4$, $H(3) \geq 13$, $H(4) \geq 20$ (see [2][3][4][18][19][20] for more details).

As to quintic planar polynomial system, there also are some results. In [5], Zhao Liqin obtained at least 23 limit cycles for a Z_3 equivariant near-Hamiltonian system of degree 5 which is the perturbation of a Z_6 equivariant quintic Hamiltonian system. Li et al.[6] studied a Z_6 equivariant perturbed Hamiltonian planar polynomial vector field of degree 5 and found that there exist at least 24 limit cycles. In [7], at least 24 limit cycles were found and two different configurations of them were given in a quintic Z_3 equivariant near-Hamiltonian system by Y.Wu. In [8, 9], it was obtained that there were at least 25 limit cycles in Z_q equivariant near-Hamiltonian system of degree 5 where $q = 2, 5$. It was shown in [10] that there were at least 28 limit cycles with four different configurations in a Z_2 equivariant quintic planar vector field.

In this paper, the following near Hamiltonian system is considered

$$\begin{aligned} \dot{x} &= \frac{\partial H(x,y)}{\partial y} + \varepsilon P_5(x,y), \\ \dot{y} &= -\frac{\partial H(x,y)}{\partial x} + \varepsilon Q_5(x,y), \end{aligned} \tag{2}$$

where $\varepsilon > 0$ and small,

$$H(x,y) = -\frac{3x^2}{2} - \frac{113x^3}{250} - \frac{3y^2}{2} - \frac{2x^6}{3} + \frac{339xy^2}{250} + 8x^2y^2 - 5x^4y^2 + 4x^4 + 4y^4 - y^6, \tag{3}$$

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the quintic polynomials $P_5(x,y)$ $Q_5(x,y)$ are given in the following forms

$$\begin{aligned} P_5(x,y) &= a_1x + a_2(x^3+xy^2) + a_3(x^5+2x^3y^2+xy^4) + a_4(x^4-6x^2y^2+y^4) \\ &+ a_6(x^4-y^4) + a_7(10x^2y^3-5x^4y-y^5) + b_4(xy^3-4x^3y) \\ &+ b_6(2xy^3+2x^3y); \end{aligned}$$

$$\begin{aligned} Q_5(x,y) &= a_1y + a_2(x^2y+y^3) + a_3(x^4y+2x^2y^3+y^5) + a_4(4x^3y-4xy^3) \\ &- a_6(2x^3y+2xy^3) + a_7(x^5-10x^3y^2+5xy^4) + b_4(x^4-6x^2y^2+y^4) \\ &+ b_6(x^4-y^4). \end{aligned}$$

Here we consider the real coefficients $a_i, b_j, i = 1,2,3,4,6,7, j = 4,6$ as parameters.

From [11], we know that the vector field defined by $(P_5(x,y),\ Q_5(x,y))$ is invariant under $2\pi/3$ rotation with respect to the origin O, so it is easy to check that system (2) is Z_3 equivariant. Furthermore, we know that system (2) is a Hamiltonian system if and only if $a_1 = 0$, $a_2 = 0$, $a_3 = 0$, $4a_4 + a_6 = 0$, $4b_4 - b_6 = 0$. It is well known that a Hamiltonian system does not have any limit cycles. Hence, in this paper we assume that

$$a_1^2 + a_2^2 + a_3^2 + (4a_4+a_6)^2 + (4b_4-b_6)^2 \neq 0. \tag{4}$$

This paper is arranged as follows. In section 2, we will give the phase portraits of the unperturbed system $(2)|_{\varepsilon=0}$ and make some descriptions of the unperturbed system. Section 3 gives some preliminary lemmas about the existence conditions of double homoclinic loops and the small homoclinic loops of system (2). We give the main results and their proof by using the method of changing the stabilities of double homoclinic loops in section 4. In section 5, we give the conclusion.

2 Phase portraits of unperturbed system

The unperturbed system of system (2) is a Hamiltonian system which has the form

$$\begin{aligned} \dot{x} &= -3y + \frac{339xy}{125} + 16x^2y - 10x^4y + 16y^3 - 6y^5, \\ \dot{y} &= 3x + \frac{339x^2}{250} - 16x^3 + 4x^5 - \frac{339y^2}{250} - 16xy^2 + 20x^3y^2. \end{aligned} \tag{5}$$

It is easy to check that unperturbed system (5) has 19 singular points by solving polynomial equations, which are 9 centers $A_i(i = 1,2,\cdots,9\ )$ and 9 saddle points $S_i(i = 1,2,\cdots,9)$ and the origin O. The coordinates of these singular points are listed as follows:

$$\begin{gathered} A_1(x_{A_1},\, y_{A_1}),\ S_1(x_{S_1},\, y_{S_1}),\ A_7(-x_{A_7},\, -y_{A_7}), \\ A_2(-x_{A_2},\, y_{A_2}),\ S_2(-x_{S_2},\, y_{S_2}),\ A_8(-x_{A_7},\, y_{A_7}), \\ A_3(-x_{A_3},\, 0),\ S_3(-x_{S_3},\, y_{S_3}),\ A_9(x_{A_9},\, 0), \\ A_4(-x_{A_2},\, -y_{A_2}),\ S_4(-x_{S_3},\, -y_{S_3}),\ S_7(-0.4,\, 0), \\ A_5(x_{A_1},\, -y_{A_1}),\ S_5(-x_{S_2},\, -y_{S_2}),\ S_8(0.2,\, y_{S_8}), \\ A_6(x_{A_6},\, 0),\ S_6(x_{S_1},\, -y_{S_1}),\ S_9(0.2,\, -y_{S_8}), \end{gathered}$$

where $x_{A_1} \doteq 0.997688$, $y_{A_1} \doteq 1.72805$, $x_{A_2} \doteq 0.950548$, $y_{A_2} \doteq 1.6464$, $x_{A_3} \doteq 1.99538$, $x_{A_6} \doteq 1.9011$, $x_{A_7} \doteq 0.24714$, $y_{A_7} \doteq 0.428059$, $x_{A_9} \doteq 0.49428$, $x_{S_1} \doteq 1.40307$, $y_{S_1} \doteq 0.70365$, $x_{S_2} \doteq 0.092154$, $y_{S_2} \doteq 1.56692$, $x_{S_3} \doteq 1.31091$, $y_{S_3} \doteq 0.863267$, $y_{S_8} \doteq 0.34641$.

Remark 1 From [11], it is easy to check that the unperturbed system (5) is Z_3-equivariant.

To plot the phase portraits of system (5), we list the representative orbits of system (5) in the following.

