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On the Conservative Finite Difference Scheme for the Novikov Equation

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Abstract: In science and technology, a lot of mathematical models can be represented by partial differential equations. But the analytical form of the solutions are difficult even impossible to be obtained.In this paper we investigate the numerical method for the Novikov equation. We propose a conservative finite difference scheme for the Novikov equation and use Brouwer fixed point theorem to obtain the existence of the solution for the corresponding difference equation. We also prove the convergence and stability of the solution by using the discrete energy method. Moreover we obtain the truncation error of the difference scheme is $R_j^n = O(k^2 + h^2)$.

Keywords: Novikov equation; finite difference scheme; conservation law; stability; convergence

1 Introduction

The Novikov equation

$$y_t + u^2 y_x + 3uu_x y = 0,\ \ y = u - u_{xx},$$

was isolated by Novikov [1] in a symmetry classification of nonlocal partial differential equations. Compared with the well-studied Camassa-Holm equation [2]

$$y_t + uy_x + 2u_x y = 0,\ \ y = u - u_{xx},$$

the Novikov equation has cubic nonlinear terms, rather than quadratic, which can be thought as a generalization of the Camassa-Holm equation. Novikov [1] found its first few symmetries and he subsequently found a scalar Lax pair for it, and proved that the equation is integrable. Hone and Wang [3] gave a matrix Lax pair for the Novikov equation, and showed how it was related by a reciprocal transformation to a negative flow in the Sawada-Kotera hierarchy. Infinitely many conserved quantities were found, as well as a bi-Hamiltonian structure. They also presented peakons for Eq.(1). Liu, Liu and Qu [4] proved such peakons were orbital stable. Hone, Lundmark and Szmigielski [5] calculated the explicit formulas for multipeakon solutions of Novikov equation, using the matrix Lax pair which was found by Hone and Wang. Very recent works are intensively devoted to studying the local well-defined, global existence of strong and weak solution, and blow up of solution of initial value problem for Novikov equation in Sobolev spaces or Besov spaces [6-13].

However, there are few studies on numerical solutions of the Novikov equation. Therefore in this paper, we will construct a finite difference scheme for the initial boundary problem of equations as follows,

$$u_t - u_{xxt} + 4u^2 u_x = 3uu_x u_{xx} + u^2 u_{xxx}, \tag{1}$$

$$u(0, x) = u_0(x),\ \ x_L \le x \le x_R, \tag{2}$$

$$u(t, x_L) = u(t, x_R),\ \ t \ge 0, \tag{3}$$

where u satisfies the following conservation law of energy:

$$E(t) = \| u \|_{L^2}^2 + \| u_x \|_{L^2}^2 = const. \tag{4}$$

We shall give an energy conservative finite scheme for equation (1)-(3) and obtain the module estimation. Moreover, we prove the convergence and stability of the finite scheme by use of discrete energy method.

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2 Preliminaries

For convenience, we take $\Omega = \{(x,t) \mid 0 \le x \le L, 0 \le t \le T\}$ in the following section. Let N, J be any positive integers and $h = \frac{L}{J+1}$, $k = \frac{T}{N}$, $x_j = jh$; $j = 0, 1, ...J+1$ and $t^n = nk$, $u_j^n = u(x_j, t^n)$, $Z_h^0 = \{u = u_j \mid u_0 = u_J = 0,\ j = 0, 1, ...J\}$.

