

Complex Dynamics of a Cournot Investment Game with Delayed Bounded Rationality

Xingdan Fan*, Zhanwen Ding

Nonlinear Science Research Center, Jiangsu University, Zhenjiang, Jiangsu, 212013. P.R. China

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Abstract: In this work, we investigate a time delay structure for the dynamic system of Cournot investment game played by two heterogeneous players. One player adjusts its investment decision by the locally marginal profit while the other player makes its strategy via its marginal profit with time delay. The existence and stability of equilibrium solutions is discussed. To show how the stability of this system depend on the model parameters, numerical simulations are used to provide evidence. It is shown that as the parameters are varied, the stability of this system may lose because of period doubling bifurcations or Neimark-Sacker bifurcations. It is also shown that a proper delay weight or a small capital depreciation has the great effect on increasing the stability domain.

Keywords: Cournot game; Investment; Heterogeneous expectation; Time delay; Dynamic system

1 Introduction

Expectations play a critical role in a complex oligopolistic game which describes a market with several players, who adopt their expectations rules of many available techniques to compute their expected outputs. That is, each player maybe behave with different expectations strategies (naive, boundedly rational, or adaptive). The earliest formal theory of oligopoly was introduced by Cournot [1]. He investigated the case with naive expectations, so that each firm guesses the opponent's output remains at the same level as that of previous period and then produces precisely the quantity of production to maximize its profit. Since Cournot, many classic research based on the Cournot model have been proposed to analyze the system stability and the complex phenomena in the oligopoly game with naive expectation [2, 3].

Recently, as point out by Bischi and Naimzada [4] there are some unrealistic assumptions in dynamical models with naive expectations that each firm not only knows the all rivals' production decisions but also a complete knowledge of the market demand function. A more sophisticated approach with bounded rationality is to be assumed. Each firm with bounded rationality increases its production if the perceived marginal profit is positive and decreases it otherwise. Such efforts have been extended for the complex dynamics of bounded rationality duopoly game played by players with homogeneous expectations [5, 6].

In the cases of most models mentioned above, the players were considered to be with the same strategy (bounded rationality) for the computed expected outputs. Several studies on the dynamics of oligopoly models with heterogeneous players have also been proposed in recent years [7–9].

Wang et al.[10] and Yuan et al.[11] considered the delayed system and M.T.Yassen et al. [12] also has considered the delayed dynamical Cournot game with bounded rationality, and in these models the bounded rationality players make time delay on the output variable and regulate their marginal profits according to the delayed data of their outputs. In these words, it is proved that time delay can enlarge the stability region and delay the occurrence of complex behaviors. However, due to incomplete information for the players in a market, they are less-known about the opponents output data, hence a more realistic case of delayed bounded rationality was studied in [13]. It is point out in [13] that any producer is

*Corresponding author. E-mail address: 820909327@qq.com

able to know his own marginal profit, which is the profit of the last unit production and can be observed in its accounting. Then Ding et al.[13] have considered a duopoly game for the case of setting time delay on the variable of marginal profit.

In the above models the product supply quantity is assumed to be the decision variable for all the players. That is, output is the key factor. The central assumption is that each player can provide sufficient quantity of good for the purpose of local profit maximization. However, it has been known that, in an emerging and immature industry under development, due to its lack of investment accumulation, the firms can not produce enough products. Hence, investment strategy is of great important for the players when an infant industry keep in the developing period. Based on this situation, a dynamical system of investment duopoly model with bounded rationality is considered in [6]. Ding et al. [6] have considered a duopoly game with homogeneous bounded rationality: each player makes investment decision according to his marginal profit. And then, Ding et al. [9] developed the investment model with heterogeneous players. In this work, we reconsider the duopoly model introduced in [6]. That is, one is boundedly rational player without delay and one player considers time delay. Hence both realistic ideas of delayed boundedly rationality and heterogeneous expectations are combined in this work.

2 The model

In our work, we consider two players producing same or homogeneous goods for sale and making investment strategies at discrete time periods $t = 0, 1, 2, \dots$. As discussed in [6, 9], we also concentrate on firms' investment strategies rather than their output strategies.

