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The Fixed Point Theorems of Ordered Contractive Mapping in Ordered Menger PM Spaces

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Abstract: In this paper, we apply the partially order in Menger PM spaces to prove some fixed point theorems for the ordered contractive self-maps in partially ordered Menger PM spaces. Our results make the improvement of the research of Jian-dong Yin.

Keywords: fixed point; ordered contractive mapping; partially ordered set

1 Introduction

Fixed point theory in probabilistic metric spaces can be considered as a part of Probabilistic Analysis, which is a very dynamic area of mathematical research. The introduction to the general concept of statistical metric spaces is owed to K. Menger[5], who dealt with probabilistic geometry. The idea of K. Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a probabilistic metric space corresponds to situations when we do not know exactly the distance between two points, but we know probabilities of possible values of this distance. A probabilistic generalization of metric spaces appears to be of interest in the investigation of physical quantities and physiological thresholds. It is also of fundamental importance in probabilistic functional analysis. The new theory of fundamental probabilistic structures was developed later on by many authors. The main influence upon the development of the theory of probabilistic metric spaces are owed to B. Schweizer, A. Sklar and their coworkers[8, 9]. Since then the theory of probabilistic metric spaces has developed in many directions [1–3, 6, 7, 10, 12–14].

In this paper, we apply not only the partially order in Menger PM spaces, but also the ordered contractive mapping which is different from the known contractive mapping to present several fixed point theorems, especially the mapping satisfying certain ordered contractive condition in partially ordered complete PM spaces. The results we have established extend many known results in Menger PM spaces as well as cone metric spaces.

2 Preliminaries

In this paper, let $\mathbb{R} = (-\infty, +\infty)$, $\mathbb{R}^+ = [0, +\infty)$ and $\mathbb{N}$ is the set of all positive integers. (Ω, Λ, P) is a complete probability metric space, (X, d) is a separable complete metric space.

Definition 1 *A mapping $F : \mathbb{R} \to \mathbb{R}^+$ is called a distribution function, if it is a nondecreasing and left continuation satisfying the following conditions:*

$$\inf_{t\in\mathbb{R}} F(t) = 0 \quad \sup_{t\in\mathbb{R}} F(t) = 1.$$

We shall denote by $\mathcal{D}$ the set of all distribution functions. We shall denote the distribution function $F(x, y)$ by $F_{x,y}$ and $F_{x,y}(t)$ will represent the value of $F_{x,y}$ at $t \in$ R. Throughout this paper H will always denote the specific distribution function defined by

$$H(t) = \begin{cases} 1, & t > 0 \\ 0, & t \le 0 \end{cases}.$$

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Definition 2 *([4]) Let E is a nonempty set. An ordered pair $(E, \mathcal{F})$ is called a PM spaces if $\mathcal{F}$ is a mapping from $E \times E$ into $\mathcal{D}$ satisfying the following conditions(for all $x, y \in E$, we denote $\mathcal{F}(x, y)$ by$F_{x,y}$):*

(PM-1) $F_{x,y}(0) = 0$;

(PM-2) $F_{x,y}(t) = H(t)$ *for all* $t \in \mathbb{R}$ *if and only if* $x = y$;

(PM-3) $F_{x,y}(t) = F_{y,x}(t)$;

(PM-4) *if* $F_{x,y}(t_1) = 1$ *and* $F_{y,z}(t_2) = 1$, *then* $F_{x,z}(t_1 + t_2) = 1$ *for all* $x, y, z \in E$ *and* $t_1, t_2 \in \mathbb{R}$.

Every metric space (E, d) can always be realized as a PM spaces by considering $F : E \times E \to \mathcal{D}$ defined by $F_{x,y}(t) = H(t - d(x, y))$ for all $x, y \in E$. So PM spaces offer a wider framework than that of metric spaces and are better suited to cover even wider statistical situations.

Definition 3 *([4]) A mapping $\triangle : [0, 1] \times [0, 1] \to [0, 1]$ is called a t-norm if:*

(1) $\triangle(a, 1) = a$, $\triangle(0, 0) = 0$;

(2) $\triangle(a, b) = \triangle(b, a)$;

(3) $\triangle(c, d) \geq \triangle(a, b)$ *for* $c \geq a$, $d \geq b$;

(4) $\triangle(\triangle(a, b), c) = \triangle(a, \triangle(b, c))$ *for all* $a, b, c \in [0, 1]$.