For simplicity, we introduce some notations as follows:

$$u_x^n = \frac{u_{j+1}^n - u_j^n}{h},\ u_{\overline{x}}^n = \frac{u_j^n - u_{j-1}^n}{h},\ u_{\widehat{x}}^n = \frac{u_{j+1}^n - u_{j-1}^n}{2h},$$

$$u_t^n = \frac{u_j^{n+1} - u_j^n}{k},\ u_{\overline{t}}^n = \frac{u_j^n - u_j^{n-1}}{k},\ u_{\widehat{t}}^n = \frac{u_j^{n+1} - u_j^{n-1}}{2k},$$

$$\overline{u} = \frac{u_j^{n+1} + u_j^{n-1}}{2}, \qquad u^n = u_j^n,$$

and define

$$(u^n, v^n)_h = h\sum_{j=0}^{J} u_j^n v_j^n,\ \ (u^n, u^n) = \parallel u^n \parallel^2,\ \ \parallel u^n \parallel_\infty = \max_{0 \le j \le J}(u_j^n).$$

In order to obtain the module estimation and investigate the convergence and stability of the finite difference scheme, we need introduce the following three lemmas.

Lemma 1 *(Discrete Sobolev inequality [14]) There exist constants c_1, c_2, satisfy*

$$\parallel u^n \parallel_\infty \le c_1 \parallel u^n \parallel + c_2 \parallel u_x^n \parallel .$$

Lemma 2 *(Discrete Gronwall inequality [15]) Suppose there exist negative functions $\omega(k)$ and $\rho(k)$, where $\rho(k)$ is decreasing. When $c > 0$ and $\forall k$, if*

$$\omega(k) \le \rho(k) + c\tau \sum_{l=0}^{k-1} \omega(1),$$

then

$$\omega(k) \le \rho(k) c^{c\tau k}.$$

Lemma 3 *For $\forall u, v \in Z_h^0$, then*
(1) $(u_x^n, v^n) = -(u^n, v_{\overline{x}}^n)$,
(2) $(u_{x\overline{x}}^n, v^n) = -(u_x^n, v_x^n)$,
(3) $\parallel u_{x\overline{x}}^n \parallel^2 \le \frac{4}{h^2} \parallel u_x^n \parallel^2$,
(4) $\parallel u_{\overline{x}}^n \parallel \le \frac{2}{h} \parallel u^n \parallel$,
(5) $\parallel u_{\widehat{x}}^n \parallel \le \parallel u_x^n \parallel$,
(6) $\parallel u_{x\overline{x}}^n \parallel^2 \le \frac{4}{h^2} \parallel u_x^n \parallel^2$.

Since the proof of Lemma 3 is clear, so here we omit it.

3 An energy conservative finite scheme

Firstly, we construct an energy conservative finite scheme for the problem (1)-(3) as follows:

$$u_t^n - u_{tx\overline{x}}^n + \left((u^{n+\frac{1}{2}})^3\right)_{\overline{x}} + (u^{n+\frac{1}{2}})^2 u_x^{n+\frac{1}{2}} = u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} + \left((u^{n+\frac{1}{2}})^2 u_{xx}^{n+\frac{1}{2}}\right)_{\overline{x}}, \tag{5}$$

$$u_j^0 = u_0(x_j), \quad 0 \le j \le J, \tag{6}$$

$$u_0^n = u_J^n = 0, \quad 0 \le n \le N. \tag{7}$$

Lemma 4 *Scheme (5) satisfies discrete conservative law as follows:*

$$E^n = \| u^n \|^2 + \| u_x^n \|^2 = E^{n-1} = \cdots = E^0. \tag{8}$$

Proof. Denoting

$$I_1 = \Big((u^{n+\frac{1}{2}})^3\Big)_{\overline{x}} + (u^{n+\frac{1}{2}})^2 u_x^{n+\frac{1}{2}},$$

$$I_2 = u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} + \Big((u^{n+\frac{1}{2}})^2 u_{xx}^{n+\frac{1}{2}}\Big)_{\overline{x}},$$

then the difference scheme (5) can be rewritten as

$$u_t - u_{tx\overline{x}} + I_1 - I_2 = 0. \tag{9}$$

Taking the inner product of (9) with $2u^{n+\frac{1}{2}}$, we obtain

$$(u_t, 2u^{n+\frac{1}{2}}) - (u_{tx\overline{x}}, 2u^{n+\frac{1}{2}}) + (I_1, 2u^{n+\frac{1}{2}}) - (I_2, 2u^{n+\frac{1}{2}}) = 0. \tag{10}$$

