ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.24(2017) No.1, pp.59-64

Null Controllability of the Weakly Dissipative Higher-order Camassa-Holm Equation with Moving Control

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(Received 6 January 2017, accepted 10 July 2017)

Abstract: In this paper, we study the null controllability of the weakly dissipative higher-order Camassa-Holm equation on the one-dimensional torus. We construct a biorhogonal family in $L^2(-T/2, T/2)$ and obtain the control function which is acting on a moving small interval with a constant velocity. Then we reduce the controllability problem to a moment problem. In the framework of the moment problem, we prove the null controllability of the higher-order CH equation in a suitable space with a regular intial data.

Keywords: higher-order Camassa-Holm equation; null controllability; moving control; moment problem

1 Introduction

The nonlinear dispersive wave equation

$$u_t + 2\lambda u_x - u_{xxt} + 3uu_x = 2u_x u_{xx} + uu_{xxx} \tag{1}$$

was derived by Camassa and Holm [1] as a model for the unidirectional propagation of shallow water waves over a flat bottom. Here $u(t,x)$ stands for the fluid velocity at time t in the spatial x direction (or equivalently the height of the free surface of water above a float bottom), λ is a constant related to the critical shallow water wave speed. Eq. (1) is the wellknown Camassa-Holm equation. It has a bi-Hamiltonian structure [2,3] and is completely integrable [1,4]. The well-posedness of the Camassa-Holm equation has been studied extensively [5-8]. Similar results for the local well-posedness of Eq. (1) were obtained by Rodriguez-Blanco [9]. In recent years, many researchers have researched the Camassa-Holm equation. They extend the studies to the generalized CH equation, higher-order CH equations and so on. L. Tian, Ch. Shen and D. Ding gave the optimal control of the viscous Camassa-Holm equation under the boundary condition and proved the existence and uniqueness of optimal solution to the viscous Camassa-Holm equation in a short interval [10]. In [11], Robert M. and X. Zhang studied the well-posedness and dynamics of a modified Camassa-Holm equation on the unit circles and obtained the result that it had some significant difference from that of the Camassa-Holm equation. Using geometrical methods, higherorder CH equations have been treated in [12] by Constantin and Kolev. The well-posedness of higher-order Camassa-Holm equations were considered in [13]. L. Tian, P. Zhang, et al have studied the global existence for the higher-order Camassa-Holm equation [14]. These works extended the research of the Camassa-Holm equation to more extensive fields.

The formulation of the higher-order Camassa-Holm equation recently derived by Coclite, Holden and et al in [15] is

$$\begin{aligned} B_k(u,u) &:= A_k^{-1} C_k(u) - uu_x, \\ A_k(u) &:= \sum_{j=0}^{k} (-1)^j \partial_x^{2j} u, \\ C_k(u) &= -uA_k(\partial_x u) + A_k(u\partial_x u) - 2\partial_x u A_k(u) \end{aligned} \tag{2}$$

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IJNS.2017.08.15/970

where k is a positive integer. In the cases of $k=0$ and $k=1$, Eq. (2) becomes the inviscid Burgers equation

$$u_t + 3uu_x = 0$$

and the Camassa-Holm equation

$$u_t - u_{xxt} + 3uu_x = 2u_x u_x x + uu_x xx.$$

We write Eq. (2) of the case of $k=2$ as following

$$m_t + 2u_x m + um_x = 0, \tag{3}$$

where $m = u - u_{xx} + u_{xxxx}, u(t,x)$ describes the horizontal velocity of the fluid.

It also can be rewritten as

$$u_t - u_{txx} + u_{txxxx} + 3uu_x - 2u_x u_{xx} - uu_{xxx} + 2u_x u_{xxxx} + uu_{xxxxx} = 0, \tag{4}$$

which is called higher-order Camassa-Holm equation.

This paper is devoted to studying null controllability for the linearing system of (4) with a moving distributed control

$$\begin{cases} y_t - y_{txx} + y_{txxxx} + \omega(y - y_{xx} + y_{xxxx}) = b(x)u(t) \\ y(x,0) = y_0 \end{cases} \tag{5}$$

where $x \in \mathbf{T} = R/(2\pi Z)$, $t \in [0,T]$, $\omega(y - y_{xx} + y_{xxxx})$ is the weakly dissipative item, $u(t)$ is the control.