Definition 4 *([4]) A Menger PM spaces $(E, \mathcal{F}, \triangle)$ is a triplet where $(E, \mathcal{F})$ is a PM spaces and $\triangle$ is a t-norm satisfying the following condition:*

$$F_{x,z}(t_1 + t_2) \geq \triangle(F_{x,y}(t_1), F_{y,z}(t_2))), \tag{1}$$

for all $x, y, z \in E$, $t_1, t_2 \geq 0$

In this paper, we suppose that t-norm $\triangle$ is continuous.

Definition 5 *([4]) A sequence $\{x_n\}$ in a Menger PM spaces $(E, \mathcal{F}, \triangle)$ is said to converge to a point x^*, if for every $\epsilon > 0$ and $\lambda > 0$, there is an plus integer $N = N(\varepsilon, \lambda)$ such that*

$$F_{x_n,x^*}(\varepsilon) > 1 - \lambda,$$

for all $n \geq N(\varepsilon, \lambda)$.

Definition 6 *([4]) A sequence $\{x_n\}$ in a Menger PM spaces $(E, \mathcal{F}, \triangle)$ is said to be a Cauchy sequence if for every $\epsilon > 0$ and $\lambda > 0$, there is an plus integer $N = N(\varepsilon, \lambda)$ such that*

$$F_{x_m,x_n}(\varepsilon) > 1 - \lambda,$$

for all $n, m \geq N(\varepsilon, \lambda)$.

Definition 7 *([4]) A Menger PM spaces $(E, \mathcal{F}, \triangle)$ is called to be complete if every Cauchy sequence in it converges to a point of it.*

Definition 8 *([3]) Let $(E, \mathcal{F}, \triangle)$ is a Menger PM spaces and $\triangle$ is a continuous t-norm with $\triangle(t, t) \geq t$, and $\varphi : E \to R$ is a mapping, a relation $'' \leq''$ in E is defined as follows:*

$$\forall x, y \in E, x \leq y \quad \textit{iff} \quad \forall t > \varphi(x) - \varphi(y), \forall \lambda > 0, F_{x,y}(t) > 1 - \lambda.$$

We can easily prove that $'' \leq''$ is a partial order in E. Call it the partial order induced by φ.

Definition 9 *Let $(E, \mathcal{F}, \triangle)$ is a Menger PM spaces and $\triangle$ is a continuous t-norm with $\triangle(t, t) \geq t$, and $\varphi : E \to \mathbb{R}$ is a mapping, $'' \leq''$ is partial order induced by φ. Let $A : E \to E$ is a mapping. Mapping A is called an ordered contractive mapping, if there exists constant $k \in (0, 1)$, such that for all $x, y \in E$, and $x \leq y$, then for each $\varepsilon > 0$ the following relation holds:*

$$F_{Ax,Ay}(\varepsilon) \geq F_{x,y}(\frac{\varepsilon}{k}).$$

Definition 10 *Let (E, d) is a metric space, $A : E \to E$ is a mapping, for each $x_0 \in E$, $\{orb(x_0A)\}$ is defined as follow:*

$$\{orb(x_0A)\} = \{x_0, Ax_0, A^2x_0, A^3x_0, \cdots, A^mx_0, \cdots\},$$

the sequence $\{orb(x_0A)\}$ is called the orbit of x_0 under A.

Definition 11 *Let $\{orb(x_0A)\}$ is an orbit and $\varphi : E \to R$ is a mapping, we call that φ is regular on $\{orb(x_0A)\}$ if the sequence*

$$\{\varphi(x_0), \varphi(Ax_0), \varphi(A^2x_0), \varphi(A^3x_0), \cdots, \varphi(A^mx_0), \cdots\},$$

is convergent.

