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The Trigonometric Cubic B-spline Algorithm for Burgers' Equation

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Abstract: The cubic Trigonometric B-spline(CTB) functions are used to set up the collocation method for finding solutions of the Burgers' equation. The effect of the CTB in the collocation method is sought by studying two text problems. The Burgers' equation is fully-discretized using the Crank-Nicholson method for the time discretizion and CTB function for discretizion of spatial variable. Numerical examples are performed to show the convenience of the method for solutions of Burgers equation

Keywords: Collocation methods, Cubic Trigonometric B-spline, Burgers' Equation

1 Introduction

Since the introduction of the Burger's equation by Bateman [1], many authors have used variety of numerical methods in attempting to solve the Burger's equation. Various forms of the finite element methods are constructed to compute the Burger's equation numerically [5–7, 12–14, 16, 24, 28]. The spline collocation procedures are also presented for getting solutions of the Burger's equation [9, 11, 15, 18–23]. The spline functions are wished to be accompanied to the numerical method to solve the differential equations since the resulting matrix system is always diagonal and can be solved easily and approximate solutions having the accuracy of the degree less than the degree of the spline functions can be set up. High order continuous differentiable approximate solutions can be produced for the differential equations of higher order. The numerical procedure for nonlinear evolution equations based on the the B-spline collocation method have been increasingly applied to various fields of science. However application of the CTB collocation method to non linear evolution problems are a few in comparison with the method the collocation based on the B-spline functions.

The numerical methods for solving a type of ordinary differential equations with quadratic and cubic CTB are given by A. Nikolis in the papers [10, 17]. The linear two-point boundary value problems of order two are solved using cubic CTB interpolation method [25]. The another numerical method employed the cubic CTB are set up to solve a class of linear two-point singular boundary value problems in the study [27]. Very recently a collocation finite difference scheme based on new cubic CTB is developed for the numerical solution of a one-dimensional hyperbolic equation (wave equation) with non-local conservation condition [29]. A new two-time level implicit technique based on the cubic CTB is proposed for the approximate solution of the nonclassical diffusion problem with nonlocal boundary condition in the study [30]. Some researches have established types of the B-spline finite element approaches for solving the Burger's equation but not with CTB as far as we know the literature.

In this paper, CTB are used to establish a collocation method and then the suggested numerical method is applied to find the numerical solutions of Burger's equation. It is also well-known that this problem arise in many branch of the science and the development of the numerical methods for the Burger's equation have been attracted for finding the steep front solutions. The use of the lower viscosity in the Burger's equation cause the appearance of the steep front and shock wave solutions. That makes difficulty in modelling solutions of the Burger's equation when solved numerically. So many authors have developed various kinds of numerical scheme in computing the equation effectively for small values of the viscosity.

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We consider the Burger's equation

$$U_t + UU_x - \lambda U_{xx} = 0,\ a \le x \le b,\ t \ge 0 \tag{1}$$

with appropriate initial conditions and the boundary conditions:$U(x,0) = f(x)$, $a \le x \le b, U(a,t) = U_a, U(b,t) = U_b$ where subscripts x and t denote differentiation, $\lambda = \dfrac{1}{Re} > 0$ and Re is the Reynolds number characterizing the strength of viscosity. U_a, U_b are the constants and $U = U(x,t)$ is a sufficiently differentiable unknown function and $f(x)$ is a bounded function.

The implementation of the proposed scheme is given in the second section. Two classical text problems are dealt with to show the robustness of the scheme.

2 C B- C

Consider a uniform partition of the problem domain $[a = x_0, b = x_N]$ at the knots $x_i, i = 0, ..., N$ with mesh spacing $h = (b-a)/N$. On this partition together with additional knots $x_{-1}, x_0, x_{N+1}, x_{N+2}, x_{N+3}$ outside the problem domain, $CTB_i(x)$ can be defined as