System (5) is a Hamiltonian system, and it has the first integral of the form $H(x, y) = h$, where function $H(x, y)$ is defined in (3). Then we have $H(O) = 0$, $H(S_i) = h_3$, $H(S_j) = h_2$, $H(A_m) = h_4$, $H(A_n) = h_5$, $H(A_j) = h_1$, $i = 1, 2, \cdots, 6$, $j = 7, 8, 9$, $m = 2, 4, 6$, $n = 1, 3, 5$, where $h_1 \doteq -0.192019$, $h_2 \doteq -0.111403$, $h_3 \doteq 5.47628$, $h_4 \doteq 12.2493$, $h_5 \doteq 18.9507$.

The level curve defined by $H(x, y) = h_3$ consists of the six saddle points S_i and six heteroclinic loops denoted by $\Gamma_{i,j}^{h_3} \cup \Gamma_{j,i}^{h_3}$ ($j = i + 1$ as $i = 1, 2, \cdots, 6$ and $j = 1$ as $i = 6$), where $\Gamma_{i,j}^{h_3}$ represents the saddle connection between S_i and S_j with the direction from point S_i to point S_j. Every heteroclinic loops $\Gamma_{i,j}^{h_3} \cup \Gamma_{j,i}^{h_3}$ embraces the focus A_i($i = 1, 2, \cdots, 6$ and $j = 1$ as $i = 6$). Similarly, the level curve which is defined by $H(x, y) = h_2$ consists of the three saddle points S_i and five heteroclinic loops which are denoted by $\Gamma_{i,j}^{h_2} \cup \Gamma_{j,i}^{h_2}$($j = i + 1$ as $i = 7, 8$, and $j = 7$ as $i = 9$) and $\Gamma_{7,9}^{h_2} \cup \Gamma_{9,8}^{h_2} \cup \Gamma_{8,7}^{h_2}$ and $\Gamma_{9,7}^{h_2} \cup \Gamma_{8,9}^{h_2} \cup \Gamma_{7,8}^{h_2}$. Every heteroclinic loops $\Gamma_{i,j}^{h_2} \cup \Gamma_{j,i}^{h_2}$ embraces the focus A_i($i = 7, 8, 9$).

Then we denote the close orbit of the unperturbed system (5) by $\Gamma : H(x, y) = h$, and these close orbits have the following properties.

(i) If $h < h_3$, the close orbit $\Gamma : H(x, y) = h$ shrinks as h increases, and surrounds all the singular points of the unperturbed system (5).

(ii) If $h_2 < h < h_3$, the close orbit $\Gamma : H(x, y) = h$ expands as h increases, and surrounds O, A_i, S_i($i = 7, 8, 9$).

(iii) If $h_2 < h < 0$, the close orbit $\Gamma : H(x, y) = h$ shrinks as h increases, and only surrounds the origin O.

(iv) If $h_1 < h < h_2$, the close orbit $\Gamma : H(x, y) = h$ expands as h increases, and only surrounds the single singular point A_i($i = 7, 8, 9$) respectively.

(v) If $h_3 < h < h_5$, the close orbit $\Gamma : H(x, y) = h$ shrinks as h increases, and only surrounds the single singular point A_i($i = 1, 2, \cdots, 6$) respectively.

From the above analysis, we obtain the phase portraits of unperturbed system (5) in Figure 1.

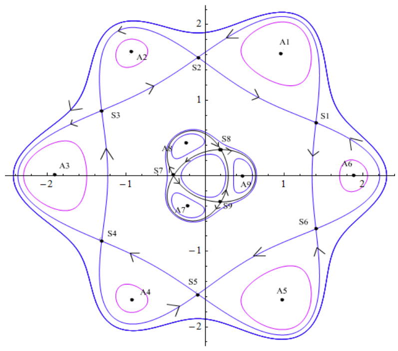


Figure 1: The phase portraits of unperturbed system (5)

3 Preliminary lemmas

As $0 < \varepsilon \ll 1$, system (2) can be regarded as perturbation of system (5). After perturbation, it is easy to find that the number of singular points is preserved, so system (2) still has 19 singular points. Denote the singular points of system (2) by $A_i(\varepsilon)$, $S_i(\varepsilon)$ which near A_i, S_i, $i = 1, 2, \cdots, 9$. Generally speaking, the saddle connections $\Gamma_{i,i+1}$, $\Gamma_{i+1,i}$, ($i = 1, 2, \cdots, 9$) of system (5) will break after perturbation. Denote $\Gamma_{S_i(\varepsilon)}^s$, $\Gamma_{S_i(\varepsilon)}^u$ the stable and unstable manifold of saddle point $S_i(\varepsilon)$. From [7], we know that the distance between $\Gamma_{S_i+1(\varepsilon)}^s$, $\Gamma_{S_i(\varepsilon)}^u$ is $d(\varepsilon, \Gamma_{i,j}) = \varepsilon N_{i,j} \cdot M(\Gamma_{i,j}) + O(\varepsilon^2)(N_{i,j} > 0$, $0 < \varepsilon \ll 1$, j=i+1), and $M(\Gamma_{i,j})$ is called Melnikov function of the saddle connections $\Gamma_{i,j}$, which is defined by the following function

$$M(\Gamma_{i,j}) = \int_{\Gamma_{i,j}} Q_5(x, y)dx - \int_{\Gamma_{i,j}} P_5(x, y)dy. \tag{6}$$

Then we obtain the following results.

Lemma 1 Melnikov functions of system (2) satisfy the following equations

$$M(\Gamma_{1,6})=M(\Gamma_{5,4})=M(\Gamma_{3,2}),\ M(\Gamma_{6,1})=M(\Gamma_{4,5})=M(\Gamma_{2,3});$$
$$M(\Gamma_{3,4})=M(\Gamma_{5,6})=M(\Gamma_{1,2}),\ M(\Gamma_{4,3})=M(\Gamma_{6,5})=M(\Gamma_{2,1});$$
$$M(\Gamma_{7,8})=M(\Gamma_{8,9})=M(\Gamma_{9,7}),\ M(\Gamma_{8,7})=M(\Gamma_{9,8})=M(\Gamma_{7,9}).$$

Proof. Noticing the fact that system (2) is Z_3-equivariant, it is easy to prove the lemma. ■

From Lemma 1, we only need to compute the following six Melnikov functions: $M(\Gamma_{1,6})$, $M(\Gamma_{6,1})$, $M(\Gamma_{3,4})$, $M(\Gamma_{4,3})$, $M(\Gamma_{7,8})$, $M(\Gamma_{8,7})$. With the aid of numeric calculation of Mathematica 8.0, we obtain the following results.