3 Main Results

Lemma 1 *Let $(E, \mathcal{F}, \triangle)$ is a Menger PM spaces, and $\varphi : E \to R$ is a mapping, $'' \leq ''$ is partial order induced by φ. If $x \leq y$ $x, y \in E$, then $\varphi(x) \geq \varphi(y)$*

Proof. Suppose not. That is, suppose that if $x \leq y$ $x, y \in E$, then $\varphi(x) \leq \varphi(y)$ such that $\varphi(x) - \varphi(y) < 0$. In this case we have $\forall t > \varphi(x) - \varphi(y), \forall \lambda > 0$

$$F_{x,y}(t) > 1 - \lambda.$$

Let $t \to 0$, then

$$F_{x,y}(0) > 1 - \lambda,$$

which is a contradiction by $F_{x,y}(0) = 0$. This contradiction proves the assertion. ■

Now we prove our main results.

Theorem 2 *Let $(E, \mathcal{F}, \triangle)$ is a Menger PM spaces and $\triangle$ is a continuous t-norm with $\triangle(t, t) \geq t$, and $\varphi : E \to R$ is a mapping, $'' \leq ''$ is partial order induced by φ. Let $A : E \to E$ is a continuous and increasing mapping, and A is an ordered contractive mapping. Suppose that the following conditions hold:*

(a) there exists $x_0 \in E$, with $x_0 \leq Ax_0$;

(b) φ has an lower bound and regulars in $\{orb(x_0A)\}$;

then, A has a fixed point in E.

Proof. Let $x_1 = Ax_0, x_2 = Ax_1, \cdots, x_n = Ax_{n-1}, \cdots, n = 1, 2, \cdots$. By (a), we have $x_0 \leq x_1$. Since A is an increasing mapping, let $x_{k-1} \leq x_k$, we have $Ax_{k-1} \leq Ax_k$ such that $x_k \leq x_{k+1}$. Continuing this process we can construct sequence $\{x_n\}$ in E with

$$x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n \cdots \tag{2}$$

By lemma1 and (2), we have

$$\varphi(x_0) \geq \varphi(x_1) \geq \varphi(x_2) \geq \cdots \geq \varphi(x_n) \cdots \tag{3}$$

By (b), which implies that $\{\varphi(x_n)\}_{n\in\mathrm{R}}$ is a bounded sequence. By (3), $\{\varphi(x_n)\}_{n\in\mathrm{R}}$ is Cauchy sequence, such that

$$\forall \varepsilon > 0, \exists N, \forall n > m > N, \varphi(x_m) - \varphi(x_n) < \varepsilon. \tag{4}$$

By (4) and $x_m \leq x_n$, then we have

$$\forall \lambda > 0, F_{x_m,x_n}(\varepsilon) > 1 - \lambda,$$

such that $\{x_n\}$ is a Cauchy sequence in E.

Since $(E, \mathcal{F}, \Delta)$ is complete, there exist $x^* \in E$ such that

$$x_n \to x^* \quad \text{as} \quad n \to \infty. \tag{5}$$

By (5) and the continuity of Δ, for all $\varepsilon > 0$, we have

$$\begin{aligned} F_{Ax^*,x^*}(\varepsilon) &\geq \Delta(F_{Ax^*,Ax_n}(\frac{\varepsilon}{2}), F_{Ax_n,x^*}(\frac{\varepsilon}{2})) \\ &= \Delta(F_{Ax^*,Ax_n}(\frac{\varepsilon}{2}), F_{x_{n+1},x^*}(\frac{\varepsilon}{2})). \end{aligned} \tag{6}$$

Let $n \to \infty$, we get

$$\Delta(F_{Ax^*,Ax_n}(\frac{\varepsilon}{2}), F_{x_{n+1},x^*}(\frac{\varepsilon}{2})) \to \Delta(F_{Ax^*,Ax^*}(\frac{\varepsilon}{2}), F_{x^*,x^*}(\frac{\varepsilon}{2})) = 1. \tag{7}$$