Computing $(u_t, 2u^{n+\frac{1}{2}})$, $(u_{tx\overline{x}}, 2u^{n+\frac{1}{2}})$, $(I_1, 2u^{n+\frac{1}{2}})$ and $(I_2, 2u^{n+\frac{1}{2}})$ respectively, we can simplify formula (10) as

$$\frac{1}{k}\Big(\| u^{n+1} \|^2 - \| u^n \|^2 \Big) + \frac{1}{k}\Big(\| u_x^{n+1} \|^2 - \| u_x^n \|^2 \Big) = 0,$$

then

$$\| u^{n+1} \|^2 + \| u_x^{n+1} \|^2 = \| u^n \|^2 + \| u_x^n \|^2 .$$

So difference scheme (5) satisfies discrete conservative law, that is

$$E^n = \| u^n \|^2 + \| u_x^n \|^2 = E^{n-1} = \cdots = E^0.$$

This completes the proof of Lemma 4. ■

Theorem 5 *The solution of difference scheme (5) satisfies,*

$$\| u^n \| \leq C, \quad \| u_x^n \| \leq C, \quad \| u^n \|_\infty \leq C.$$

Proof. From Lemma 4, we know $\| u^n \| \leq C$, $\| u_x^n \| \leq C$, then by Lemma 1, we obtain $\| u^n \|_\infty \leq C$. This completes the proof of Theorem 5. ■

Next, in order to show the existence of the approximations for scheme (5), we need to use Brouwer fixed point theorem [16].

Theorem 6 *There exists $u^n \in Z_h^0$ which satisfies difference scheme (5).*

Proof. We shall use mathematical induction to prove Theorem 5.

When $n = 1$, we know that there exists u^1 which satisfies scheme (5) from initial condition (6). Next, we need to prove the case when $n > 1$.

Assume there exists u^n satisfies scheme (5) when $n < N$, then we need to prove that there exists u^{n+1} also satisfies scheme (5).

Defining a operator $\omega(v)$ in Z_h^0 as follows:

$$\omega(v) = 2v - 2u^n - (2v_{x\overline{x}} - 2u_{x\overline{x}}^n) + k\Big((v^3)_{\overline{x}} + v^2 v_x\Big) - k\Big(v v_x v_{xx} + (v^2 v_{xx})_{\overline{x}}\Big). \tag{11}$$

it is clear that ω is continuous.

Taking the inner product of (11) with v, we obtain

$$(2v, v) = 2 \| v \|^2,$$

$$(2v_{x\overline{x}}, v) = -2 \| v_x \|^2,$$

$$\Big((v^3)_{\overline{x}}+v^2v_x,v\Big)=h\sum_{j=0}^{J}\Big(\big((v_j^3)_{\overline{x}}+v_j^2v_{xj}\big)\cdot v_j\Big)=0$$

$$\Big(vv_xv_{xx}+(v^2v_{xx})_{\overline{x}},v\Big)=h\sum_{j=0}^{J}\Big(\big((v_jv_{xj}v_{xxj}+(v_j^2v_{xxj})_{\overline{x}}\big)\cdot v_j\Big)=0$$

By using Cauchy-Schwartz inequality, we obtain

$$(2u^n,v)\le 2\parallel u^n\parallel\cdot\parallel v\parallel,$$

$$(2u^n_{x\overline{x}},v)=-(2u^n_x,v_x)\ge -2\parallel u^n_x\parallel\cdot\parallel v_x\parallel.$$

From above discussions, we get

$$\begin{aligned}(\omega(v),v)&=(2v-2u^n-(2v_{x\overline{x}}-2u^n_{x\overline{x}})+k\Big((v^3)_{\overline{x}}+v^2v_x\Big)-k\Big(vv_xv_{xx}+(v^2v_{xx})_{\overline{x}}\Big),v)\\&\ge 2\parallel v\parallel^2-2\parallel u^n\parallel\cdot\parallel v\parallel+2\parallel v_x\parallel^2-2\parallel u^n_x\parallel\cdot\parallel v_x\parallel\ge\parallel v\parallel^2-\parallel u^n\parallel^2-\parallel u^n_x\parallel^2.\end{aligned}$$