Lemma 2 For $0<\varepsilon\ll 1$, Melnikov functions $M(\Gamma_{1,6})$, $M(\Gamma_{6,1})$, $M(\Gamma_{3,4})$ and $M(\Gamma_{4,3})$ respectively have the following forms

$$\begin{aligned}
M(\Gamma_{1,6}) &\doteq -0.04316a_1+3.8659a_2+7.98226a_3+1.99840a_4+5.18956a_6+1.72086a_7;\\
M(\Gamma_{6,1}) &\doteq -0.76515a_1-13.7964a_2-56.3786a_3-32.8209a_4-12.8951a_6-1.72086a_7;\\
M(\Gamma_{3,4}) &\doteq -1.22092a_1-20.2829a_2-92.36317a_3+49.0403a_4+16.9500a_6+1.72086a_7;\\
M(\Gamma_{4,3}) &\doteq -0.08185a_1+3.8703a_2+7.30355a_3+0.16772a_4-4.64803a_6-1.720863a_7.
\end{aligned}$$

Proof. By using Mathematica 8.0, we obtain the following functions

$$\begin{aligned}
\Gamma_{1,6}:y &= y_{16}(x),\ x_2\le x\le x_1;\ x=x_{16}(y),\ -y_1\le y\le y_1;\\
\Gamma_{4,3}:y &= y_{43}(x),\ x_3\le x\le x_4;\ x=x_{43}(y),\ -y_2\le y\le y_2;\\
\Gamma_{6,1}:y &= y_{60}(x),\ x_1\le x\le x_1^*;\ y=y_{01}(x),\ x_1^*\le x\le x_2^*;\ x=x_{61}(y),\ -y_1\le y\le y_1;\\
\Gamma_{3,4}:y &= y_{30}(x),\ x_3^*\le x\le x_3;\ y=y_{04}(x),\ x_4^*\le x\le x_3^*;\ x=x_{34}(y),\ -y_2\le y\le y_2.
\end{aligned}$$

All the above functions are determined by the equation $H(x,y)=h_3$, where $x_1=x_{S_1}$, $x_2\doteq 1.35480$, $x_3\doteq -x_{S_3}$, $x_4\doteq -1.23340$, $x_1^*\doteq 1.46390$, $x_2^*\doteq 2.21126$, $x_3^*\doteq -1.34914$, $x_4^*\doteq -2.372014$, $y_1=y_{S_1}$, $y_2=y_{S_3}$.

From (6), Melnikov function of the saddle connection $\Gamma_{1,6}$ is calculated as follows:

$$\begin{aligned}
\int_{\Gamma_{1,6}}Q_5(x,y)dx &= \int_{x_1}^{x_2}Q_5(x,y_{16}(x))dx+\int_{x_2}^{x_1}Q_5(x,-y_{16}(x))dx\\
&= a_1k_{1,1}+a_2k_{2,1}+a_3k_{3,1}+a_4k_{4,1}+a_6k_{6,1}+a_7k_{7,1};\\
\int_{\Gamma_{1,6}}P_5(x,y)dy &= \int_{y_1}^{-y_1}P_5(x_{16}(y),y)dy\\
&= a_1k_{1,2}+a_2k_{2,2}+a_3k_{3,2}+a_4k_{4,2}+a_6k_{6,2}+a_7k_{7,2}.
\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{6,1}$ is calculated as follows:

$$\begin{aligned}
\int_{\Gamma_{6,1}}Q_5(x,y)dx &= \int_{x_1}^{x_1^*}Q_5(x,y_{60}(x))dx+\int_{x_1^*}^{x_2*}Q_5(x,y_{01}(x))dx+\\
&\quad \int_{x_2^*}^{x_1*}Q_5(x,-y_{01}(x))dx+int_{x_1^*}^{x_1}Q_5(x,-y_{60}(x))dx\\
&= a_1k_{1,3}+a_2k_{2,3}+a_3k_{3,3}+a_4k_{4,3}+a_6k_{6,3}+a_7k_{7,3};\\
\int_{\Gamma_{6,1}}P_5(x,y)dy &= \int_{-y_1}^{y_1}P_5(x_{61}(y),y)dy\\
&= a_1k_{1,4}+a_2k_{2,4}+a_3k_{3,4}+a_4k_{4,4}+a_6k_{6,4}+a_7k_{7,4}.
\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{3,4}$ is calculated as follows:

$$\begin{aligned}
\int_{\Gamma_{3,4}} Q_5(x,y)dx &= \int_{x_3}^{x_3^*} Q_5(x,y_{30}(x))dx + \int_{x_3^*}^{x_4^*} Q_5(x,y_{04}(x))dx + \\
&\quad \int_{x_4^*}^{x_3^*} Q_5(x,-y_{04}(x))dx + int_{x_3^*}^{x_3} Q_5(x,-y_{30}(x))dx \\
&= a_1k_{1,5} + a_2k_{2,5} + a_3k_{3,5} + a_4k_{4,5} + a_6k_{6,5} + a_7k_{7,5}; \\
\int_{\Gamma_{3,4}} P_5(x,y)dy &= \int_{y_2}^{-y_2} P_5(x_{34}(y),y)dy \\
&= a_1k_{1,6} + a_2k_{2,6} + a_3k_{3,6} + a_4k_{4,6} + a_6k_{6,6} + a_7k_{7,6}.
\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{4,3}$ is calculated as follows:

$$\begin{aligned}
\int_{\Gamma_{4,3}} Q_5(x,y)dx &= \int_{x_3}^{x_4} Q_5(x,y_{43}(x))dx + \int_{x_4}^{x_3} Q_5(x,-y_{43}(x))dx \\
&= a_1k_{1,7} + a_2k_{2,7} + a_3k_{3,7} + a_4k_{4,7} + a_6k_{6,7} + a_7k_{7,7}; \\
\int_{\Gamma_{4,3}} P_5(x,y)dy &= \int_{-y_2}^{y_2} P_5(x_{43}(y),y)dy \\
&= a_1k_{1,8} + a_2k_{2,8} + a_3k_{3,8} + a_4k_{4,8} + a_6k_{6,8} + a_7k_{7,8}.
\end{aligned}$$

After a lot of numerical calculations, we obtain the following results by using Mathematica 8.0:

$$\begin{aligned}
k_{1,1} &\doteq -0.04316137,\ k_{2,1} \doteq -0.09482117,\ k_{3,1} \doteq -0.2095090,\ k_{4,1} \doteq -0.39051156, \\
&\quad k_{6,1} \doteq 0.26267075,\ k_{7,1} \doteq 0.562356668; \\
k_{1,2} &\doteq 0,\ k_{2,2} \doteq -3.9608195,\ k_{3,2} \doteq -8.19177,\ k_{4,2} \doteq -2.38891,\ k_{6,2} \doteq -4.9268956, \\
&\quad k_{7,2} \doteq -1.158506660; \\
k_{1,3} &\doteq -0.76515175,\ k_{2,3} \doteq -2.54847278,\ k_{3,3} \doteq -8.78324015,\ k_{4,3} \doteq -15.0207547, \\
&\quad k_{6,3} \doteq 9.0094163,\ k_{7,3} \doteq 31.1831989; \\
k_{1,4} &\doteq 0,\ k_{2,4} \doteq 11.247930,\ k_{3,4} \doteq 47.595420,\ k_{4,4} \doteq 17.800185,\ k_{6,4} \doteq 21.9046152, \\
&\quad k_{7,4} \doteq 32.9040622; \\
k_{1,5} &\doteq -1.22092592,\ k_{2,5} \doteq -4.31925507,\ k_{3,5} \doteq -16.0690833,\ k_{4,5} \doteq 24.0859745, \\
&\quad k_{6,5} \doteq -15.546579,\ k_{7,5} \doteq 50.3377220; \\
k_{1,6} &\doteq 0,\ k_{2,6} \doteq 15.963738,\ k_{3,6} \doteq 76.29409,\ k_{4,6} \doteq -24.954327,\ k_{6,6} \doteq -32.496619, \\
&\quad k_{7,6} \doteq 48.6168589; \\
k_{1,7} &\doteq -0.08185760,\ k_{2,7} \doteq -0.16738669,\ k_{3,7} \doteq -0.34742379,\ k_{4,7} \doteq 0.51804027, \\
&\quad k_{6,7} \doteq -0.4296418,\ k_{7,7} \doteq 0.569370864; \\
k_{1,8} &\doteq 0,\ k_{2,8} \doteq -4.0377181,\ k_{3,8} \doteq -7.6509821,\ k_{4,8} \doteq 0.350312,\ k_{6,8} \doteq 4.218390, \\
&\quad k_{7,8} \doteq 2.29023383.
\end{aligned}$$

From above numeric results and equation (6), the lemma is proved. ■

Lemma 3 For $0 < \varepsilon \ll 1$, Melnikov functions $M(\Gamma_{7,8})$, $M(\Gamma_{8,7})$ respectively have the following forms

$$\begin{aligned}
M(\Gamma_{7,8}) &\doteq 0.348711099a_1 + 0.20254578a_2 + 0.06900199a_3 + 0.07863389a_4 + 0.01965847a_6; \\
M(\Gamma_{8,7}) &\doteq -0.147538383a_1 - 0.02527178a_2 - 0.00294147a_3 + 0.00449010a_4 + 0.00112252a_6.
\end{aligned}$$

Proof. By using Mathematica 8.0, we obtain the following functions

$$\begin{aligned}\Gamma_{7,8}: y &= y_{70}(x),\ x_5 \le x \le x_6;\ y = y_{08}(x),\ x_6 \le x \le x_7;\\ x &= x_{70}(y),\ y_3 \le y \le y_4;\ x = x_{08}(y),\ y_4 \le x \le y_5;\\ \Gamma_{8,7}: y &= y_{87}(x),\ x_5 \le x \le x_7;\ x = x_{87}(y),\ y_3 \le y \le y_5.\end{aligned}$$

All the above functions are determined by the equation $H(x,y) = h_2$, where $x_5 = -0.4$, $x_6 \doteq -0.481108$, $x_7 = 0.2$, $y_3 = 0$, $y_4 \doteq 0.59688$, $y_5 \doteq y_{S_8}$.

From (6), Melnikov function of the saddle connection $\Gamma_{7,8}$ is calculated as follows:

$$\begin{aligned}\int_{\Gamma_{7,8}} Q_5(x,y)dx &= \int_{x_5}^{x_6} Q_5(x,y_{70}(x))dx + \int_{x_6}^{x_7} Q_5(x,y_{08}(x))dx\\ &= a_1k_{1,9} + a_2k_{2,9} + a_3k_{3,9} + a_4k_{4,9} + a_6k_{6,9} + a_7k_{7,9} + +b_4k_{8,9} + b_6k_{9,9};\\ \int_{\Gamma_{7,8}} P_5(x,y)dy &= \int_{y_3}^{y_4} P_5(x_{70}(y),y)dy + \int_{y_4}^{y_5} P_5(x_{08}(y),y)dy\\ &= a_1k_{1,10} + a_2k_{2,10} + a_3k_{3,10} + a_4k_{4,10} + a_6k_{6,10} + a_7k_{7,10} + b_4k_{8,10} + b_6k_{9,10}.\end{aligned}$$

Melnikov function of the saddle connection $\Gamma_{8,7}$ is calculated as follows:

$$\begin{aligned}\int_{\Gamma_{8,7}} Q_5(x,y)dx &= \int_{x_7}^{x_5} Q_5(x,y_{87}(x))dx\\ &= a_1k_{1,11} + a_2k_{2,11} + a_3k_{3,11} + a_4k_{4,11} + a_6k_{6,11} + a_7k_{7,11} + b_4k_{8,11} + b_6k_{9,11};\\ \int_{\Gamma_{8,7}} P_5(x,y)dy &= \int_{y_5}^{y_3} P_5(x_{87}(y),y)dy\\ &= a_1k_{1,12} + a_2k_{2,12} + a_3k_{3,12} + a_4k_{4,12} + a_6k_{6,12} + a_7k_{7,12} + b_4k_{8,12} + b_6k_{9,12}.\end{aligned}$$

After a lot of numerical calculations, we obtain the following results by using Mathematica 8.0:

$$\begin{aligned}k_{1,9} &\doteq 0.348711,\ k_{2,9} \doteq 0.12041,\ k_{3,9} \doteq 0.043109,\ k_{4,9} \doteq 0.03843,\ k_{6,9} \doteq 0.041111\\ &\quad k_{7,9} \doteq 0.0077271,\ k_{8,9} \doteq 0.001536,\ k_{9,9} \doteq -0.0541830;\\ k_{1,10} &\doteq 0,\ k_{2,10} \doteq -0.08213,\ k_{3,10} \doteq -0.02589,\ k_{4,10} \doteq -0.040203,\ k_{6,10} \doteq 0.02145\\ &\quad k_{7,10} \doteq 0.0077271,\ k_{8,10} \doteq 0.001536,\ k_{9,10} \doteq -0.0541830;\\ k_{1,11} &\doteq -0.1475,\ k_{2,11} \doteq -0.0161,\ k_{3,11} \doteq -0.0018,\ k_{4,11} \doteq 0.0031,\ k_{6,11} \doteq -0.0004\\ &\quad k_{7,11} \doteq 0.0003180,\ k_{8,11} \doteq -0.001536,\ k_{9,11} \doteq 0.0018087;\\ k_{1,12} &\doteq 0\ ,\ k_{2,12} \doteq 0.00915,\ k_{3,12} \doteq 0.001111,\ k_{4,12} \doteq -0.001358,\ k_{6,12} \doteq -0.001605\\ &\quad k_{7,12} \doteq 0.0003180,\ k_{8,12} \doteq -0.001536,\ k_{9,12} \doteq 0.0018087.\end{aligned}$$

From above numeric results and equation (6), the lemma is proved. ■

Next, from [12], we give the existence conditions of double homoclinic loops and the small homoclinic loops of system (2).