From (6) and (7), we have

$$F_{Ax^*,x^*}(\varepsilon) \geq 1.$$

Therefore, we have $Ax^* = x^*$ such that A has a fixed point x^* in E. ■

Theorem 3 *Let $(E, \mathcal{F}, \triangle)$ is a Menger PM spaces and $\triangle$ is a continuous t-norm with $\triangle(t,t) \geq t$, and $\varphi : E \to R$ is a mapping, $'' \leq''$ is partial order induced by φ. Let $A : E \to E$ is a continuous and increasing mapping, and A is an ordered contractive mapping. Assume that there exists $x_0 \leq y_0$ with $x_0 \leq Ax_0, Ay_0 \leq y_0$, then A has a fixed point x^* in E. For all $u_0, v_0 \in E$ with $x_0 \leq u_0 \leq v_0 \leq y_0$, then the sequence $u_n = Au_{n-1}, v_n = Av_{n-1}$ satisfy $u_n \to x^*, v_n \to x^*$ as $n \to \infty$.*

Proof. Let $x_1 = Ax_0, x_2 = Ax_1, \cdots, x_n = Ax_{n-1}, \cdots$ and $y_1 = Ay_0, y_2 = Ay_1, \cdots, y_n = Ay_{n-1}, \cdots, n = 1, 2, \cdots$, since $x_0 \leq Ax_0$ and using the increasing property of A, we have $Ax_0 \leq Ax_1$ such that $x_1 \leq x_2$. Let $x_{k-1} \leq x_k$, then $Ax_{k-1} \leq Ax_k$ such that $x_k \leq x_{k+1}$. Similarly, we have $y_k \leq y_{k+1}$.

Assume that there exists $x_0 \leq y_0$ with $x_0 \leq Ax_0, Ay_0 \leq y_0$,by the increasing of A, we have $x_0 \leq Ax_0 \leq Ay_0 \leq y_0$, and $x_0 \leq Ax_0 \leq Ax_1 \leq Ay_1 \leq Ay_0 \leq y_0$, such that $x_0 \leq x_1 \leq x_2 \leq y_2 \leq y_1 \leq y_0$. Continuing this process, we have

$$x_0 \leq x_1 \leq x_2 \cdots \leq x_n \leq \cdots \leq y_n \leq \cdots \leq y_2 \leq y_1 \leq y_0 \tag{8}$$

By Lemma1, we have

$$\varphi(x_0) \geq \varphi(x_1) \geq \cdots \geq \varphi(x_n) \geq \cdots \geq \varphi(y_n) \geq \cdots \geq \varphi(y_1) \geq \varphi(y_0). \tag{9}$$

Therefore $\{\varphi(x_n)\}_{n\in \mathrm{R}}, \{\varphi(y_n)\}_{n\in \mathrm{R}}$ are Cauchy sequences such that

$$\forall \varepsilon > 0, \exists N, \forall n > m > N, \varphi(x_m) - \varphi(x_n) < \varepsilon.$$

Since the definition of the partial order, let $x_m \leq x_n$

$$\forall \varepsilon > 0, \forall \lambda > 0, F_{x_m,x_n}(\varepsilon) > 1 - \lambda.$$

Therefore $\{x_n\}$ is a Cauchy sequence in E.
Since $(E, \mathcal{F}, \Delta)$ is complete, there exist $x^* \in E$ such that

$$x_n \to x^* \quad \text{as} \quad n \to \infty.$$

Similarly, we can prove that $y_n \to y^*$ as $n \to \infty$.
Next we prove $x^* = y^*$.
Since the continuity of Δ, for all $\varepsilon > 0$, we have

$$\begin{aligned} F_{x^*,y^*}(\varepsilon) &\geq \Delta(F_{x^*,x_{n+1}}(\frac{\varepsilon}{3}), F_{Ax_n,y^*}(\frac{2\varepsilon}{3})) \\ &\geq \Delta(F_{x^*,x_{n+1}}(\frac{\varepsilon}{3}), \Delta(F_{Ax_n,Ay_n}(\frac{\varepsilon}{3}, F_{Ay_n,y^*}(\frac{\varepsilon}{3}))) \\ &\geq \Delta(F_{x^*,x_{n+1}}(\frac{\varepsilon}{3}), \Delta(F_{x_n,y_n}(\frac{\varepsilon}{3k}, F_{y_{n+1},y^*}(\frac{\varepsilon}{3}))) \\ &\cdots \\ &\geq \Delta(F_{x^*,x_{n+1}}(\frac{\varepsilon}{3}), \Delta(F_{x_0,y_0}(\frac{\varepsilon}{3k^n}, F_{y_{n+1},y^*}(\frac{\varepsilon}{3}))) \to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus we prove $x^* = y^*$ such that $x_n \to x^*$,$y_n \to y^*$ as $n \to \infty$.
Next we prove x^* is the fixed point of A.
By (1), we have, for all $\varepsilon > 0$,

$$F_{Ax^*,x^*}(\varepsilon) \geq \Delta(F_{Ax^*,Ax_n}(\frac{\varepsilon}{2}), F_{x_{n+1},x^*}(\frac{\varepsilon}{2})). \tag{10}$$