For $\forall\, v\in Z_h^0$, then

$$\parallel v\parallel^2=\parallel u^n\parallel^2+\parallel u^n_x\parallel^2+1,$$

so we get $(\,\omega(v)\,,\,v\,)>0$. By Brouwer fixed point theorem, there exists $v^*\in H$ such that $\omega(v^*)=0$ and $\parallel v^*\parallel\le\alpha$.

Let $u^{n+1}=2v^*-u^n$, and from scheme (5), we have

$$\begin{aligned}&2v^*-2v^*_{tx\overline{x}}+2\Big((v^{*3})_{\overline{x}}+v^{*2}v^*_x\Big)-2\Big(v^*v^*_xv^*_{xx}+(v^{*2}v^*_{xx})_{\overline{x}}\Big)-\Big(u^n_t-u^n_{tx\overline{x}}+\Big((u^{n+\frac12})^3\Big)_{\overline{x}}+(u^{n+\frac12})^2u_x^{n+\frac12}\Big)\\&=u^{n+\frac12}u_x^{n+\frac12}u_{xx}^{n+\frac12}+\Big((u^{n+\frac12})^2u_{xx}^{n+\frac12}\Big)_{\overline{x}},\end{aligned}$$

it is easy to verify that u^{n+1} satisfies difference equation (5). So we complete the proof of Theorem 6. ■

4 Convergence and stability of the difference scheme

In order to investigate the convergence and stability of the difference scheme, we need to obtain the truncation error of the scheme (5).

Theorem 7 *Suppose the solution $u(x,t)$ of equation (1)-(3) is continuous, then the truncation error of scheme (5) is $R_j^n=O(k^2+h^2)$.*

Proof. Firstly, we can use the Taylor expansion of u_j^{n+1}, u_j^n, u_{j+k}^{n+1}, u_{j-k}^{n+1}, u_{j+1}^{n+1}, u_{j-1}^{n+1}, u_{j+1}^n, u_{j-1}^n at the point $(x_j,t_{n+\frac12})$. Secondly, From above Tayor expansions we reorganize difference equation (5) at point $(x_j,t_{n+\frac12})$, then

$$\begin{aligned}u_t^n&=\frac{u_j^{n+1}-u_j^n}{k}\\&=\frac1k\Big(\big(u_j^{n+\frac12}+\frac k2(\frac{\partial u}{\partial t})_j^{n+\frac12}+\frac{k^2}{8}(\frac{\partial^2 u}{\partial t^2})_j^{n+\frac12}+\frac{k^3}{48}(\frac{\partial^3 u}{\partial t^3})_j^{n+\frac12}+\cdots\big)\\&-\big(u_j^{n+\frac12}-\frac k2(\frac{\partial u}{\partial t})_j^{n+\frac12}+\frac{k^2}{8}(\frac{\partial^2 u}{\partial t^2})_j^{n+\frac12}+\cdots\big)\Big)=(\frac{\partial u}{\partial t})_j^{n+\frac12}+\frac{k^2}{24}(\frac{\partial^3 u}{\partial t^3})_j^{n+\frac12}+\cdots,\\u^n_{tx\overline{x}}&=\frac{(u_{x\overline{x}})_j^{n+1}-(u_{x\overline{x}})_j^n}{k}\\&=\frac{\big((u_{\overline{x}})_{j+1}^{n+1}-(u_{\overline{x}})_j^{n+1}\big)-\big((u_{\overline{x}})_{j+1}^{n}-(u_{\overline{x}})_j^{n}\big)}{kh}\\&=\frac{1}{h^2}\Big(\big((\frac{\partial u}{\partial t})_{j+1}^{n+\frac12}-2(\frac{\partial u}{\partial t})_j^{n+\frac12}+(\frac{\partial u}{\partial t})_{j-1}^{n+\frac12}\big)+\big(\frac{k^2}{24}(\frac{\partial^3 u}{\partial t^3})_{j+1}^{n+\frac12}-\frac{k^2}{12}(\frac{\partial^3 u}{\partial t^3})_j^{n+\frac12}+\frac{k^2}{24}(\frac{\partial^3 u}{\partial t^3})_{j-1}^{n+\frac12}\big)+\cdots\Big)\\&=(\frac{\partial^3 u}{\partial t\partial x^2})_j^{n+\frac12}+\cdots,\end{aligned}$$