Lemma 4 As $0 < \varepsilon \ll 1$, there exist functions

$$\begin{aligned}d(\varepsilon,\Gamma_{6,1},\Gamma_{1,6}) &= \varepsilon N_1(M(\Gamma_{6,1}) + M(\Gamma_{1,6})) + O(\varepsilon^2),\ N_1 > 0\\ d(\varepsilon,\Gamma_{3,4},\Gamma_{4,3}) &= \varepsilon N_2(M(\Gamma_{3,4}) + M(\Gamma_{4,3})) + O(\varepsilon^2),\ N_2 > 0\\ d(\varepsilon,\Gamma_{7,8},\Gamma_{8,7}) &= \varepsilon N_3(M(\Gamma_{7,8}) + M(\Gamma_{8,7})) + O(\varepsilon^2),\ N_3 > 0\end{aligned}$$

such that

(i) when $d(\varepsilon, \Gamma_{6,1}, \Gamma_{1,6}) = 0$, then system (2) has a homoclinic loop denoted by $\Gamma_{6,1}(S_1(\varepsilon))$ (resp., $\Gamma_{6,1}(S_6(\varepsilon))$) near $\Gamma_{6,1} \cup \Gamma_{1,6}$, which passes through the saddle point $S_1(\varepsilon)$ (resp., $S_6(\varepsilon)$) if $d(\varepsilon, \Gamma_{6,1}) < 0$(resp., $d(\varepsilon, \Gamma_{6,1}) > 0$);

(ii) when $d(\varepsilon, \Gamma_{3,4}, \Gamma_{4,3}) = 0$, then system (2) has a homoclinic loop denoted by $\Gamma_{3,4}(S_3(\varepsilon))$ (resp., $\Gamma_{3,4}(S_4(\varepsilon))$) near $\Gamma_{3,4} \cup \Gamma_{4,3}$, which passes through the saddle point $S_3(\varepsilon)$(resp., $S_4(\varepsilon)$) if $d(\varepsilon, \Gamma_{3,4}) > 0$(resp., $d(\varepsilon, \Gamma_{3,4}) < 0$);

(iii) when $d(\varepsilon, \Gamma_{7,8}, \Gamma_{8,7}) = 0$, then system (2) has a homoclinic loop denoted by $\Gamma_{7,8}(S_7(\varepsilon))$ (resp., $\Gamma_{7,8}(S_8(\varepsilon))$) near $\Gamma_{7,8} \cup \Gamma_{8,7}$, which passes through the saddle point $S_7(\varepsilon)$(resp., $S_8(\varepsilon)$) if $d(\varepsilon, \Gamma_{7,8}) < 0$(resp., $d(\varepsilon, \Gamma_{7,8}) > 0$).

From Lemma 4, we got the following result.

Lemma 5 There exist functions φ_1, φ_2, φ_3, ϕ_1

$$\begin{aligned}
\varphi_1(a_2, a_3, a_4, a_6, \varepsilon) &\doteq -12.2853439a_2 - 59.873330a_3 - 38.131921a_4 - 9.5329803a_6 + O(\varepsilon); \\
\varphi_2(a_3, a_4, a_6, \varepsilon) &\doteq -17.318557a_3 + 242.653402a_4 + 60.663349a_6 + O(\varepsilon); \\
\varphi_3(a_4, a_6, \varepsilon) &\doteq 20.3320277a_4 + 5.083007a_6 + O(\varepsilon); \\
\phi_1(a_3, a_4, a_6) &\doteq 38.10311a_3 - 622.0189a_4 - 158.23007a_6;
\end{aligned}$$

such that the following conclusions hold.

(i) If $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_7 > \phi_1$, then the system (2) has three double homoclinic loops $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$, $\Gamma_{2,3}(S_3(\varepsilon)) \cup \Gamma_{3,4}(S_3(\varepsilon))$, $\Gamma_{4,5}(S_5(\varepsilon)) \cup \Gamma_{5,6}(S_5(\varepsilon))$, where $\Gamma_{i,j}(S_i(\varepsilon))$ is the homoclinic loop passing through saddle point $S_i(\varepsilon)$ and tending to $\Gamma_{i,j} \cup \Gamma_{j,i}$, as $\varepsilon \to 0$, $i = 1, 3, 5$;

(ii) Assume the condition (1) is hold, further if $a_3 = \varphi_3$, then the system (2) has three more small homoclinic loops $\Gamma_{7,8}(S_7(\varepsilon))$, $\Gamma_{8,9}(S_8(\varepsilon))$, $\Gamma_{9,7}(S_9(\varepsilon))$, where $\Gamma_{i,j}(S_i(\varepsilon))$ is the homoclinic loop passing through saddle point $S_i(\varepsilon)$ and tending to $\Gamma_{i,j} \cup \Gamma_{j,i}$, as $\varepsilon \to 0$, $i = 7, 8, 9$.

Proof. From Lemma 4, we can study the conditions that the system (2) have three double homoclinic loops and three small homoclinic loops. Let $d(\varepsilon, \Gamma_{6,1}, \Gamma_{1,6}) = d(\varepsilon, \Gamma_{3,4}, \Gamma_{4,3}) = 0$ and $d(\varepsilon, \Gamma_{7,8}, \Gamma_{8,7}) = 0$. Then from the implicit function theorem, we know that there exist functions φ_1, φ_2, to let the following equation hold

$$d(\varepsilon, \Gamma_{6,1}, \Gamma_{1,6}) = d(\varepsilon, \Gamma_{3,4}, \Gamma_{4,3}) = 0 \Leftrightarrow a_1 = \varphi_1,\ a_2 = \varphi_2$$

When equations $a_1 = \varphi_1$, $a_2 = \varphi_2$ hold, we have

$$\begin{aligned}
M(\Gamma_{6,1}) &\doteq 65.57024a_3 - 1070.4094a_4 - 272.29232a_6 - 1.7208633a_7, \\
M(\Gamma_{3,4}) &\doteq 72.24030a_3 - 1186.4623a_4 - 291.9256a_6 + 1.7208633a_7.
\end{aligned}$$

Let $M(\Gamma_{6,1}) = 0$, we get $a_7 = \phi_1(a_3, a_4, a_6)$, where ϕ_1 is given in (7). Therefore, if $a_7 > \phi_1$, then $d(\varepsilon, \Gamma_{6,1}) < 0$, $d(\varepsilon, \Gamma_{1,6}) > 0$, and if $a_7 > \phi_1$, we also get $d(\varepsilon, \Gamma_{3,4}) > 0$.

Hence, from Z_3-equivariance of system (2) and Lemma 4, we prove the first part of the lemma.

From the implicit function theorem, we can also know that there exist function φ_3 to let the following equation hold

$$d(\varepsilon, \Gamma_{7,8}, \Gamma_{8,7}) = 0 \Leftrightarrow a_3 = \varphi_3$$

Then in the same way, we prove that the second part of the lemma is true. ■

In the above context, we have already got the existence conditions of double homoclinic loops and the small homoclinic loops of system (2). Next, the stabilities of the double homoclinic loops and the small homoclinic loops of system (2) need to be studied. Denote the divergence quantity of equilibrium point P of system (2) by $div(P) = (\frac{\partial P_5}{\partial x} + \frac{\partial Q_5}{\partial y})(P)$. Then we will give the criteria to determine the stabilities of the double homoclinic loops and the small homoclinic loops in the following lemma.