$$CTB_i(x) = \frac{1}{\theta}\begin{cases} \omega^3(x_{i-2}), & x \in [x_{i-2}, x_{i-1}] \\ \omega(x_{i-2})(\omega(x_{i-2})\phi(x_i) + \phi(x_{i+1})\omega(x_{i-1})) + \phi(x_{i+2})\omega^2(x_{i-1}), & x \in [x_{i-1}, x_i] \\ \omega(x_{i-2})\phi^2(x_{i+1}) + \phi(x_{i+2})(\omega(x_{i-1})\phi(x_{i+1}) + \phi(x_{i+2})\omega(x_i)), & x \in [x_i, x_{i+1}] \\ \phi^3(x_{i+2}), & x \in [x_{i+1}, x_{i+2}] \\ & 0, \text{otherwise} \end{cases} \tag{2}$$

where $\omega(x_i) = \sin(\frac{x - x_i}{2}), \phi(x_i) = \sin(\frac{x_i - x}{2}), \theta = \sin(\frac{h}{2})\sin(h)\sin(\frac{3h}{2})$.

$CTB_i(x)$ are twice continuously differentiable piecewise trigonometric cubic B-spline on the interval$[a, b]$. The iterative formula

$$T_i^k(x) = \frac{\sin(\frac{x - x_i}{2})}{\sin(\frac{x_{i+k-1} - x_i}{2})}T_i^{k-1}(x) + \frac{\sin(\frac{x_{i+k} - x}{2})}{\sin(\frac{x_{i+k} - x_{i+1}}{2})}T_{i+1}^{k-1}(x),\ k = 2, 3, 4, ... \tag{3}$$

gives the cubic B-spline trigonometric functions starting with the CTB-splines of order 1

$$T_i^1(x) = \begin{cases} 1,\ x \in [x_i, x_{i+1}) \\ 0 \ \ \text{,otherwise.} \end{cases}$$

Each $CTB_i(x)$ is twice continuously differentiable and the values of $CTB_i(x), CTB_i^{'}(x)$ and $CTB_i^{''}(x)$ at the knots x_i 's can be computed from Eq.(3) as

Table 1: Values of $B_i(x)$ and its principle two derivatives at the knot points

	$T_i(x_k)$	$T_i'(x_k)$	$T_i''(x_k)$
x_{i-2}	0	0	0
x_{i-1}	$\sin^2(\frac{h}{2})\csc(h)\csc(\frac{3h}{2})$	$\frac{3}{4}\csc(\frac{3h}{2})$	$\frac{3(1+3\cos(h))\csc^2(\frac{h}{2})}{16[2\cos(\frac{h}{2})+\cos(\frac{3h}{2})]}$
x_i	$\frac{2}{1+2\cos(h)}$	0	$\frac{-3\cot^2(\frac{3h}{2})}{2+4\cos(h)}$
x_{i+1}	$\sin^2(\frac{h}{2})\csc(h)\csc(\frac{3h}{2})$	-$\frac{3}{4}\csc(\frac{3h}{2})$	$\frac{3(1+3\cos(h))\csc^2(\frac{h}{2})}{16[2\cos(\frac{h}{2})+\cos(\frac{3h}{2})]}$
x_{i+2}	0	0	0

$CTB_i(x)$, $i = -1, ..., N+1$ are a basis for the trigonometric spline space. An approximate solution U_N to the unknown U is written in terms of the expansion of the CTB as

$$U_N(x,t) = \sum_{i=-1}^{N+1} \delta_i CTB_i(x) \tag{4}$$

where δ_i are time dependent parameters to be determined from the collocation points $x_i, i = 0, ..., N$ and the boundary and initial conditions. The nodal values U and its first and second derivatives at the knots can be found from the (4) as

$$\begin{aligned} U_i &= \alpha_1\delta_{i-1} + \alpha_2\delta_i + \alpha_1\delta_{i+1} \\ U_i' &= \beta_1\delta_{i-1} + \beta_2\delta_{i+1} \\ U_i'' &= \gamma_1\delta_{i-1} + \gamma_2\delta_i + \gamma_1\delta_{i+1} \end{aligned} \tag{5}$$

$$\begin{array}{ll} \alpha_1 = \sin^2(\frac{h}{2})\csc(h)\csc(\frac{3h}{2}) & \alpha_2 = \dfrac{2}{1+2\cos(h)} \\ \beta_1 = -\frac{3}{4}\csc(\frac{3h}{2}) & \beta_2 = \frac{3}{4}\csc(\frac{3h}{2}) \\ \gamma_1 = \dfrac{3((1+3\cos(h))\csc^2(\frac{h}{2}))}{16(2\cos(\frac{h}{2})+\cos(\frac{3h}{2}))} & \gamma_2 = -\dfrac{3\cot^2(\frac{h}{2})}{2+4\cos(h)} \end{array}$$

