ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.24(2017) No.1, pp.9-23

Existence of the Mild Solution for Neutral Fractional Integro-differential Equations with Nonlocal Conditions

Alka Chadha[1], Dwijendra N Pandey[2] *
[1] Department of Mathematics, Indian Institute of Technology Guwahati, India,
[2]Department of Mathematics, Indian Institute of Technology Roorkee, India

(Received Received 21 October 2014, accepted 15 March 2017)

Abstract: This paper studies the existence of the mild solution for the neutral fractional integro-differential equation with nonlocal initial conditions in a Banach space. We obtain the sufficient condition for the existence results via fixed point theorem and approximate technique without assuming noncompactness or Lipschitz continuity of the nonlocal functions. We give an example for explaining the applicability of the abstract results developed.

Keywords: Fractional calculus; Caputo derivative; Resolvent operator; Neutral fractional differential equation; Nonlocal conditions
2010 Mathematics Subject Classification: 26A33, 34K37, 34K40, 34K45, 35R11, 45J05, 45K05.

1 Introduction

During the few years, the fractional calculus, which permits for us to study differentiation and integration of any order, has gotten a lot of consideration from many researchers and scientists due to its incredible applications in various fields such as material sciences, mechanics, seepage flow in porous media, in fluid dynamic traffic models, population dynamics, economics, chemical technology, medicine and many others. The fractional differential equations are found to be more suitable than classical differential equations for describing the some real world problems and phenomena which arise in engineering and science, such as physics, biology, viscoelasticity, electrochemistry, electromagnetic, control. Indeed, the memory and genetic properties of different materials and process can be described by the differential equation involving fractional derivative. For more details on fractional calculus and applications, we refer to monographs [1]-[4].

The existence of the solution of the differential equations with nonlocal conditions has been investigated widely by many authors as, the nonlocal conditions are more realistic than the classical initial conditions such as in dealing with many physical problems. The differential equation with nonlocal conditions has been firstly considered by Byszewski [5]. In [7], authors have studied the existence of the mild solution to fractional integro-differential equations of Sobolev type with nonlocal conditions by applying a fixed point theorem for condensing operators. In [8], authors have established the existence and uniqueness of the mild solutions for fractional differential equations with nonlocal conditions. The existence of solutions to semi-linear neutral fractional differential equations have been proved by the authors [10]. The existence of the η-mild solution for nonlocal Cauchy fractional stochastic differential equation has been obtained by authors in [16] by using Schauder fixed point and approximate techniques with compact semigroup. Shaochun ji [17] has considered the approximate controllability of the nonlocal fractional differential equation by virtue of compact semigroup via approximating method. The recent results of the nonlocal fractional differential equation can be found in [11]-[15] and [18]-[19].

By the motivation of above considered work, we study the following neutral integro-differential equation with nonlocal

* **Corresponding author**. *E-mail address*: alkachadda23@gmail.com

IJNS.2017.08.15/965

conditions in a Banach space $(X, \|\cdot\|_X)$,

$$^cD_t^\alpha[y(t) - F(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))ds)] = Ay(t) + \int_0^t f(t-s)y(s)ds$$
$$+G(t, y(h_3(t))),\ t \in J = [0,T],\ \ 0 < T < \infty, \tag{1.1}$$
$$y(0) = u_0 + h(y) \in X,\ \ y'(0) = 0, \tag{1.2}$$

where $^cD_t^\alpha$ means the Caputo fractional derivative of order α, $1 < \alpha < 2$, $h_1 : [0,T] \to (0,T]$, $h_j \in C([0,T],[0,T])$ for $j = 2,3$, A and $f(t),\ t \geq 0$ are closed, densely linear operators defined on a common domain in a Banach space X. The nonlocal function $h : C([0,T],X) \to X$ is continuous function which satisfies the some local growth conditions and functions F, G and a_1 are appropriate continuous functions to be specified later.

The article is organized as follows. Section 2 recalls some fundamental definitions of fractional calculus, theorems, lemmas and some facts about the resolvent operator. Section 3 gives the existence result of the mild solutions by using compact, analytic resolvent operator and Sadovskii' fixed point theorem via approximating technique. Section 4 provides an example to show the effectiveness of the obtained abstract result.

2 Preliminaries and Assumptions

In this section, we discuss some elementary definitions of fractional calculus, theorems, lemma and some facts about α-resolvent operators.

In this work, X is assumed to be a Banach space with supremum norm. The notation $C([0,T],X)$ stands for the Banach space of continuous functions from $[0,T]$ into X with following norm $\|y\|_{[0,T]} = \sup_{s\in[0,T]}\|y(s)\|$ for each $y \in C([0,T],X)$ and $L^1([0,T],X)$ denotes the Banach space of functions $y : [0,T] \to X$ which are Bochner integrable normed by $\|y\|_{L^1} = \int_0^T \|y(s)\|ds$, for each $y \in L^1([0,T],X)$. A measurable function $y : [0,T] \to X$ is Bochner integrable if and only if $\|y\|$ is Lebesgue integrable. For more details of Bochner integral, we refer to Yosida [28]. Moreover, $L(X)$ is a Banach space of all linear bounded operator from X into itself with norm

$$\|F\|_{L(X)} = \sup\{\|F(y)\| :\ \|y\| \leq 1\},$$

and $B_r(y,X)$ denotes a closed ball with center at y and radius $r > 0$ in X i.e. $B_r(y,X) = \{z \in X : \|z-y\|_X \leq r\}$.