We consider a Cournot investment game where $K_i(t)$ denotes the capital stock supplied by firm i , $x_i(t)$ represents the investment of firm i during period $t, i = 1, 2$. In addition the capital depreciation rate is $\theta, 0 < \theta < 1$. Then the capital stock of firm i is given by (see e.g.[6])

$$K_i(t) = (1 - \theta)K_i(t - 1) + x_i(t), i = 1, 2. \tag{1}$$

Let $q_i(t)$ denotes the output of firm i at time t . As done in [6], the output of i th in period t is determined by its capital stock $K_i(t)$ through a linear output function

$$q_i(t) = B_i K_i(t), i = 1, 2, \tag{2}$$

where the positive parameters B_i is the output transformation rate. From Eq.(1)and Eq.(2) we have

$$q_i(t) = B_i((1 - \theta)K_i(t - 1) + x_i(t)), i = 1, 2.$$

The market price $p(t)$ in period t , a linear inverse demand function, is given by [6, 7, 9]:

$$p(t) = a - bQ(t), i = 1, 2,$$

where $Q(t) = q_1(t) + q_2(t)$ is the total supply and a, b are both positive. It is also assumed that two firms have liner cost functions [6, 9]:

$$C_i(q_i(t)) = c_i q_i(t), i = 1, 2,$$

where c_i is a positive constant which is the marginal cost of firm i .

At last, resulting from the above assumptions, the profit of firm i in period t is given by

$$\begin{aligned} \pi_i(x_1(t), x_2(t)) &= q_i(t)p(t) - C_i(q_i(t)) - x_i(t) = (aB_i - B_i c_i)((1 - \theta)K_i(t - 1) \\ &\quad + x_i(t)) - bB_i^2((1 - \theta)K_i(t - 1) + x_i(t))^2 - bB_i B_j((1 - \theta) \\ &\quad K_i(t - 1) + x_i(t))((1 - \theta)K_j(t - 1) + x_j(t)) - x_i(t), i \neq j, i, j = 1, 2. \end{aligned}$$

Then the marginal profit of firm i is given by

$$\frac{\partial \pi_1}{\partial x_1} \Big|_t = aB_1 - 1 - B_1 c_1 - 2bB_1^2((1 - \theta)K_1(t - 1) + x_1(t)) - bB_1 B_2((1 - \theta)K_2(t - 1) + x_2(t)), \tag{3}$$

$$\frac{\partial \pi_2}{\partial x_2} \Big|_t = aB_2 - 1 - B_2 c_2 - 2bB_2^2((1 - \theta)K_2(t - 1) + x_2(t)) - bB_1 B_2((1 - \theta)K_1(t - 1) + x_1(t)). \tag{4}$$

In order to obtain more competitive market shares than opponents, every player carries out same investment strategies. In this work, we consider two boundedly rational firms with different decision-making. one player may be with bounded rationality and modify his investment strategy during period $t + 1$ according to the marginal profit $\frac{\partial \pi_i(x_1, x_2)}{\partial x_i}$ in period t [6, 9] or by making use of the marginal profit in the previous periods $t, t - 1, \dots, t - T$.

Suppose that firm 1, put time delay on the marginal profit $\frac{\partial \pi_1}{\partial x_1}$, and computes its investment at time $t + 1$ by weighting previous marginal profit information

$$\frac{\partial \pi_1}{\partial x_1} \Big|_t, \frac{\partial \pi_1}{\partial x_1} \Big|_{t-1}, \dots, \frac{\partial \pi_1}{\partial x_1} \Big|_{t-T}.$$

That is, firm i makes an expected marginal profit by averaging the previous information with different weight coefficients:

$$\phi_1^D = \sum_{L=0}^T W_L \frac{\partial \pi_1}{\partial x_1} \Big|_{t-L}.$$

where W_L is the weight coefficient given to period $t - L, W_L \geq 0, \sum_{L=0}^T W_L = 1$. As studied in [6], we also consider the case of one step delay ($T = 1$), then the above equation can be written as

$$\phi_1^D = (1 - \omega) \frac{\partial \pi_1}{\partial x_1} \Big|_t + \omega \frac{\partial \pi_1}{\partial x_1} \Big|_{t-1}, \quad (5)$$

where $\omega (0 \leq \omega \leq 1)$ is the weight to the delayed period $t - 1$ while $1 - \omega$ is assigned to the non-delayed period t . Then the dynamic investment equation of firm 1 has the form:

$$x_1(t + 1) = x_1(t) + \alpha_1 x_1(t) \phi_1^D, \quad (6)$$

where α_1 is a positive constant representing the speed of adjustment. Hence the dynamic equation of firm 1 is obtained from taking (5) into (6), which has the form

$$x_1(t + 1) = x_1(t) + \alpha_1 x_1(t) \left((1 - \omega) \frac{\partial \pi_1}{\partial x_1} \Big|_t + \omega \frac{\partial \pi_1}{\partial x_1} \Big|_{t-1} \right). \quad (7)$$

On the other hand, we assume that firm 2 adjusts its investment at time $t + 1$ only according to its marginal profit $\frac{\partial \pi_2}{\partial x_2} \Big|_t$ as discussed in [6]. Therefore, the investment dynamics for firm 2 is described by

$$x_2(t + 1) = x_2(t) + \alpha_2 x_2(t) \frac{\partial \pi_2}{\partial x_2} \Big|_t. \quad (8)$$