The time derivative and space derivatives can be approximated by using the standard finite difference formula and the Crank–Nicolson scheme respectively to have the time-integrated Burger's equation:

$$\frac{U^{n+1}-U^n}{\Delta t} + \frac{(UU_x)^{n+1}+(UU_x)^n}{2} - \lambda\frac{U_{xx}^{n+1}+U_{xx}^n}{2} = 0 \tag{6}$$

where $U^{n+1} = U(x,t)$ is the solution of the equation at the $(n+1)$th time level. Here $t^{n+1} = t^n + \Delta t$, and Δt is the time step, superscripts denote n th time level , $t^n = n\Delta t$

The nonlinear term $(UU_x)^{n+1}$ in Eq. (6) is linearized by using the following form [3, 4]:

$$(UUx)^{n+1} = U^{n+1}U_x^n + U^nU_x^{n+1} - U^nU_x^n \tag{7}$$

So linearized time-integrated Burgers' equation have the following form:

$$U^{n+1} - U^n + \frac{\Delta t}{2}(U^{n+1}U_x^n + U^nU_x^{n+1}) - \lambda\frac{\Delta t}{2}(U_{xx}^{n+1} + U_{xx}^n) = 0 \tag{8}$$

Substitution 4 into 8 and evaluation resulting equation at knots leads to the fully-discretized equation:

$$\begin{aligned} &\left(\alpha_1 + \frac{\Delta t}{2}(\alpha_1L_2 + \beta_1L_1 - \lambda\gamma_1)\right)\delta_{i-1}^{n+1} + \left(\alpha_2 + \frac{\Delta t}{2}(\alpha_2L_2 - \lambda\gamma_2)\right)\delta_i^{n+1} + \\ &\left(\alpha_1 + \frac{\Delta t}{2}(\alpha_1L_2 + \beta_2L_1 - \lambda\gamma_1)\right)\delta_{i+1}^{n+1} = (\alpha_1 - \lambda\frac{\Delta t}{2}\gamma_1)\delta_{i-1}^n + \\ &(\alpha_2 - \lambda\frac{\Delta t}{2}\gamma_2)\delta_i^n + (\alpha_1 - \lambda\frac{\Delta t}{2}\gamma_1)\delta_{i+1}^n \end{aligned} \tag{9}$$

where

$$\begin{aligned} L_1 &= \alpha_1\delta_{i-1}^n + \alpha_2\delta_i^n + \alpha_1\delta_{i+1}^n \\ L_2 &= \beta_1\delta_{i-1}^n + \beta_2\delta_{i+1}^n \end{aligned}$$

The system consist of $N+1$ linear equation in $N+3$ unknown parameters $\mathbf{d}^{n+1} = (\delta_{-1}^{n+1}, \delta_0^{n+1}, \ldots, \delta_{N+1}^{n+1})$. The above system can be made solvable by elimination the time parameters $\delta_{-1}, \delta_{N+1}$with help of the boundary conditions

$U(x,a) = U_0, U(x,b) = U_N$ when written as

$$\begin{aligned} \delta_{-1} &= \frac{1}{\alpha_1}(U_0 - \alpha_2\delta_0 - \alpha_1\delta_1), \\ \delta_{N+1} &= \frac{1}{\alpha_1}(U_n - \alpha_1\delta_{N-1} - \alpha_2\delta_N). \end{aligned} \tag{10}$$

A variant of Thomas algorithm is used to solve the system.

Initial parameters $d^0 = \delta^0_{-1}, \delta_0^0, \ldots, \delta_{N+1}^0$ must be found to start the iteration process. To do so, initial condition and boundary values of derivative of initial conditions gives the following equation

1. $U_N(x_i, 0) = U(x_i, 0), i = 0, ..., N$

2. $(U_x)_N(x_0, 0) = U'(x_0)$

3. $(U_x)_N(x_N, 0) = U'(x_N)$.