$$\begin{aligned}
\Big((u^{n+\frac{1}{2}})^3\Big)_{\overline{x}} &= \frac{\big((u^{n+\frac{1}{2}})^3\big)_j - \big((u^{n+\frac{1}{2}})^3\big)_{j-1}}{h} \\
&= 3(u^{n+\frac{1}{2}})^2(\frac{\partial u}{\partial x})_j^{n+\frac{1}{2}} - \frac{h}{2}\Big((3u^{n+\frac{1}{2}})^2(\frac{\partial^2 u}{\partial x^2})_j^{n+\frac{1}{2}} + 6u^{n+\frac{1}{2}}(\frac{\partial u}{\partial x})_j^{n+\frac{1}{2}} + \cdots\Big) \\
(u^{n+\frac{1}{2}})^2 u_x^{n+\frac{1}{2}} &= (u^{n+\frac{1}{2}})^2 \cdot \frac{u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}}{h} \\
&= (u^{n+\frac{1}{2}})^2 \cdot \Big((\frac{\partial u}{\partial x})_j^{n+\frac{1}{2}} + \frac{h}{2}(\frac{\partial^2 u}{\partial x^2})_j^{n+\frac{1}{2}} + \frac{h^2}{6}(\frac{\partial^3 u}{\partial x^3})_j^{n+\frac{1}{2}} + \cdots\Big) \\
u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} &= u^{n+\frac{1}{2}} \cdot \frac{u_{j+1}^{n+\frac{1}{2}} - u_j^{n+\frac{1}{2}}}{h} \cdot \frac{u_{j+2}^{n+\frac{1}{2}} - 2u_{j+1}^{n+\frac{1}{2}} + u_j^{n+\frac{1}{2}}}{h^2} \\
&= u^{n+\frac{1}{2}} \cdot \Big((\frac{\partial u}{\partial x})_j^{n+\frac{1}{2}} + \frac{h}{2}(\frac{\partial^2 u}{\partial x^2})_j^{n+\frac{1}{2}} + \cdots\Big) \cdot \Big((\frac{\partial^2 u}{\partial x^2})_j^{n+\frac{1}{2}} + h(\frac{\partial^3 u}{\partial x^3})_j^{n+\frac{1}{2}} + \cdots\Big) \\
\Big((u^{n+\frac{1}{2}})^2 u_{xx}^{n+\frac{1}{2}}\Big)_{\overline{x}} &= \frac{u_{xxj}^{n+\frac{1}{2}} \cdot (u_j^{n+\frac{1}{2}})^2 - u_{xx(j-1)}^{n+\frac{1}{2}} \cdot (u_{j-1}^{n+\frac{1}{2}})^2}{h} \\
&= 2u^{n+\frac{1}{2}} \cdot \Big((\frac{\partial u}{\partial x})_j^{n+\frac{1}{2}} \cdot (\frac{\partial^2 u}{\partial x^2})_j^{n+\frac{1}{2}} + (\frac{\partial^3 u}{\partial x^3})_j^{n+\frac{1}{2}} + h(\frac{\partial^3 u}{\partial x^3})_j^{n+\frac{1}{2}} \cdots\Big)
\end{aligned}$$