From Eqs.(1),(3),(4),(7) and (8), the dynamical system with one step delay is finally given by

$$\begin{cases} x_1(t + 1) = x_1(t) + \alpha_1 x_1(t) \left((1 - \omega) (aB_1 - 1 - B_1 c_1 - 2bB_1^2 ((1 - \theta)K_1(t - 1) + x_1(t)) - bB_1 B_2 ((1 - \theta)K_2(t - 1) + x_2(t))) + \omega (aB_1 - 1 - B_1 c_1 - 2bB_1^2 ((1 - \theta)K_1(t - 2) + x_1(t - 1)) - bB_1 B_2 ((1 - \theta)K_2(t - 2) + x_2(t - 1))) \right) \\ x_2(t + 1) = x_2(t) + \alpha_2 x_2(t) (aB_2 - 1 - B_2 c_2 - bB_1 B_2 ((1 - \theta)K_1(t - 1) + x_1(t)) - 2bB_1^2 ((1 - \theta)K_2(t - 1) + x_2(t))) \\ K_1(t) = (1 - \theta)K_1(t - 1) + x_1(t) \\ K_2(t) = (1 - \theta)K_2(t - 1) + x_2(t) \end{cases} \quad (9)$$

So we denote $K_i(t - 1)$ by $I_i(t)$ and hence $K_i(t)$ by $I_i(t + 1), K_i(t - 2)$ by $I_i(t - 1)$ and set $x_3(t) = x_1(t - 1), x_4(t) =$

$x_2(t-1), I_3(t) = I_1(t-1), I_4(t) = I_2(t-1)$, then we can rewrite (9) in the form

$$\begin{cases} x_1(t+1) = x_1(t) + \alpha_1 x_1(t) ((1-\omega)(aB_1 - 1 - B_1 c_1 - 2bB_1^2((1-\theta)K_1(t-1) + x_1(t)) - bB_1 B_2((1-\theta)K_2(t-1) + x_2(t))) + \omega(aB_1 - 1 - B_1 c_1 - 2bB_1^2((1-\theta)K_1(t-2) + x_1(t-1)) - bB_1 B_2((1-\theta)K_2(t-2) + x_2(t-1)))) \\ x_2(t+1) = x_2(t) + \alpha_2 x_2(t) (aB_2 - 1 - B_2 c_2 - bB_1 B_2((1-\theta)K_1(t-1) + x_1(t)) - 2bB_1^2((1-\theta)K_2(t-1) + x_2(t))) \\ I_1(t+1) = (1-\theta)I_1(t) + x_1(t) \\ I_2(t+1) = (1-\theta)I_2(t) + x_2(t) \\ x_3(t+1) = x_1(t) \\ x_4(t+1) = x_2(t) \\ I_3(t+1) = I_1(t) \\ I_4(t+1) = I_2(t) \end{cases} \tag{10}$$

3 The equilibrium points and stability

By setting $x_i(t+1) = x_i(t)$ and $I_i(t+1) = I_i(t)$ in (10), we obtain four equilibrium points:

$$\begin{aligned} E_0 &= (0, 0, 0, 0, 0, 0, 0, 0), \\ E_1 &= \left(\frac{\theta(aB_1 - B_1 c_1 - 1)}{2bB_1^2}, 0, \frac{aB_1 - B_1 c_1 - 1}{2bB_1^2}, 0, \frac{\theta(aB_1 - B_1 c_1 - 1)}{2bB_1^2}, 0, \frac{aB_1 - B_1 c_1 - 1}{2bB_1^2}, 0 \right), \\ E_2 &= \left(0, \frac{\theta(aB_2 - B_2 c_2 - 1)}{2bB_2^2}, 0, \frac{aB_2 - B_2 c_2 - 1}{2bB_2^2}, 0, \frac{\theta(aB_2 - B_2 c_2 - 1)}{2bB_2^2}, 0, \frac{aB_2 - B_2 c_2 - 1}{2bB_2^2} \right), \\ E^* &= (x_1^*, x_2^*, I_1^*, I_2^*, x_3^*, x_4^*, I_3^*, I_4^*), \end{aligned}$$

where

$$\begin{aligned} x_1^* &= x_3^* = \frac{\theta(B_1 - 2B_2 + B_1 B_2(a - 2c_1 + c_2))}{3bB_1^2 B_2}, \\ x_2^* &= x_4^* = \frac{\theta(B_2 - 2B_1 + B_1 B_2(a - 2c_2 + c_1))}{3bB_2^2 B_1}, \\ I_1^* &= I_3^* = \frac{B_1 - 2B_2 + B_1 B_2(a - 2c_1 + c_2)}{3bB_1^2 B_2}, \\ I_2^* &= I_4^* = \frac{B_2 - 2B_1 + B_1 B_2(a - 2c_2 + c_1)}{3bB_2^2 B_1}. \end{aligned}$$