The system (9) yields an $(N + 3) \times (N + 3)$ matrix system, which can be solved by use of the Thomas algorithm.

Once the initial parameters d^0 has been obtained from the initial and boundary conditions, the recurrence relation gives time evolution of vector d^n,from the time evolution of the approximate solution $U_N(x, t)$ can be computed via the equation (6).

3

( ) Analytical solution of the Burger's equation with the problem sine wave initial condition $U(x, 0) = sin(\pi x)$ and boundary conditions $U(0, t) = U(1, t) = 0$ can be expressed as an infinite series [2]

$$U(x,t) = \frac{4\pi\lambda \sum_{j=1}^{\infty} j\mathbf{I}_j(\frac{1}{2\pi\lambda}) \sin(j\pi x) \exp(-j^2\pi^2\lambda t)}{\mathbf{I}_0(\frac{1}{2\pi\lambda}) + 2\sum_{j=1}^{\infty} \mathbf{I}_j(\frac{1}{2\pi\lambda}) \cos(j\pi x) \exp(-j^2\pi^2\lambda t)} \tag{11}$$

where $\mathbf{I}_j$ are the modified Bessel functions. This problem gives the decay of sinusoidal disturbance. The convergence of the solution [8] is slow for small values of λso that the numerical solutions of the Burger's equation are looked for. Using the parameters $N = 40$, $\Delta t = 0.0001, \lambda = 1, 0.1, 0.01, 0.00$, graphical solutions at different times are depicted in the Figs 1-4 The amplitude of the solution decays as time pass, seen in Fig 1-2 clearly and the sharpness through the right boundary develops when the smaller viscosities are used. he same incidents also exist for studies given in the paper [21, 23]