From above expansions, we obtain the linear part of equation (5) $u_t^n - u_{tx\overline{x}}^n$ at the point $(x_j, t_{n+\frac{1}{2}})$ satisfying

$$u_t^n - u_{tx\overline{x}}^n = (\frac{\partial u}{\partial t} - \frac{\partial^3 u}{\partial t \partial x^2})_{(x_j, t_{n+\frac{1}{2}})} + O(k^2 + h^2)$$

On the other hand, the nonliear parts at the point $(x_j, t_{n+\frac{1}{2}})$ satisfy

$$\Big((u^{n+\frac{1}{2}})^3\Big)_{\overline{x}} + (u^{n+\frac{1}{2}})^2 u_x^{n+\frac{1}{2}} = 4\big(u^2\frac{\partial u}{\partial x}\big)_j^{n+\frac{1}{2}} + O(k^2 + h^2)$$

$$u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} + \Big((u^{n+\frac{1}{2}})^2 u_{xx}^{n+\frac{1}{2}}\Big)_{\overline{x}} = 3\Big(u \cdot \frac{\partial u}{\partial x} \cdot \frac{\partial^2 u}{\partial x^2}\Big)_j^{n+\frac{1}{2}} + 2\Big(u \cdot \frac{\partial^3 u}{\partial x^3}\Big)_j^{n+\frac{1}{2}} + O(k^2 + h^2).$$

Then the truncation error of scheme (5) is $R_j^n = O(k^2 + h^2)$.

■

Theorem 8 *The solution of difference scheme (5) approaches to the solution of original differential equation (1)-(3) in norm L_∞, and the corresponding truncation error is $O(k^2 + h^2)$.*

Proof. Assume that u_j^n is the solution of difference scheme (5), U_j^n is the solution of original differential equation (1)-(3). We can easily have $e_j^n = U_j^n - u_j^n$.

For difference scheme (5),

$$R^n = U_t^n - U_{tx\overline{x}}^n + \Big((U^{n+\frac{1}{2}})^3\Big)_{\overline{x}} + (U^{n+\frac{1}{2}})^2 U_x^{n+\frac{1}{2}} - U^{n+\frac{1}{2}} U_x^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} - \Big((U^{n+\frac{1}{2}})^2 U_{xx}^{n+\frac{1}{2}}\Big)_{\overline{x}}, \tag{12}$$

then let (12) minus (5), we have

$$R^n = e_t^n - e_{tx\overline{x}}^n + I_1 + I_2, \tag{13}$$

where

$$I_1 = \Big((U^{n+\frac{1}{2}})^3\Big)_{\overline{x}} + (U^{n+\frac{1}{2}})^2 U_x^{n+\frac{1}{2}} - \Big((u^{n+\frac{1}{2}})^3\Big)_{\overline{x}} - (u^{n+\frac{1}{2}})^2 u_x^{n+\frac{1}{2}},$$

$$I_2 = u^{n+\frac{1}{2}} u_x^{n+\frac{1}{2}} u_{xx}^{n+\frac{1}{2}} + \Big((u^{n+\frac{1}{2}})^2 u_{xx}^{n+\frac{1}{2}}\Big)_{\overline{x}} - U^{n+\frac{1}{2}} U_x^{n+\frac{1}{2}} U_{xx}^{n+\frac{1}{2}} - \Big((U^{n+\frac{1}{2}})^2 U_{xx}^{n+\frac{1}{2}}\Big)_{\overline{x}}.$$

Taking the inner product of (13) with $2e^{n+\frac{1}{2}}$, we obtain

$$(R^n\,,\, 2e^{n+\frac{1}{2}}) = (e_t^n\,,\, 2e^{n+\frac{1}{2}}) - (e_{tx\overline{x}}^n\,,\, 2e^{n+\frac{1}{2}}) + (I_1\,,\, 2e^{n+\frac{1}{2}}) + (I_2\,,\, 2e^{n+\frac{1}{2}}).$$