The equilibrium E_0, E_1 and E_2 are the boundary points which have an economic meaning if

$$aB_1 - B_1 c_1 - 1 > 0, \tag{11}$$

$$aB_2 - B_2 c_2 - 1 > 0. \tag{12}$$

And E^* is a unique interior equilibrium. In this work, we only study the nonnegative case. Since $a, b, c_1, c_2, B_1, B_2, \omega$ and θ are all positive parameters, nonnegative equilibrium solution E^* should satisfy the following inequalities:

$$B_1 - 2B_2 + B_1 B_2(a - 2c_1 + c_2) > 0, \tag{13}$$

$$B_2 - 2B_1 + B_1 B_2(a - 2c_2 + c_1) > 0. \tag{14}$$

The following study is based on all the assumptions in the above inequalities (11)-(14).

3.1 Stability of boundary equilibrium

At an equilibrium point $(x_1^*, x_2^*, I_1^*, I_2^*, x_3^*, x_4^*, I_3^*, I_4^*)$, we work out the Jacobian matrix:

$$\begin{pmatrix} \xi_1 & \psi_1(1-\omega) & 2\psi_2(1-\theta)(1-\omega) & \psi_1(1-\theta)(1-\omega) & 2\psi_2\omega & \psi_1\omega & 2\psi_2(1-\theta) & \psi_1\omega(1-\theta) \\ \psi_3 & \xi_2 & \psi_3(1-\theta) & 2\psi_4(1-\theta) & 0 & 0 & 0 & 0 \\ 1 & 0 & 1-\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1-\theta & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}, \quad (15)$$

where

$$\begin{aligned} \xi_1 &= 1 - \alpha_1 + \alpha_1 B_1(a - c_1 - bB_2x_2) + 2bB_1^2 I_1 \alpha_1 (\theta - 1 - \omega - \theta\omega) + bB_1 B_2 I_2 \alpha_1 (\theta - 1 + \omega - \theta\omega) \\ &\quad + 4bB_1^2 x_1 \alpha_1 (\omega - 1) + 2bB_1^2 \alpha_1 \omega (I_3 \theta - I_3 - x_3) + bB_1 B_2 \alpha_1 \omega (I_4 \theta - I_4), \\ \xi_2 &= bB_1 B_2 \alpha_2 (I_1 \theta - x_1 - I_1) + 2bB_2^2 \alpha_2 (I_2 \theta - 2x_2 - I_2) + 1 - \alpha_2 + B_2 \alpha_2 (a - c_2), \\ \psi_1 &= -bB_1 B_2 x_1 \alpha_1, \psi_2 = -bB_1^2 x_1 \alpha_1, \psi_3 = -bB_1 B_2 x_2 \alpha_2, \psi_4 = -bB_2^2 x_2 \alpha_2. \end{aligned}$$

We know that the local stability of the fixed points is based on the eigenvalues of the Jacobian matrix. That is to say, an equilibrium point $(x_1^*, x_2^*, I_1^*, I_2^*, x_3^*, x_4^*, I_3^*, I_4^*)$ will be locally asymptotically stable if each eigenvalue λ of $J(x_1^*, x_2^*, I_1^*, I_2^*, x_3^*, x_4^*, I_3^*, I_4^*)$ is to satisfy the inequality $|\lambda| < 1$. Otherwise if there exists an eigenvalue λ such that $|\lambda| > 1$, it will be unstable.

Proposition 1 *The boundary equilibriums E_0, E_1 and E_2 are all unstable.*

Proof. At the equilibrium E_0 , the Jacobian matrix (15) takes the form

$$\begin{pmatrix} \gamma_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \gamma_2 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1-\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1-\theta & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$\begin{aligned} \gamma_1 &= 1 + \alpha_1(aB_1 - B_1c_1 - 1), \\ \gamma_2 &= 1 + \alpha_2(aB_2 - B_2c_2 - 1). \end{aligned}$$

The Jacobian matrix $J(E_0)$ has eight eigenvalues $\lambda_{1,2,3,4} = 0, \lambda_{5,6} = 1 - \theta, \lambda_7 = 1 + \alpha_1(aB_1 - B_1c_1 - 1)$ and $\lambda_8 = 1 + \alpha_2(aB_2 - B_2c_2 - 1)$. From the inequality (11)-(14) and the condition that α_1, α_2 are both positive parameters, we see that the eigenvalue λ_7, λ_8 are greater than 1. Hence the E_0 is an unstable fixed point.