The concept of moving point control was firstly introduced by J. L. Lions in [16] for the wave equation. One important motivation for this kind of control is that the exact controllability of the wave equation with a pointwise control and Dirichlet boundary conditions fails if the point is a zero of some eigenfunction of the Dirichlet Laplacian, while it holds when the point is moving under some (much more stable) conditions . The controllability of the wave equation (resp. of the heat equation) with a moving point control was investigated in [16-18] . See also [19] for Maxwells equations.

Let $z(x,t) = y(x - ct, t)$, then (5) turns into

$$\begin{cases} z_t - z_{txx} + z_{txxxx} + c(z_x - z_{xxx} + z_{xxxxx}) + \omega(z - z_{xx} + z_{xxxx}) = b(x)u(t) \\ z(x,0) = z_0 \end{cases} \tag{6}$$

For convenience, we assume that $c = -1$, while for any c it can be derived in the same way.

Namely, we consider the following system

$$\begin{cases} z_t - z_{txx} + z_{txxxx} - (z_x - z_{xxx} + z_{xxxxx}) + \omega(z - z_{xx} + z_{xxxx}) = b(x)u(t) \\ z(x,0) = z_0 \end{cases} \tag{7}$$

2 Preliminaries

2.1 Spectral decomposition

The free evolution equation as sociated with (7) reads

$$z_t - z_{txx} + z_{txxxx} - (z_x - z_{xxx} + z_{xxxxx}) + \omega(z - z_{xx} + z_{xxxx}) = 0. \tag{8}$$

Let us first consider the operator

$$z_t = Az := (I - \partial_x^2 + \partial_x^4)^{-1}((z_x - z_{xxx} + z_{xxxxx}) - \omega(z - z_{xx} + z_{xxxx})). \tag{9}$$

The eigenvalues of A can be obtained by solving

$$\lambda(z - z_{xx} + z_{xxxx}) = (z_x - z_{xxx} + z_{xxxxx}) - \omega(u - u_{xx} + u_{xxxx}) \tag{10}$$

Expanding z as a Fourier series $z = \sum_{k\in Z} z_k(t)e^{ikx}$, we obtain that (10) is satisfied provided that for each $k \in Z$

$$(\lambda_k(1 + k^2 + k^4) - ik(1 + k^2 + k^4) + \omega(1 + k^2 + k^4))z_k(t) = 0. \tag{11}$$

Since $z_k(t) \neq 0$, the eigenvalues of A read as

$$\lambda_k = -\omega + ik. \tag{12}$$

The expression of λ_k implies that $\lambda_k \neq \lambda_l$, for $k \neq l$.

To each λ_k corresponds an eigenfunction e^{ikx}, if we consider an initial data $y_0 \in L^2(\mathbf{T})$ with $y_0 = \sum_{k\in Z} a_k e^{ikx}$, the solution of (11) corresponding to this initial data can be written as

$$z(x,t) = \sum_{k\in Z} a_k e^{\lambda_k t} e^{ikx} \tag{13}$$

As an easy consequence of the above representation formula, we have the following result.

Proposition 1 *([20]) Let $s \in R$. If $y_0 \in H^s(\boldsymbol{T})$, then $z \in C([0,\infty]; H^s(\boldsymbol{T}))$.*

Proof. If $y_0 \in H^s(T)$, we have $\sum_{k\in Z} |a_k|^2 k^{2s} < \infty$. Then

$$|a_k e^{\lambda_k t}|^2 k^{2s} \leq |a_k|^2 k^{2s} e^{2\lambda_k t} \leq |a_k|^2 k^{2s} e^{-2t} \tag{14}$$

and hence $z \in C([0,\infty), H^s(\mathbf{T}))$ ■

2.2 Construction of a biorthogonal family

This section is motivated by [20]. The main idea is to construct a suitable family $\{\Phi_k\}_{k\in Z}$ of entire functions of exponential type, then Ψ_k can be defined as the inverse Fourier transform of Φ_k, which has the following characteristic.

Proposition 2 *Let $T > 2\pi$. There exists a family $\{\Psi_k\}_{k\in Z}$ of functions in $L^2(-T/2, T/2)$ such that*

$$\int_{-T/2}^{T/2} \Psi_k(t) e^{\lambda_l t} dt = \delta_k^l, \quad k, l \in Z, \tag{15}$$

where δ_k^l is the Kronecker symbol. Moreover

$$\|\Psi_k\|_{L^2(-T/2,T/2)} \leq C|k|, \tag{16}$$

where C is a positive constant.