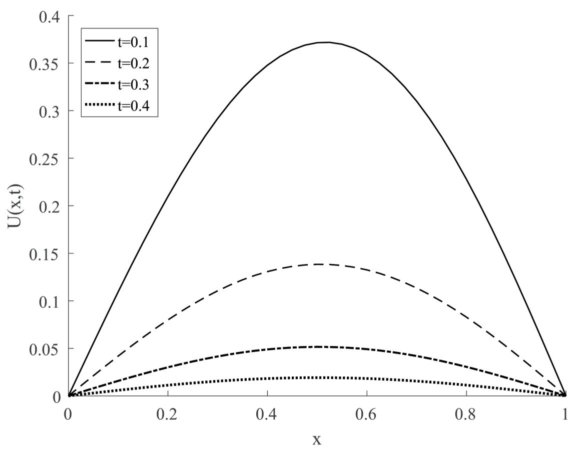

Fig. 1: Solutions at different times for $\lambda = 1$

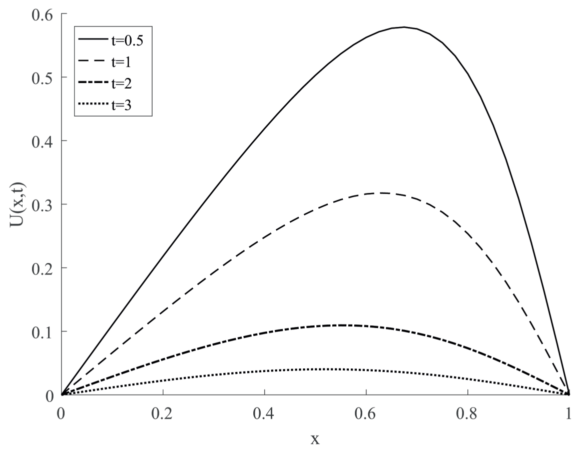

Fig. 2: Solutions at different times for $\lambda = 0.1$

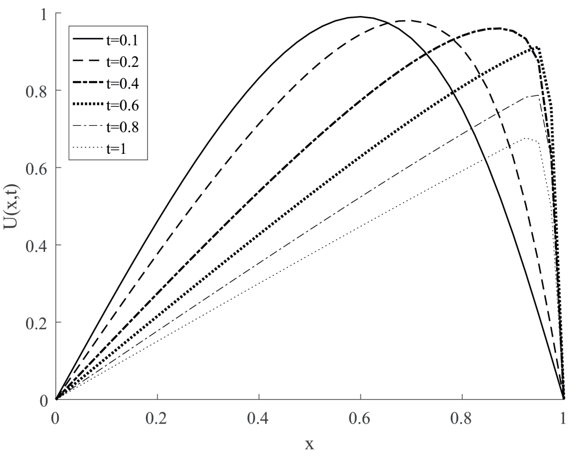

Fig. 3: Solutions at different times for $\lambda = 0.01$

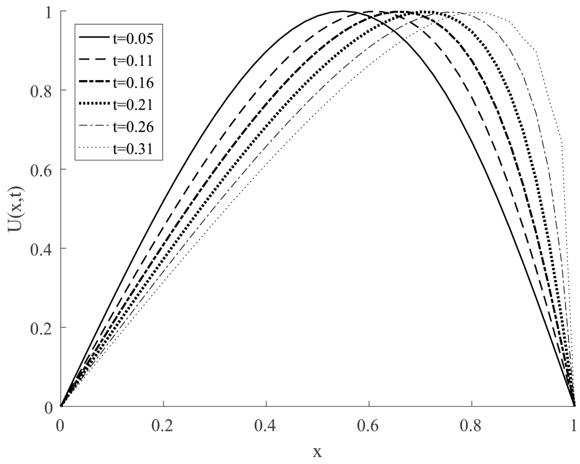

Fig. 4: Solutions at different times for $\lambda = 0.001$

The results of proposed numerical methods are compared with the cubic B-spline collocation, cubic B-spline Galerkin. Galerkin procedure are seen to produce slightly same results with the CTB collocation method. Our advantage is that the cost of the CTB procedure is less than the Galerkin methods given in the tables 1-3.

Table 2: Comparison of the numerical solutions of Problem 1 obtained for $\lambda = 1$.and $N = 40$, $\Delta t = 0.0001$ at different times with the exact solutions

x	t	Present	Ref.[15] ($N = 80$)	[16]	Exact
0.25	0.4	0.01355	0.01357	0.01357	0.01357
	0.6	0.00188	0.00189	0.00189	0.00189
	0.8	0.00026	0.00026	0.00026	0.00026
	1.0	0.00004	0.00004	0.00004	0.00004
	3.0	0.00000	0.00000	0.00000	0.00000
0.50	0.4	0.01920	0.01923	0.01924	0.01924
	0.6	0.00266	0.00267	0.00267	0.00267
	0.8	0.00037	0.00037	0.00037	0.00037
	1.0	0.00005	0.00005	0.00005	0.00005
	3.0	0.00000	0.00000	0.00000	0.00000
0.75	0.4	0.01361	0.01362	0.01363	0.01363
	0.6	0.00188	0.00189	0.00189	0.00189
	0.8	0.00026	0.00026	0.00026	0.00026
	1.0	0.00004	0.00004	0.00004	0.00004
	3.0	0.00000	0.00000	0.00000	0.00000

Table 3: Comparison of the numerical solutions of Problem 1 obtained for $\lambda = 0.1$ and $N = 40$, $\Delta t = 0.0001$ at different times with the exact solutions

x	t	Present	Ref.[15] $(N = 80)$	Ref.[20]	Ref.[16]	Exact
0.25	0.4	0.30892	0.30890	0.30891	0.30890	0.30889
	0.6	0.24078	0.24075	0.24075	0.24074	0.24074
	0.8	0.19572	0.19569	0.19568	0.19568	0.19568
	1.0	0.16261	0.16258	0.16257	0.16257	0.16256
	3.0	0.02718	0.02720	0.02721	0.02720	0.02720
0.50	0.4	0.56971	0.56965	0.56969	0.56964	0.56963
	0.6	0.44730	0.44723	0.44723	0.44721	0.44721
	0.8	0.35932	0.35925	0.35926	0.35924	0.35924
	1.0	0.29197	0.29192	0.29193	0.29191	0.29192
	3.0	0.04017	0.04019	0.04021	0.04020	0.04021
0.75	0.4	0.62524	0.62538	0.62543	0.62541	0.62544
	0.6	0.48698	0.48715	0.48723	0.48719	0.48721
	0.8	0.37369	0.37385	0.37394	0.37390	0.37392
	1.0	0.28727	0.28741	0.28750	0.28746	0.28747
	3.0	0.02974	0.02976	0.02978	0.02977	0.02977