The Jacobian matrix at E_1 takes a form as

$$\begin{pmatrix} 1 + \zeta_1 & \frac{B_2\zeta_1}{2B_1} & (1-\theta)\zeta_1 & \frac{B_2(1-\theta)\zeta_1}{2B_1} & \zeta_2 & \frac{B_2\zeta_2}{2B_1} & (1-\theta)\zeta_2 & \frac{B_2\zeta_2(1-\theta)}{2B_1} \\ 0 & 1 + \alpha_2\zeta_3 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1-\theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1-\theta & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

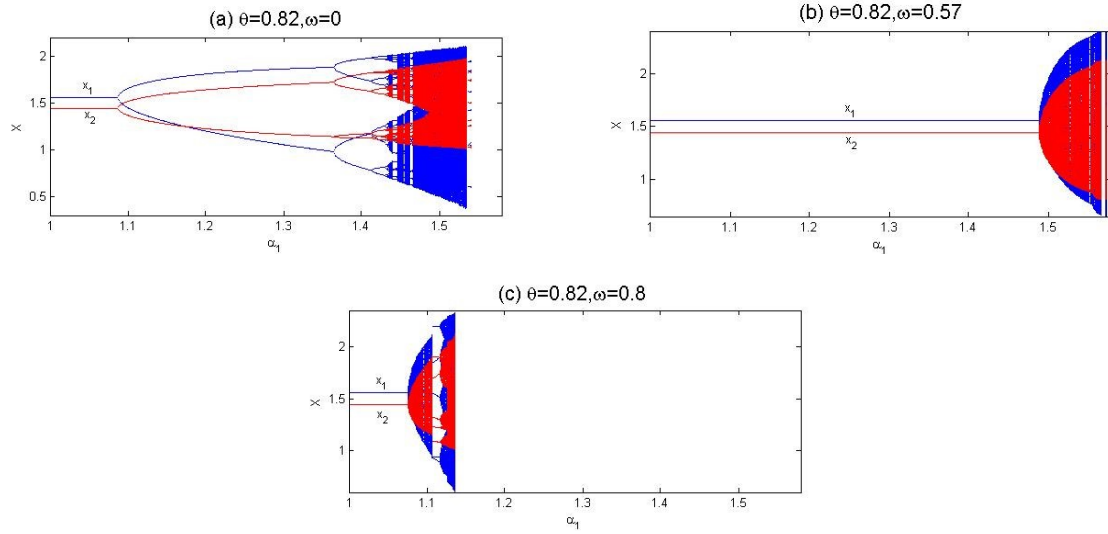


Figure 1: Bifurcation diagrams with respect to the adjustment speed α_1 for a same θ and different ω

where

$$\begin{aligned} \zeta_1 &= \alpha_1 \theta (1 + B_1 c_1 - B_1 a), \\ \zeta_2 &= \alpha_1 \theta \omega (1 - B_1 a + B_1 c_1), \\ \zeta_3 &= 1 - \alpha_2 + \frac{B_2 \alpha_2}{2B_1} + \frac{B_2 \alpha_2 (a + c_1 - 2c_2)}{2}. \end{aligned}$$

In the eight eigenvalues of the matrix $J(E_1)$ there is an eigenvalue $\lambda = 1 + \frac{\alpha_2(B_2 - 2B_1 + B_1 B_2(a - 2c_2 + c_1))}{2B_1}$. Obviously, the inequality (14) and the fact that B_1 and α_2 are both positive tells that $|\lambda| > 1$. So the equilibrium point E_1 is unstable. E_2 is as similar as E_1 , hence E_2 is also unstable. ■

3.2 Local stability of the interior equilibrium

In order to analyze the local stability of the interior equilibrium E^* , we write the matrix $J(E^*)$ as

$$J(E^*) = \begin{pmatrix} 1 + \frac{2D_1}{B_2^2(\theta-1)} & \frac{D_1}{B_1 B_2(\theta-1)} & \frac{-2D_1}{B_2^2} & \frac{-D_1}{B_1 B_2} & 2D_2 B_1 & D_2 B_2 & \frac{2D_1 \omega}{B_2^2(\omega-1)} & \frac{D_1 \omega}{B_1 B_2(\omega-1)} \\ \frac{-D_3}{B_1 B_2(\theta-1)} & 1 - \frac{2D_3}{B_2^2(\theta-1)} & \frac{D_3}{B_1 B_2} & \frac{2D_2}{B_1^2} & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 - \theta & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 - \theta & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix},$$

where

$$D_1 = bB_2^2 B_1^2 \alpha_1 x_1^* (\omega - 1) (\theta - 1), D_2 = -x_1^* \alpha_1 B_1 b \omega, D_3 = x_2^* \alpha_2 b B_1^2 B_2^2 (\theta - 1).$$