Proof. We note

$$P(z) = \frac{\sin[(z+i\omega)\pi]}{(z+i\omega)\pi}, z \in \boldsymbol{C}, \tag{17}$$

which has simple zeros exactly at $\{i\lambda_k\}_{k\in Z\setminus\{0\}}$

It is easy to see that

$$\mid P(z) \mid \leq e^{|((z+i\omega)\pi|} \leq e^{\pi|z|}. \tag{18}$$

This implies that P is an entire function of exponential type π, and

$$P'(i\lambda_k) = (-1)^{k+1}\frac{1}{k}. \tag{19}$$

Now, we can construct the functions in the biorhogonal family.

We set

$$\Phi_k(z) = \frac{P(z)}{P'(i\lambda_k)(z - i\lambda_k)}, k \in Z \setminus \{0\}. \tag{20}$$

Obviously, Φ_k is an entire function of exponential type $\frac{T}{2}$ and

$$\Phi_k(i\lambda_l) = \delta_k{}^l. \tag{21}$$

Furthermore, we have that

$$|\Phi_k(x)| = |\frac{P(x)}{P'(i\lambda_k)(x - i\lambda_k)}| \leq \frac{C|k|}{|(x+i\omega)(x+k+i\omega)} = \frac{C|k|}{\sqrt{(x^2+xk-\omega^2)^2 + \omega^2(2x+k)^2}}. \tag{22}$$

Therefore,

$$\begin{aligned}\|\Phi_k\| &\leq C|k|(\int_{-\infty}^{+\infty}\frac{1}{(x^2+xk-\omega^2)^2+\omega^2(2x+k)^2}dx)^{\frac{1}{2}}\\ &= C|k|(\int_{0}^{+\infty}\frac{1}{s^4+(2\omega^2-\frac{k^2}{2})s^2+(\omega^2+\frac{k^2}{2})^2}ds)^{\frac{1}{2}}\\ &= C|k|(\int_{0}^{+\infty}\frac{1}{(s^2-(\frac{k^2}{2}-2\omega^2))^2+2\omega^2k^2}ds)^{\frac{1}{2}} \leq C|k|.\end{aligned} \tag{23}$$

Similiar to $\Phi_k(z)$, we may define

$$\Phi_0(z) = \frac{P(z)}{P'(-i\omega)(z+i\omega)} \tag{24}$$

It is clear that Φ_0 is an entire function of exponential type $T/2$ and it belongs to $L^2(R)$. Now, we construct the family Ψ_k by inverse Fourier transform of Φ_k. Then it follows from the Paley-Wiener theorem that Ψ_k also belongs to $L^2(R)$ and is supported in $[-\frac{T}{2},\frac{T}{2}]$. Thus, we obtain

$$\int_{-T/2}^{T/2}\Psi_k(t)e^{\lambda_l t}dt = \Phi_k(i\lambda_l), k,l\in Z \tag{25}$$

And (15) follows from (21) and (25). The proof of Proposition 2.2 is completed. ■

3 Null controllability

The precise result of null controllability for (7) is given in the following theorem.

Theorem 3 *Let $b(x)\in L^2(\boldsymbol{T})$ be such that*

$$\rho_k = \int_{\boldsymbol{T}} b(x)e^{-ikx}dx \neq 0 \quad for \quad k\in Z. \tag{26}$$

For any $T>2\pi$ and any $y_0\in L^2(\boldsymbol{T})$ decomposed as $y_0=\sum_{k\in Z}a_ke^{ikx}$ with

$$\sum_{k\in Z\setminus\{0\}}|\rho_k|^{-1}|a_k||k|^5<\infty, \tag{27}$$

then there exists a control $u\in L^2(0,T)$ such that (7) is null controllable.

Proof. Let us recall the control system

$$z_t - z_{txx} + z_{txxxx} - (z_x - z_{xxx} + z_{xxxxx}) + \omega(z - z_{xx} + z_{xxxx}) = b(x)u(t), \tag{28}$$

where $b\in L^2(\mathbf{T})$, $supp\ \ b\subset\mathbf{T}$ and $u\in L^2(0,T)$ is the control decided in the later.