Table 4: Comparison of the numerical solutions of Problem 1 obtained for $\lambda = 0.01$ and $N = 40$, $\Delta t = 0.0001$ at different times with the exact solutions

x	t	Present	Ref.[15] $(N = 80)$	Ref.[20]	Ref.[16]	Exact
0.25	0.4	0.34191	0.34192	0.34192	0.34192	0.34191
	0.6	0.26896	0.26897	0.22894	0.26897	0.22896
	0.8	0.22148	0.22148	0.22144	0.22148	0.22148
	1.0	0.18819	0.18819	0.18815	0.18819	0.18819
	3.0	0.07511	0.07511	0.07509	0.07511	0.07511
0.50	0.4	0.66071	0.66071	0.66075	0.66071	0.66071
	0.6	0.52942	0.52942	0.52946	0.52942	0.52942
	0.8	0.43914	0.43914	0.43917	0.43914	0.43914
	1.0	0.37442	0.37442	0.37444	0.37442	0.37442
	3.0	0.15017	0.15018	0.15016	0.15018	0.15018
0.75	0.4	0.91029	0.91027	0.91023	0.91027	0.91026
	0.6	0.76725	0.76725	0.76728	0.76724	0.76724
	0.8	0.64740	0.64740	0.64744	0.64740	0.64740
	1.0	0.55605	0.55605	0.55609	0.55605	0.55605
	3.0	0.22489	0.22483	0.22481	0.22481	0.22481

( ) Well-known other solution of the Burger's equation is

$$U(x,t) = \frac{\alpha + \mu + (\mu - \alpha)\exp\eta}{1 + \exp\eta},\ 0 \le x \le 1,\ t \ge 0, \tag{12}$$

where $\eta = \dfrac{\alpha(x - \mu t - \gamma)}{\lambda}$. α, μ and γ are arbitrary constants. Initial conditions are $U(0,t) = 1$, $U(1,t) = 0.2$ or $U_x(0,t) = 0$, $U_x(1,t) = 0$, for $t \ge 0$ This form of the solution is known as the travelling wave equation and respresent the propogation of the wave front through the right. Parameter λ determine the sharpness of the solution

The initila solutions are taken from the analytical solution when $t = 0$. The program is run for the parameters $\alpha = 0.4$, $\mu = 0.6$, $\gamma = 0.125$ and $\lambda = 0.01, h = 1/36, \Delta t = 0.001$. solutions at some space values x are presented in Table 5 and compared with those obtained in the studies [15, 16, 16] using Cubic B-spline collocation, quadratic/Cubic

B-spline Galerkin methods. Solution behaviours are illustrated in Fig 5-6 for the coefficient $\lambda = 0.01$ and 0.001 at times $t = 0, 0.4, 0.8, 1.2$. With smaller $\lambda = 0.001$,the sharp front is formed and propogates to right during run of the program. Graphical presentation of the absolute errors at time $t = 0.4$ is drawn im Figs 7-8.

Table 5: Comparison of the results at time $t = 0.5$, $h = 1/36$, $\Delta t = 0.01$, $\lambda = 0.01$

x	Present	Ref. [15] $\Delta t = 0.025$	Ref. [16] (QBGM)	Ref. [16] (CBGM)	Exact
0.000	1.	1.	1.	1.	1.
0.056	1.	1.	1.	1.	1.
0.111	1.	1.	1.	1.	1.
0.167	1.	1.	1.	1.	1.
0.222	1.	1.	1.	1.	1.
0.278	0.999	0.999	0.998	0.998	0.998
0.333	0.983	0.986	0.980	0.980	0.980
0.389	0.845	0.850	0.841	0.842	0.847
0.444	0.456	0.448	0.458	0.457	0.452
0.500	0.237	0.236	0.240	0.241	0.238
0.556	0.203	0.204	0.205	0.205	0.204
0.611	0.2	0.2	0.2	0.2	0.2
0.667	0.2	0.2	0.2	0.2	0.2
0.722	0.2	0.2	0.2	0.2	0.2
0.778	0.2	0.2	0.2	0.2	0.2
0.833	0.2	0.2	0.2	0.2	0.2
0.889	0.2	0.2	0.2	0.2	0.2
0.944	0.2	0.2	0.2	0.2	0.2
1.000	0.2	0.2	0.2	0.2	0.2