The characteristic polynomial $p(\lambda)$ of $J(E^*)$ has the form:

$$p(\lambda) = \lambda^8 + p_1 \lambda^7 + p_2 \lambda^6 + p_3 \lambda^5 + p_4 \lambda^4 + p_5 \lambda^3 + p_6 \lambda^2 + p_7 \lambda + p_8.$$

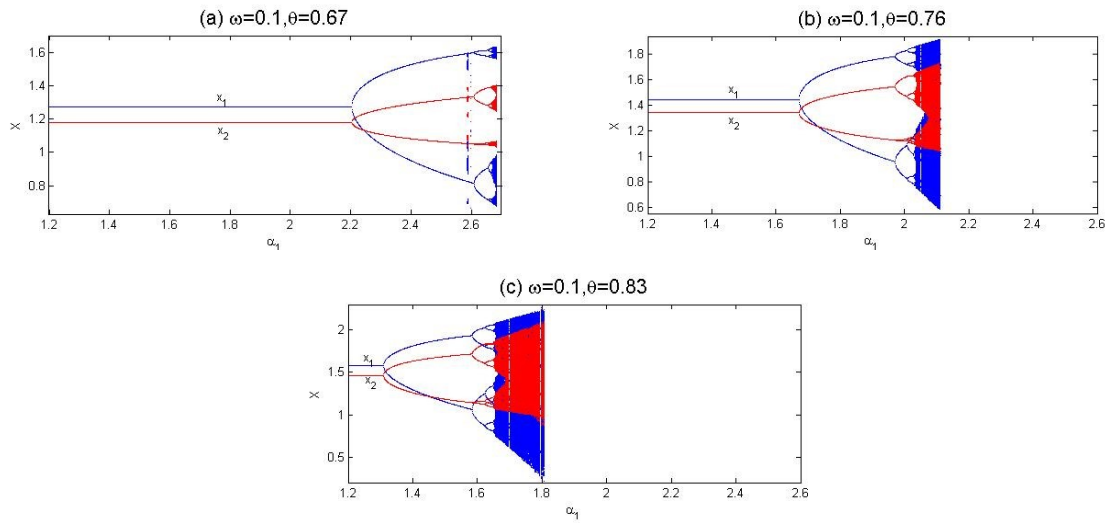


Figure 2: Bifurcation diagrams with respect to the adjustment speed α_1 for a same ω and different θ

By calculation we get

$$\begin{aligned}
 p_1 &= -4 + 2x_1^* \alpha_1 b B_1^2 (1 - \omega) + 2(x_2^* \alpha_2 b B_2^2 + \theta), \\
 p_2 &= 6 - 6\theta + \theta^2 + x_1^* B_1^2 \alpha_1 b (\omega - 1) (4 - 2\theta - 3\alpha_2 x_2^* b B_2^2) + 2x_1^* B_1^2 \alpha_1 b \omega - 2\alpha_2 x_2^* b B_2^2 (2 - \theta), \\
 p_3 &= 3b^2 B_1^2 B_2^2 x_1 x_2 \alpha_1 \alpha_2 \omega - 2(2 - 3\theta + \theta^2) + 2b(-B_2^2 x_2 \alpha_2 (-1 + \theta) + B_1^2 x_1 \alpha_1 (1 - 3\omega - \theta + 2\theta\omega)), \\
 p_4 &= (-1 + \theta)(-1 - 2x_1 B_1^2 \alpha_1 b \omega + \theta), \\
 p_5 &= 0, p_6 = 0, p_7 = 0, p_8 = 0.
 \end{aligned}$$

By using Schur-Cohn Criterion(see e.g.[14]), we can list the local stability stability of E^* as follows

- (i) $p(1) = 1 + p_1 + p_2 + p_3 + p_4 > 0$;
- (ii) $(-1)^8 p(-1) = 1 - p_1 + p_2 - p_3 + p_4 > 0$;
- (iii) the determinants of the 1×1 matrices B_1^\pm and the 7×7 matrices B_7^\pm are all positive, where

$$B_1^\pm = (1 \pm p_8),$$

$$B_7^\pm = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ p_1 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & & \vdots & \vdots \\ p_5 & p_4 & \cdots & 1 & 0 \\ p_6 & p_5 & \cdots & p_1 & 1 \end{pmatrix} \pm \begin{pmatrix} 0 & 0 & \cdots & 0 & p_8 \\ 0 & 0 & \cdots & p_8 & p_7 \\ \vdots & \vdots & & \vdots & \vdots \\ 0 & p_8 & \cdots & p_4 & p_3 \\ p_8 & p_7 & \cdots & p_3 & p_2 \end{pmatrix}.$$

Simply speaking, if all the roots of the polynomial $p(\lambda)$ lie inside the unit disk, then it must satisfy the above equivalent conditions. In this case that the interior equilibrium E^* is locally asymptotically stable.