The adjoint equation to the above system reads

$$\varphi_t - \varphi_{txx} + \varphi_{txxxx} - (z_x - z_{xxx} + z_{xxxxx}) + \omega(z - z_{xx} + z_{xxxx}) = 0 \tag{29}$$

Multiplying (28) by φ and integrating by parts, we obtain

$$\int_{\mathbf{T}} zm|_0^T dx + \int_{\mathbf{T}} z(T)(m_{xxxx}(T) - m_{xx}(T))dx + \int_{\mathbf{T}} z(0)(m_{xx}(0) - m_{xxxx}(0))dx = \int_0^T\int_{\mathbf{T}} bu\varphi dxdt, \tag{30}$$

where $m=\varphi-\varphi_{xx}+\varphi_{xxxx}$.

Note that $\varphi(x,t)=\hat{z}(2\pi-x,T-t)$ is the solution of (29) if $\hat{z}(2\pi-x,T-t)=\sum_{k\in Z}a_k e^{\lambda_k t}e^{ikx}$ is the solution of (28) for $u(t)\equiv 0$. Then from (30), we get

$$\sum_{k\in Z}a_k(1+k^2+k^4)^2 < z(T),e^{-ikx} > -\sum_{k\in Z}a_k(1+k^2+k^4)^2 e^{\lambda_k(T)} < z(0),e^{-ikx} > \\ =\sum_{k\in Z}a_k\int_0^T e^{\lambda_k(T-t)}u(t)dt\int_{\mathbf{T}}b(x)e^{-ikx}dx \tag{31}$$

Recall $\rho_k=\int_T b(x)e^{-ikx}dx$. Since the identity (31) is true for each $a_k\in l^2$, it follows that

$$(1+k^2+k^4)^2 < z(T),e^{-ikx} > -(1+k^2+k^4)^2 e^{\lambda_k T} < z(0),e^{-ikx} >=\rho_k\int_0^T e^{\lambda_k(T-t)}u(t)dt \tag{32}$$

Let us denote $\alpha_k=-\rho_k^{-1}(1+k^2+k^4)e^{\lambda_k\frac{T}{2}} < z(0),e^{-ikx} >$ and introduce a function using the biorthogonal family which we construct in the section 2.2.

$$\Psi(t)=\sum_{k\in Z\backslash\{0\}}\alpha_k\Psi_k(t) \tag{33}$$

Then we choose the control function

$$u(t)=\Psi(\frac{T}{2}-t) \tag{34}$$

It is clear that

$$\|u\|_{L^2(0,T)}=\|\Psi\|_{L^2(-\frac{T}{2},\frac{T}{2})}\le C(\sum_{k\in Z\backslash\{0\}}\rho_k^{-1}|a_k|(1+k^2+k^4)|k|)<\infty \tag{35}$$

Hence $u\in L^2(0,T)$.

Then, from (26),(27),(33), (34) and Proposition 2.2, for any $k\in Z$, we have

$$\rho_k\int_0^T e^{\lambda_k(T-t)}u(t)dt=\rho_k e^{\lambda_k\frac{T}{2}}\int_0^T e^{\lambda_k t}\Psi(t)dt=\rho_k e^{\lambda_k\frac{T}{2}}\alpha_k=-(1+k^2+k^4)^2e^{\lambda_k T} < z(0),e^{-ikx} > \tag{36}$$

Combining with (32) and (36), we include that $< z(T),e^{-ikx} >=0$, for any $k\in Z$, which implies that $z(T)=0$. This completes the proof of Theorem 3.1. ■

4 Conclusion

In this paper, the equation $z_t-z_{txx}+z_{txxxx}-(z_x-z_{xxx}+z_{xxxxx})+\omega(z-z_{xx}+z_{xxxx})=b(x)u(t)$ is proved to be null controllable on the torus (i.e. with periodic boundary conditions). Of course we also can gain the similar result with a moving point control, such as $u(t)\delta(x)$. Null controllability can also be studied by means of Calman estimation. Once null controllability is obtained, we can consider the exact controllability and local exact controllability to a trajectory for this kind of equation in the future .

Acknowledgements

The research is supported by the National Natural Science Foundation of China (No: 11171135, 71690242), and the Major Project of Natural Science Foundation of Jiangsu Province Colleges and Universities (14KJA110001).

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