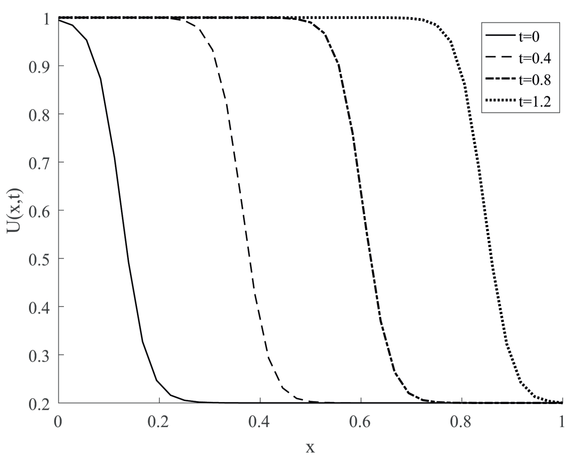

Fig. 5: Solutions at different times for $\lambda = 0.01$

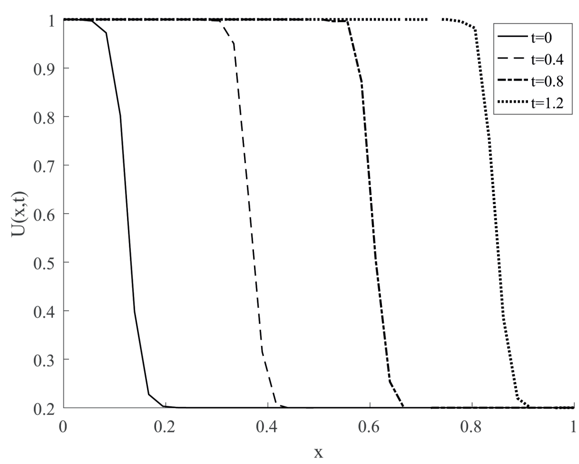

Fig. 6: Solutions at different times for $\lambda = 0.005$

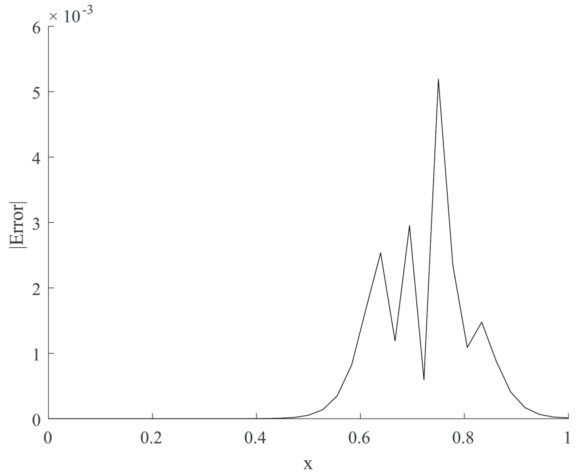

Fig. 7: L_∞ error norm for $\lambda = 0.01$

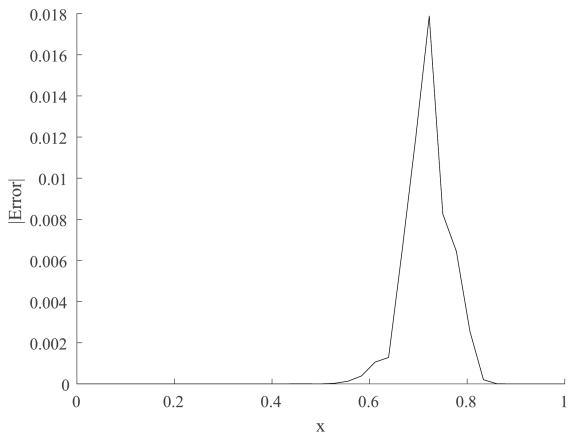

Fig. 8: L_∞ error norm for $\lambda = 0.005$

The collocation methods with trigonometric B-spline functions is made up to find solutions the Burger's equation. We have hown that methods is capable of producing solutions of the Burgers equation fairly. The method can be used as an alternative to the methods accompanied B-spline functions.

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