In our model, it is clear that $Det(B_1^\pm) = 1$ is positive since $p_8 = 0$. And by calculating we have

$$p(1) = 1 + p_1 + p_2 + p_3 + p_4 = 3x_1^* x_2^* \alpha_1 \alpha_2 b^2 B_1^2 B_2^2,$$

which implies that the first condition ((i)) is satisfied. So we have that, the equilibrium E^* of system (10) will be asymptotically stable, if the conditions $1 - p_1 + p_2 - p_3 + p_4 > 0$, $Det(B_7^+) > 0$ and $Det(B_7^-) > 0$ are all satisfied. These can be reduced to the following two inequalities:

$$\begin{aligned}
 p_1 + p_3 &< 1 + p_2 + p_4, \\
 p_2 - p_2 p_4 + p_4^3 - p_1 p_3 + p_1^2 p_4 &< |1 - p_4^2 + p_2 p_4 - p_2 p_4^2 + p_1 p_3 p_4 - p_3^2|.
 \end{aligned} \tag{16}$$

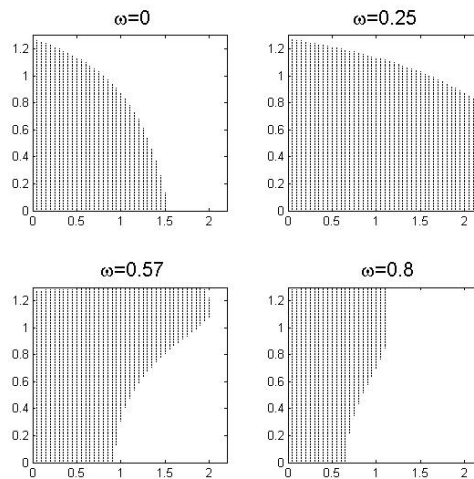


Figure 3: Stability region in the (α_1, α_2) -plane for different levels of ω

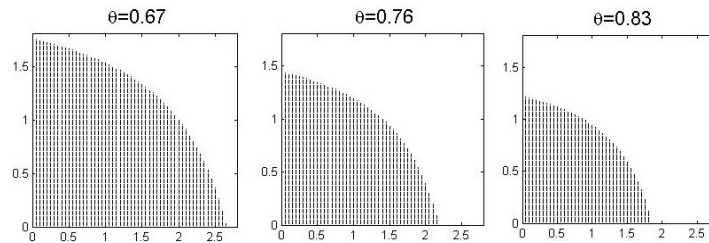


Figure 4: Stability region in the (α_1, α_2) -plane for different levels of θ

4 Numerical simulation

In this section, we perform numerical simulations to show the complex dynamical behaviors of system (10) when losing stability and investigate how stability changes in case of different combinations of the delay weight ω and the depreciation rate θ . In all the numerical simulations, we choose the following parameters values: $a = 6, b = 1, B_1 = 0.7, B_2 = 0.8, c_1 = 0.5$ and $c_2 = 0.6$.

Figs.1 and 2 show the bifurcation diagram with respect to the adjustment speed α_1 , keeping the value of the other one α_2 at 0.8. The influence of the delay weight (by player1 for the marginal profit) on the dynamical behaviors of the delayed Cournot duopoly game are shown in Fig.1. In Fig.1, the depreciation rate θ is fixed as 0.82 and three cases for ω are considered: $\omega = 0, 0.57, 0.8$. On the other hand, taking the delay weight $\omega = 0.1$ and plotting the bifurcation diagrams for different values of $\theta = 0.67, 0.76, 0.83$, we obtain Fig.2 in which it shows the influence of the depreciation rate on the delayed model. From both Fig.1 and Fig.2, we can see that the system starts with equilibrium state, then appears bifurcation and finally ends with chaotic state as the adjustment speed α_1 increases. That is, the system will be stable for small values of the adjustment speed.

Fig.1(a) shows the bifurcation diagrams for the case $\omega = 0$: when $0 < \alpha_1 < 1.088$, the system is stable at Nash equilibrium point. With α_1 increasing, the equilibrium becomes unstable, period doubling bifurcations occur and finally the system falls into chaos as $\alpha_1 = 1.437$ approximately. Fig.1(b) ($\omega = 0.57$) shows that the system keeps stable for α_1 taking the value $\alpha_1 = 1.489$. Fig.1(c) (for $\omega = 0.8$) shows the bifurcation appears at around $\alpha_1 = 1.077$. Comparing the bifurcation Figs.1(a),(b) and (c), it is observed that when bifurcation appears it is later under the proper delay effect than those respected for the no delay case and the excessive delayed case. It means that a proper delay can reduce the bifurcation appearance and promote the system stability. It can be also shown in Fig.3 that the delay weight has great influence of increasing the stability domain. By the stability conditions (11)-(14) which is given at the end of Section 3 we can draw the stability region in the plane of adjustment (α_1, α_2) . Fig.3 shows the stability region for different values