Lemma 6 Suppose that the parameters of system (2) satisfy the following conditions $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_3 = \varphi_3$ and $a_7 > \phi_1$. Then we have

(i) if $a_4 - \varphi_4 < 0(resp., a_4 - \varphi_4 > 0)$, then the double homoclinic loop $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$ of system (2) is inner and outer stable(unstable), where φ_4 is given in (7).

(ii) if $a_4 = \varphi_4$ and $div(S_2(\varepsilon)) < 0(resp., div(S_2(\varepsilon)) > 0)$, then the double homoclinic loop $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$ of system (2) is inner and outer stable(unstable).

(iii) if $div(S_7(\varepsilon)) < 0(resp., div(S_7(\varepsilon)) > 0)$, then the small homoclinic loop $\Gamma_{7,8}(S_7(\varepsilon))$ of system (2) is inner and outer stable(unstable).

Proof. By direct calculation, we have the following result:

$$div(S_1(\varepsilon)) \doteq \varepsilon(a_1 + 9.854a_2 + 36.419a_3 + 5.424a_4 + 1.356a_6 - 30.457b_4 + 7.614b_6) + O(\varepsilon^2).$$

From [13, 14, 17], we know that the stabilities of the double homoclinic loop $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$ and the small homoclinic loop $\Gamma_{7,8}(S_7(\varepsilon))$ are determined by the sign of divergence quantities of the saddle point $S_1(\varepsilon)$ and $S_7(\varepsilon)$ respectively, so it easy to prove the first and last part of the lemma is true.

Let $div(S_1(\varepsilon)) = 0$, then we get $a_4 = \varphi_4$(where φ_4 is given in (7)). From [13, 14, 17], we also know if $a_4 = \varphi_4$, the double homoclinic loop $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$ of system (2) is degenerated, and its stability is determined by the sign of the integrals of the following divergence quantities

$$\begin{aligned} \sigma_1 &= \varepsilon \oint_{L_{1,2}(S_1(\varepsilon))} \left(\frac{\partial P_5}{\partial x} + \frac{\partial Q_5}{\partial y}\right)dt; \\ \sigma_2 &= \varepsilon \oint_{L_{6,1}(S_1(\varepsilon))} \left(\frac{\partial P_5}{\partial x} + \frac{\partial Q_5}{\partial y}\right)dt. \end{aligned}$$

From [15], we can know that σ_1, σ_2 and $div(S_2(\varepsilon))$ have the same sign as $div(S_2(\varepsilon)) \neq 0$. So from [13–15, 17], we prove that the second part of the lemma is true.

From the above analysis, the proof is completed. ■

We denote the Melnikov function of the close orbit $\Gamma : H(x, y) = h$ by $M(\Gamma) = \oint_\Gamma Q_5(x, y)dx - P_5(x, y)dy$. The equation $H(x, y) = -0.09$ determines the close orbits Γ_s, Γ_m and Γ_l, where Γ_s only surrounds the origin O, Γ_m surrounds O, A_i, $S_i(i = 7, 8, 9)$ and Γ_l surrounds all the singular points of the unperturbed system (5). As $0 < \varepsilon \ll 1$, we study the breaking way of the close orbits Γ_s, Γ_m and Γ_l. From [16], we know that the breaking ways of the close orbits Γ_s, Γ_m and Γ_l are closely related with the sign of $M(\Gamma_s)$, $M(\Gamma_m)$ and $M(\Gamma_l)$. So we compute the Melnikov functions of these three close orbits in the following lemma.

Lemma 7 For $\varepsilon > 0$ and small, then Melnikov function $M(\Gamma_s)$, $M(\Gamma_m)$ and $M(\Gamma_l)$ respectively have the following expressions

$$\begin{aligned} M(\Gamma_s) &\doteq -0.2386675a_1 - 0.0365510a_2 - 0.0028328a_3 + 0.0028260a_4 + 0.0007194a_6; \\ M(\Gamma_m) &\doteq 1.0469147a_1 + 0.7281667a_2 + 0.2627759a_3 + 0.2377991a_4 + 0.05944979a_6; \\ M(\Gamma_l) &\doteq -14.105015a_1 - 130.05528a_2 - 615.35708a_3 + 41.974227a_4 + 10.493556a_6. \end{aligned}$$

Proof. By using Mathematica 8.0, we get the following functions:

$$\begin{aligned} \Gamma_s : y &= y_{s0}(x),\ x_8 \le x \le x_9;\ y = y_{0s}(x),\ x_8 \le x \le x_9; \\ x &= x_{s0}(y),\ y_6 \le y \le y_7;\ x = x_{0s}(y),\ y_6 \le y \le y_7; \\ \Gamma_m : y &= y_{m0}(x),\ x_{10} \le x \le x_{11};\ y = y_{01}(x),\ x_{10} \le x \le x_{12}; \\ y &= y_{12}(x),\ x_{10} \le x \le x_{12};\ y = y_{2m}(x),\ x_{10} \le x \le x_{11}; \\ x &= x_{m0}(y),\ y_8 \le y \le y_9;\ x = x_{01}(y),\ y_9 \le y \le y_{10}; \\ x &= x_{12}(y),\ y_{10} \le y \le y_{11};\ x = x_{2m}(y),\ y_8 \le y \le y_{11}. \end{aligned}$$

All the above functions are determined by the equation $H(x, y) = -0.09$, where $x_8 = -0.29753$, $x_9 \doteq 0.25786$, $x_{10} \doteq -0.50423$, $x_{11} \doteq -0.48259$, $x_{12} \doteq 0.6529$, $y_6 \doteq 0.27302$, $y_7 \doteq -0.27302$, $y_8 \doteq -0.61442$, $y_9 = 0.61442$, $y_{10} = 0.27929$, $y_{11} = -0.27929$.