Since the continuity of A and $x_{n+1} \to x^*$, as $n \to \infty$, we have

$$Ax_n \to Ax^* \quad \text{as} \quad n \to \infty.$$

and

$$\begin{aligned} F_{Ax^*,x^*}(\varepsilon) &\geq \Delta(F_{Ax^*,Ax_n}(\frac{\varepsilon}{2}), F_{x_{n+1},x^*}(\frac{\varepsilon}{2})) \\ &\to \Delta(F_{Ax^*,Ax^*}(\frac{\varepsilon}{2}), F_{x^*,x^*}(\frac{\varepsilon}{2})) = 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus we have $Ax^* = x^*$.

Let $u_1 = Au_0, u_2 = Au_1, \cdots, u_n = Au_{n-1}, \cdots$ and $v_1 = Av_0, v_2 = Av_1, \cdots, v_n = Av_{n-1}, \cdots, n = 1, 2, \cdots$. If there exists $u_0, v_0 \in E$ with $u_0 \leq Au_0, v_0 \leq Av_0$ and $x_0 \leq u_0 \leq v_0 \leq y_0$, by the increasing of A, we have

$$u_0 \leq u_1 \leq u_2 \cdots \leq u_n \leq \cdots \leq v_n \leq \cdots \leq v_2 \leq v_1 \leq v_0$$

By the increasing of A and $x_0 \leq u_0 \leq v_0 \leq y_0$, we have

$$Ax_0 \leq Au_0 \leq Av_0 \leq Ay_0,$$

such that

$$x_1 \leq u_1 \leq v_1 \leq y_1. \tag{11}$$

Since (11) and the increasing of A, we have

$$Ax_1 \leq Au_1 \leq Av_1 \leq Ay_1,$$

such that

$$x_2 \leq u_2 \leq v_2 \leq y_2.$$

Continuing this process, we have

$$x_n \leq u_n \leq v_n \leq y_n, \forall n \in \mathbb{N}.$$

Similar to the proof of the above process, there exists $\bar{x} \in E$ with

$$u_n \to \bar{x}, v_n \to \bar{x} \quad \text{as} \quad n \to \infty,$$

and

$$A\bar{x} = \bar{x}.$$

Since that $A : E \to E$ is a continuous and increasing mapping, and A is an ordered contractive mapping, we have

$$\begin{aligned} F_{x^*,\bar{x}}(\varepsilon) &\geq \Delta(F_{Ax^*,Ay_n}(\frac{\varepsilon}{3}), F_{Ax_n,A\bar{x}}(\frac{2\varepsilon}{3})) \\ &\geq \Delta(F_Ax^*, Ay_n(\frac{\varepsilon}{3}), \Delta(F_{Ay_n,Au_n}(\frac{\varepsilon}{3}, F_{Au_n,A\bar{x}}(\frac{\varepsilon}{3}))) \\ &\geq \Delta(F_{x^*,y_{n+1}}(\frac{\varepsilon}{3}), \Delta(F_{y_n,u_n}(\frac{\varepsilon}{3k}, F_{u_{n+1},\bar{x}}(\frac{\varepsilon}{3}))) \\ &\cdots \\ &\geq \Delta(F_{x^*,y_{n+1}}(\frac{\varepsilon}{3}), \Delta(F_{y_0,u_0}(\frac{\varepsilon}{3k^n}, F_{u_{n+1},\bar{x}}(\frac{\varepsilon}{3}))) \to 1 \quad \text{as} \quad n \to \infty. \end{aligned}$$

Thus $u_n \to x^*, v_n \to x^*$ as $n \to \infty$. ■

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