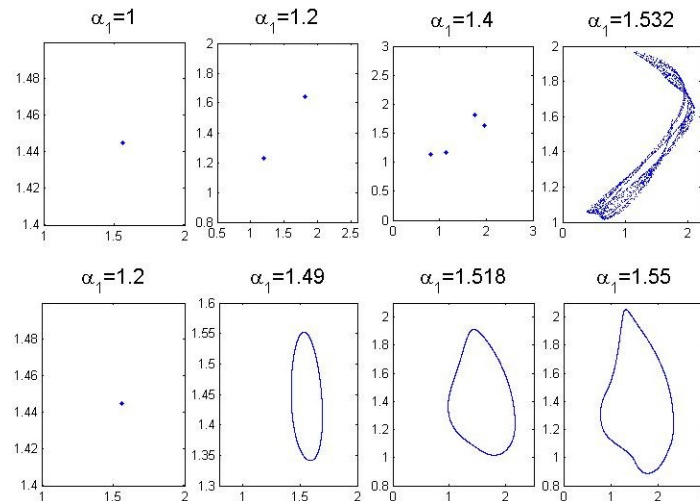


Figure 5: Phase portrait for Fig.1 (a),Fig.1(b) with various ω , θ and α_1

of ω : $\omega = 0.025, 0.57, 0.8$. Comparing fig.3(a),(b),(c) and (d), one can deduce that the stability region under a proper delay ($\omega = 0.25$) is larger than those that in other cases. Hence, we conclude that an intermediate delay weight can expand the domain of stability. And Fig.2(a) is done for ($\theta = 0.67$): the behavior of the system is changed from stable fixed point to bifurcate into period-2 points when α_1 takes its value around 2.206. Fig.2(b) (*for* $\theta = 0.76$) shows that the Nash equilibrium point is locally stable for α_1 taking its value up to about 1.674. Fig.2(c) shows the bifurcation diagrams for the case $\theta = 0.83$. The Nash equilibrium point converges as $\alpha_1 < 1.311$. By observing these diagrams, we can conclude that stability for a case of smaller depreciation rate remains larger region than that for a bigger case. That is, similar to Ding et al.[6] that, a smaller depreciation rate has a stronger stabilization effect on the system dynamical evolution. Through numerically solving the inequalities (11)-(14), the stability regions which agree to Fig.2 is plotted in Fig.4. It shows the (α_1, α_2) -stability regions for different values of θ ($\theta = 0.67, 0.76, 0.83$). It is obvious from the numerical simulation that a small depreciation rate can increase stability domain.

As described in Figs.1 and 2, it is demonstrated that the system displays complex dynamics via different processes of bifurcation when the system stability is lost. We can see that in Fig.1(a) and Fig.2(a)(b)(c), the system loses its stability through a period doubling bifurcation. Beyond that, the stability loss is evidently due to Neimark-Sacker is observed in fig.1(b)(c). Furthermore, a two-dimensional phase portraits is plotted in fig.5 in order to depict detail the system trajectories in relation to the variables x_1 and x_2 . Five phase portraits presented in fig.5 on the first line corresponds to fig.1(a) and shows that in the stability loss process there are periods 2, 4, \dots , which is a doubling bifurcation process and finally drives the system to chaos. Another ones associated with fig.1(b), plotted on the second line, show that typical Neimark-Sacker bifurcation processes, where closed invariant curves take place when the Nash equilibrium is unstable and chaos occur with α_1 increasing fast. It is obvious from above analysis that the system displays complicated dynamics after the Neimark-Sacker bifurcation which is different from what happens after the period doubling bifurcation.

5 Conclusion

In this paper, we reconsider the Cournot investment game in [6] for the case of playing by heterogeneous players instead of homogeneous players. We established the delayed Cournot duopoly game with bounded rationality, which is played by two kinds of heterogeneous players: one with bounded rationality without delay adjusts its investment decision according to the local estimate of the marginal profit; the other arranges its investment depend on averaging his own marginal profits with different weights.

The local stability of the boundary equilibrium points and Nash equilibrium are analyzed in this system. Three boundary equilibrium points are proved to be unstable and the local stability of the Nash equilibrium is given by using Schur-Cohn criterion. Through numerical simulations we have seen that the stability and complicated behaviors of the system are sensitive to the systems parameters, mainly considering the depreciation rate, the delay weight and the adjustment speed. The results show that a smaller depreciation rate has a greater impact on the system stability; an intermediate

delay weight can enlarge the stability region of the system and delays the occurrence of bifurcation; and the fast adjustment speed will lead to the instability of the system through period doubling bifurcations or Neimark-Sacker bifurcation, and then the system go to chaos.

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