By direct calculation, Melnikov function of the close orbit Γ_s is calculated as follows:

$$\begin{aligned} M(\Gamma_s) &= \int_{x_8}^{x_9} Q_5(x, y_{s0}(x))dx + \int_{x_9}^{x_8} Q_5(x, y_{0s}(x))dx \\ &- \int_{y_6}^{y_7} P_5(x_{s0}(y), y)dy - \int_{y_7}^{y_6} P_5(x_{0s}(y), y)dy. \end{aligned}$$

Melnikov function of the close orbit Γ_m is calculated as follows:

$$\begin{aligned} M(\Gamma_m) &= \int_{x_{11}}^{x_{10}} Q_5(x, y_{m0}(x))dx + \int_{x_{10}}^{x_{12}} Q_5(x, y_{01}(x))dx \\ &+ \int_{x_{12}}^{x_{10}} Q_5(x, y_{12}(x))dx + \int_{x_{10}}^{x_{11}} Q_5(x, y_{2m}(x))dx \\ &- \int_{y_8}^{y_9} P_5(x_{m0}(y), y)dy - \int_{y_9}^{y_{10}} P_5(x_{01}(y), y)dy \\ &- \int_{y_{10}}^{y_{11}} P_5(x_{12}(y), y)dy - \int_{y_{11}}^{y_8} P_5(x_{2m}(y), y)dy. \end{aligned}$$

Hence, using numeric computation, we get the epression of $M(\Gamma_s)$ and $M(\Gamma_m)$. Then using the same way, we can compute the expression of $M(\Gamma_l)$. ■

It is well known that as $0 < \varepsilon \ll 1$, the stabilities of singular points $A_i(\varepsilon)$, $i = 1, 2, \cdots, 9$ of system (2) are closely related with the sign of divergence quantity of the points and the first order focus quantity of the points. Denote $V_3(P)$ the first order focus quantity of the singular point P of system (2). Then from focus quantity formulae given in [16] and computing, we get the following lemma.

Lemma 8 For $\varepsilon > 0$ and small, we get the following formula

$$\begin{aligned} div(A_1(\varepsilon)) &\doteq (a_1 + 15.92614a_2 + 95.11514a_3 - 63.55727a_4 - 15.88952a_6)\varepsilon + O(\varepsilon^2); \\ div(A_6(\varepsilon)) &\doteq (a_1 + 14.45662a_2 + 78.37387a_3 + 54.96742a_4 + 13.74183a_6)\varepsilon + O(\varepsilon^2); \\ div(A_7(\varepsilon)) &\doteq (a_1 + 0.977251a_2 + 0.358132a_3 + 0.966071a_4 + 0.241518a_6)\varepsilon + O(\varepsilon^2); \\ div(S_7(\varepsilon)) &\doteq (a_1 + 0.64000a_2 + 0.153623a_3 - 0.5127346a_4 - 0.12864a_6)\varepsilon + O(\varepsilon^2); \\ div(S_2(\varepsilon)) &\doteq (a_1 + 9.854a_2 + 36.419a_3 + 5.424a_4 + 1.356a_6 + 30.457b_4 - 7.614b_6)\varepsilon + O(\varepsilon^2); \\ V_3(A_1(\varepsilon)) &\doteq (-34.486a_2 - 328.64a_3 + 260.48a_4 + 65.12a_6)\varepsilon + O(\varepsilon^2), when\ div(A_1(\varepsilon)) = 0. \end{aligned}$$

4 Main results and their proof

Now by using the method of changing the the stabilities of homoclinic loops and the Poincare-Bendixson Theorem, we give the main results and their proof of this paper.

Our main results are stated as follows.

Theorem 1 *There exist functions*

$$\begin{aligned} \varphi_1(a_2, a_3, a_4, a_6, \varepsilon) &\doteq -12.2853a_2 - 59.8733a_3 - 38.1319a_4 - 9.5329a_6 + O(\varepsilon); \\ \varphi_2(a_3, a_4, a_6, \varepsilon) &\doteq -17.318557a_3 + 242.653402a_4 + 60.663349a_6 + O(\varepsilon); \\ \varphi_3(a_4, a_6, \varepsilon) &\doteq 20.332027a_4 + 5.083007a_6 + O(\varepsilon); \\ \varphi_4(a_6, b_4, b_6, \varepsilon) &\doteq -0.25a_6 - 0.125076b_4 + 0.031269b_6 + O(\varepsilon); \\ \phi_1(a_3, a_4, a_6) &\doteq 38.10311a_3 - 622.0189a_4 - 158.23007a_6, \end{aligned} \tag{7}$$

such that for fixed $4b_4 - b_6 > 0$, $a_7 > \phi_1$ and $\varepsilon > 0$ and small, the following two conclusions hold.

(i) If $0 < a_1 - \varphi_1 \ll a_2 - \varphi_2 \ll \varphi_3 - a_3 \ll \varphi_4 - a_4 \ll \varepsilon^2$, then the system (2) at least has 19 limit cycles with the configuration given in Figure 2(a).

(ii) If $0 < a_1 - \varphi_1 \ll \varphi_2 - a_2 \ll \varphi_3 - a_3 \ll \varphi_4 - a_4 \ll \varepsilon^2$, then the system (2) at least has 19 limit cycles with the configuration given in Figure 2(b).

Proof. First, we suppose that the parameters of system (2) satisfy the following conditions: $4b_4 - b_6 > 0$, $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_3 = \varphi_3$, $a_7 > \phi_1$ and $a_4 = \varphi_4$. From Lemma 5 and Lemma 6, we can know that for $0 < \varepsilon \ll 1$, the system (2) has 3 double homoclinic loops and 3 small homoclinic loops. The 3 double homoclinic loops are degenerated and their stabilities are determined by the sign of $div(S_2(\varepsilon))$, while the 3 small homoclinic loops's stabilities are determined by the sign of $div(S_7(\varepsilon))$.

Noticing the above assumptions and Lemma 8, we know $div(S_2(\varepsilon)) > 0$, $div(S_7(\varepsilon)) < 0$. That means the double homoclinic loop $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$ is inner and outer unstable and the small homoclinic loop $\Gamma_{7,8}(S_7(\varepsilon))$ is inner and outer stable. From Lemma 8, we also know $div(A_1(\varepsilon)) < 0$, $div(A_6(\varepsilon)) < 0$ and $div(A_7(\varepsilon)) > 0$, it mean that $A_1(\varepsilon)$ and $A_6(\varepsilon)$ are stable, and $A_7(\varepsilon)$ is unstable. So by applying Poincaré Bendixson theorem, we are not certain that system (2) has any limit cycle surrounding $A_i(\varepsilon)$, $i = 1, 2, \cdots, 9$. From Lemma 2 and Lemma 7, we also get $M(\Gamma_{1,6}) + M(\Gamma_{4,3}) > 0$ and $M(\Gamma_m) < 0$. By applying Poincaré Bendixson theorem again, we get that system (2) has one limit cycles surrounding $S_i(\varepsilon)$, $A_i(\varepsilon)$, O, $i = 7, 8, \cdots, 9$. From the above analysis, we conclude that system (2) only has one limit cycle under the above assumptions.

In the following, by using the disturbing skill, we prove that system (2) has 18 more limit cycles. In first step, fix the value of $4b_4 - b_6 > 0$, slightly change the value of a_4 to satisfy that $0 < \varphi_4 - a_4 \ll \varepsilon^2$. At the same time, let $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_3 = \varphi_3$ and $a_7 > \phi_1$. So the 3 double homoclinic loops of system (2) still existing but $div(S_1(\varepsilon)) < 0$. Hence the double homoclinic loop $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$ have changed from a unstable one to the stable one. By applying Poincaré Bendixson theorem, we get 3 limit cycles near $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$.