Similarly to the method in [17], we compute $(e_t^n\ ,\ 2e^{n+\frac{1}{2}})\ ,\ (e_{tx\overline{x}}^n\ ,\ 2e^{n+\frac{1}{2}})\ ,$ $(I_1\ ,\ 2e^{n+\frac{1}{2}})\ ,\ (I_2\ ,\ 2e^{n+\frac{1}{2}})$ respectively, and from Lemma 3 and Cauchy-Schwartz inequality, we have

$$(e_t^n\ ,\ 2e^{n+\frac{1}{2}}) = h\sum_{j=0}^{J} e_{tj}\cdot 2e_j^{n+\frac{1}{2}} = \frac{1}{k}\Big(\parallel e^{n+1}\parallel^2 - \parallel e^n\parallel^2\Big),$$

$$(e_{tx\bar{x}}, 2e^{n+\frac{1}{2}}) = h\sum_{j=0}^{J} e_{tx\bar{x}j}\cdot 2e_j^{n+\frac{1}{2}} = \frac{1}{k}\Big(\parallel e_x^{n+1}\parallel^2 - \parallel e_x^n\parallel^2\Big),$$

$$\begin{aligned}(I_1, 2e^{n+\frac{1}{2}}) &= ([(U^{n+\frac{1}{2}})^3]_{\bar{x}} + (U^{n+\frac{1}{2}})^2U_x^{n+\frac{1}{2}} - [(u^{n+\frac{1}{2}})^3]_{\bar{x}} - (u^{n+\frac{1}{2}})^2u_x^{n+\frac{1}{2}}, 2e^{n+\frac{1}{2}})\\ &= 2h\sum_{j=0}^{J}\{e_{\bar{x}j}^{n+\frac{1}{2}}(u_j^{n+\frac{1}{2}})^2_{\bar{x}}e_j^{n+\frac{1}{2}} + (e_j^{n+\frac{1}{2}})^3u_{xj}^{n+\frac{1}{2}} + 2(e_j^{n+\frac{1}{2}})^2u_j^{n+\frac{1}{2}}u_{xj}^{n+\frac{1}{2}}\}\\ &\le C\Big(\parallel e_x^{n+1}\parallel^2 + \parallel e_x^n\parallel^2 + \parallel e^{n+1}\parallel^2 + \parallel e^n\parallel^2\Big),\end{aligned}$$

$$\begin{aligned}(I_2, 2e^{n+\frac{1}{2}}) &= \Big[u^{n+\frac{1}{2}}u_x^{n+\frac{1}{2}}u_{xx}^{n+\frac{1}{2}} + \Big((u^{n+\frac{1}{2}})^2u_{xx}^{n+\frac{1}{2}}\Big)_{\overline{x}} - U^{n+\frac{1}{2}}U_x^{n+\frac{1}{2}}U_{xx}^{n+\frac{1}{2}} - \Big((U^{n+\frac{1}{2}})^2U_{xx}^{n+\frac{1}{2}}\Big)_{\overline{x}}, 2e^{n+\frac{1}{2}}\Big]\\ &\le C\Big(\parallel e_x^{n+1}\parallel^2 + \parallel e_x^n\parallel^2 + \parallel e^{n+1}\parallel^2 + \parallel e^n\parallel^2\Big).\end{aligned}$$

Therefore,

$$\begin{aligned}(R^n, 2e^{n+\frac{1}{2}}) &= (e_t^n, 2e^{n+\frac{1}{2}}) - (e_{tx\overline{x}}^n, 2e^{n+\frac{1}{2}}) + (I_1, 2e^{n+\frac{1}{2}}) + (I_2, 2e^{n+\frac{1}{2}})\\ &= \frac{1}{k}\Big(\parallel e^{n+1}\parallel^2 - \parallel e^n\parallel^2\Big) + \frac{1}{k}\Big(\parallel e_x^{n+1}\parallel^2 - \parallel e_x^n\parallel^2\Big) + (I_1, 2e^{n+\frac{1}{2}}) + (I_2, 2e^{n+\frac{1}{2}}).\end{aligned}$$