In second step, fix the value of a_4 and continue to let $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_7 > \phi_1$ and slightly change a_3 to satisfy $0 < \varphi_3 - a_3 \ll \varphi_4 - a_4$. Then the small homoclinic loop $\Gamma_{7,8}(S_7(\varepsilon))$ is broken, and a stable limit cycle appears.

The last step, fix the value of a_4, a_3, and keep $a_1 = \varphi_1$, $a_7 > \phi_1$ change a_2 slightly to satisfy that $|a_2 - \varphi_2| \ll \varphi_3 - a_3$. Then there are two cases(see [13] more details):

Case i. $0 < a_1 - \varphi_1 \ll a_2 - \varphi_2$. Then there are 2 limit cycles;

Case ii. $0 < a_1 - \varphi_1 \ll \varphi_2 - a_2$. Then there are 2 limit cycles.

Noticing the fact that system (2) is Z_3-equivariant, we can conclude that we get 18 more limit cycles by using disturbing skill. From the above analysis, we prove the system (2) totally at least has 19 limit cycles whose configurations are given in Figure 2.

The proof is completed. ■

Theorem 2 *There exist functions φ_1, φ_2, ϕ_1 which are given in (7) and*

$$\begin{aligned} \varphi_5(a_4, a_6, b_4, b_6, \varepsilon) &\doteq 33.39732a_4 + 8.34933a_6 + 1.63416b_4 - 0.40853b_6 + O(\varepsilon); \\ \varphi_6(a_6, b_4, b_6, \varepsilon) &\doteq -0.25a_6 - 0.30905b_4 + 0.07726b_6 + O(\varepsilon), \end{aligned} \tag{8}$$

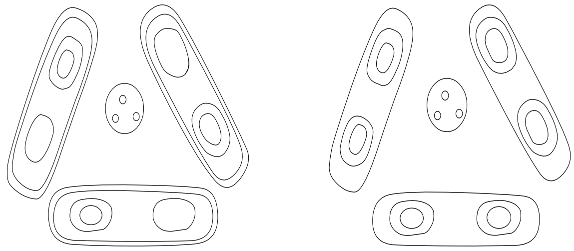
such that for fixed $4b_4 - b_6 > 0$, $a_7 > \phi_1$ and $\varepsilon > 0$ and small, the following two conclusions hold.

(i) If $0 < a_1 - \varphi_1 \ll a_2 - \varphi_2 \ll \varphi_5 - a_3 \ll a_4 - \varphi_6 \ll \varepsilon^2$, then the system (2) at least has 23 limit cycles with the configuration given in Figure 3(a).

(ii) If $0 < a_1 - \varphi_1 \ll \varphi_2 - a_2 \ll \varphi_5 - a_3 \ll a_4 - \varphi_6 \ll \varepsilon^2$, then the system (2) at least has 23 limit cycles with the configuration given in Figure 3(b).

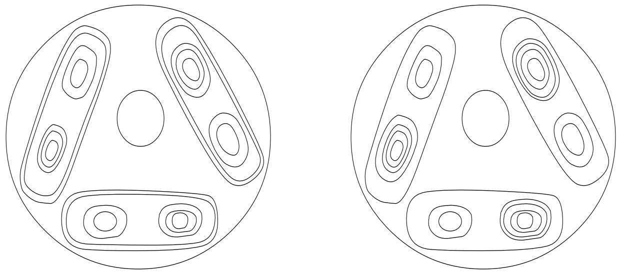
Proof. First, we suppose that the parameters of system (2) satisfy the following conditions: $4b_4 - b_6 > 0$, $a_1 = \varphi_1$, $a_2 = \varphi_2$, $a_7 > \phi_1$. It mean that the system (2) only has 3 double homoclinic loops. Let $div(S_1(\varepsilon)) = div(A_1(\varepsilon)) = 0$, then we get $a_3 = \varphi_5(a_4, a_6, b_4, b_6, \varepsilon)$, $a_4 = \varphi_6(a_6, b_4, b_6, \varepsilon)$, where φ_5, φ_6 are given in (8). From Lemma 6, we know the system (2) has 3 degenerated double homoclinic loops and their stabilities are determined by the sign of $div(S_2(\varepsilon))$.

Suppose all the above assumptions are hold, from Lemma 8, we know $div(S_2(\varepsilon)) > 0$, it mean that the double homoclinic loop $\Gamma_{1,2}(S_1(\varepsilon)) \cup \Gamma_{6,1}(S_1(\varepsilon))$ is inner and outer unstable. From lemma 8, we also can have $V_3(A_1(\varepsilon)) > 0$, it mean that $A_1(\varepsilon)$ is a unstable fine focus. So by applying Poincaré Bendixson theorem and noticing the Z_3-equivariance of system (2), we get that system (2) has 3 limit cycles repectively surrounding $A_i(\varepsilon)$, $i = 1, 3, 5$. Under the assumptions,



(a) case 1 (b) case 2

Figure 2: The two configurations of 19 limit cycles in system (2)



(a) case 1 (b) case 2

Figure 3: The two configurations of 23 limit cycles in system (2)

we also get $M(\Gamma_l) > 0$ and $M(\Gamma_{6,1}) + M(\Gamma_{3,4}) < 0$, $M(\Gamma_{1,6}) + M(\Gamma_{4,3}) > 0$ and $M(\Gamma_m) < 0$. By applying Poincar$\acute{e}$ Bendixson theorem again, we get that system (2) has two limit cycles, the first one of them is surrounding all the singular points, the second one is surrounding $S_i(\varepsilon),\ A_i(\varepsilon), O,\ i = 7, 8, \cdots, 9$. From the above analysis, we conclude that system (2) has 5 limit cycles under the above given assumptions.

Next, by using same disturbing skill just as the one given in the proof of theorem 1, we can prove that system (2) has 3 more limit cycles which respectively surround $A_i(\varepsilon),\ i = 1, 3, 5$ and 15 more limit cycles near the double homoclinic loops of system (2) with two different configurations. From the above analysis, we prove that the system (2) totally at least has 23 limit cycles whose configurations are given in Figure 3.

The proof is completed. ■

Remark 2 The configurations of these limit cycles are new and different from the configurations obtained by Zhao liqin [5], where the unperturbed systems is a Z_6 equivariant quintic Hamiltonian system.

5 Conclusion

In this paper, the qualitative method of differential equation is used to study the number and distribution of limit cycles of a perturbed quintic Hamiltonian system. The existence and stability theory of heteroclinic loop and homoclinic loop are applied to study the heteroclinic loop and homoclinic loop bifurcation of such system under Z_3-equivariant quintic perturbation. By using the method of changing the the stabilities of homoclinic loops and the Poincare-Bendixson Theorem, we find the perturbed system has at least 23 limit cycles with two different configurations.

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