Taking Schwartz inequality, it is follows that

$$(R^n, 2e^{n+\frac{1}{2}}) \le 2\parallel R^n\parallel\cdot\parallel e^{n+\frac{1}{2}}\parallel\le\parallel R^n\parallel^2 + 2\Big(\parallel e^{n+1}\parallel^2 + \parallel e^n\parallel^2\Big).$$

From above discussion, we have

$$\begin{aligned}&\parallel e^{n+1}\parallel^2 - \parallel e^n\parallel^2 + \parallel e_x^{n+1}\parallel^2 - \parallel e_x^n\parallel^2\\ &\le Ck\parallel R^n\parallel^2 + Ck(\parallel e_x^{n+1}\parallel^2 + \parallel e_x^n\parallel^2 + \parallel e^{n+1}\parallel^2 + \parallel e^n\parallel^2).\end{aligned} \tag{14}$$

Let $\varphi_n = \parallel e^n\parallel^2 + \parallel e_x^n\parallel^2$, then (14) becomes to

$$\varphi_{n+1} - \varphi_n \le Ck(\varphi_{n+1} + \varphi_n) + Ck\parallel R^n\parallel^2,$$

hence we have

$$\begin{aligned}\varphi_n - \varphi_{n-1} &\le Ck(\varphi_n + \varphi_{n-1}) + Ck\parallel R^{n-1}\parallel^2,\\ \varphi_{n-1} - \varphi_{n-2} &\le Ck(\varphi_{n-1} + \varphi_{n-2}) + Ck\parallel R^{n-2}\parallel^2,\\ &\cdots\cdots\\ \varphi_1 - \varphi_0 &\le Ck(\varphi_1 + \varphi_0) + Ck\parallel R^0\parallel^2.\end{aligned}$$

From above inequalities, one has

$$\varphi_n \le \varphi_0 + Ck\sum_{i=0}^{n-1}\varphi_i + Ck\sum_{i=0}^{n-1}\parallel R^i\parallel^2,$$

where

$$k\sum_{i=0}^{n-1}\parallel R^i\parallel^2\le nk\max_{0\le i\le n-1}\parallel R^i\parallel^2\le TO(k^2+h^2)^2.$$

Since $\varphi_0 = 0$, then

$$\varphi_n \leq Ck \sum_{i=0}^{n-1} \varphi_i + CTO(k^2 + h^2)^2$$

From discrete Gronwall inequality, then $\varphi_n \leq O(k^2 + h^2)^2$, that is

$$\| e_x^n \|^2 + \| e^n \|^2 \leq O(k^2 + h^2)^2,$$

then we have

$$\| e^n \| \leq O(k^2 + h^2) \ , \ \| e_x^n \| \leq O(k^2 + h^2),$$

From Theorem 5, we assert

$$\| e_x^n \|_\infty \leq O(k^2 + h^2).$$

We complete the proof of the theorem. ■

Theorem 9 *The solution of difference scheme (5) is stable in norm L_∞.*

Proof. Assume that u_j^n is the solution of difference scheme (5), U_j^n is the solution of original differential equation (1)-(3). We can easily obtain that $e_j^n = U_j^n - u_j^n$.

Then from Theorem 8, following inequality holds true

$$\| e^n \|^2 \leq C \| U_0 - u_0 \|^2 \ .$$

Thus, we complete the proof of Theorem 9. ■

5 Conclusions

In this paper we give a difference scheme for the Novikov equation. In Section 2 we give some preparation knowledges. In Section 3 we propose a conservative finite difference scheme for the Novikov equation and use Brouwer fixed point theorem to obtain the existence of the solution for the corresponding difference equation. In Section 4 we prove the convergence and stability of the solution by using the discrete energy method.

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