Throughout the work, we assume that $A, f(t),\ t \geq 0$ are closed densely linear operators defined on a common domain $D(A)$ on Banach space X. Let $D(A)$ denotes the domain of A with the graph norm. For $0 < \eta \leq 1$, the notation $(-A)^\eta$ represents the fractional power of the operator $-A$ with dense domain $D((-A)^\eta)$ in X. It is easy to verify that $D((-A)^\eta)$ is a Banach space with the norm

$$\| y\|_\eta = \| (-A)^\eta y\|,\ \text{ for }\ y \in D((-A)^\eta). \tag{2.1}$$

Hence, we signify the space $D((-A)^\eta)$ by X_η endowed with the η-norm $(\|\cdot\|_\eta)$ and this norm is equivalent to the graph norm of $(-A)^\eta$, that is, $\| y\|_\eta = (\| y\|^2 + \| A^\eta y\|^2)^{1/2}$. Also, we have that $X_\kappa \hookrightarrow X_\eta$ for $0 < \eta < \kappa$ and therefore, the embedding is continuous. Then, we define $X_{-\eta} = (X_\eta)^*$, for each $\eta > 0$. The space $X_{-\eta}$ stands for the dual space of X_η, is a Banach space with the norm $\| z\|_{-\eta} = \| A^{-\eta}z\|$, $z \in X_{-\eta}$. For more details on the fractional powers of closed linear operator, we refer to Pazy [27].

To set the structure for our primary existence results, we give the following definitions.

Definition 1 *[4]The Riemann-Liouville fractional integral operator J of order $\alpha > 0$ is defined by*

$$J_t^\alpha F(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}F(s)ds, \tag{2.2}$$

where $F \in L^1((0,T),X)$.

Definition 2 *[4] The Riemann-Liouville fractional derivative is given as*

$$^{RL}D_t^\alpha F(t) = D_t^m J_t^{m-\alpha}F(t),\ m-1 < \alpha < m,\ m \in \mathbb{N}, \tag{2.3}$$

where $D_t^m = \frac{d^m}{dt^m},\ F \in L^1((0,T),X),\ J_t^{m-\alpha}F \in W^{m,1}((0,T),X)$. *Here, the notation* $W^{m,1}((0,T),X)$ *stands for the Sobolev space defined by*

$$\begin{aligned} W^{m,1}((0,T);X) &= \{y \in X : \exists z \in L^1((0,T);X) : y(t) = \sum_{k=0}^{m-1} d_k \frac{t^k}{k!} \\ &\quad + \frac{t^{m-1}}{(m-1)!} * z(t),\ \ t \in (0,T)\}. \end{aligned} \tag{2.4}$$

Note that $z(t) = y^m(t)$, $d_k = y^k(0)$.

Definition 3 *[4]The Caputo fractional derivative is given as*

$${}^cD_t^\alpha F(t) = \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-s)^{m-\alpha-1} F^m(s)ds,\ \ m-1 < \alpha < m, \tag{2.5}$$

where $F \in C^{m-1}((0,T),X) \cap L^1((0,T),X)$.

Now, we present α-resolvent operator appeared in [20].

Definition 4 *[20] A one-parameter family of bounded linear operators* $S_\alpha(t),\ t \geq 0$ *on* X *is said to be an* α*-resolvent operator for*

$${}^cD_t^\alpha y(t) = Ay(t) + \int_0^t f(t-s)y(s)ds, \tag{2.6}$$

$$y(0) = x,\ \ y'(0) = 0, \tag{2.7}$$

if
(i) *The function* $S_\alpha(\cdot) : [0,\infty) \to L(X)$ *is strongly continuous;*
(ii) $S_\alpha(0)x = x$, $\forall\ x \in X$ *and* $\alpha \in (0,1)$*;*
(iii) *For* $x \in D(A)$, $S_\alpha(\cdot)x \in C([0,\infty);[D(A)]) \cap C^1((0,\infty);X)$ *and*

$$\begin{aligned} {}^cD_t^\alpha S_\alpha(t)x &= AS_\alpha(t)x + \int_0^t f(t-s)S_\alpha(s)xds, \\ &= S_\alpha(t)Ax + \int_0^t S_\alpha(t-s)f(s)xds,\ \ t \geq 0. \end{aligned} \tag{2.8}$$

In this manner, we consider the following assumptions:

$(P1)$ The operator $A : D(A) \subset X \to X$ is a closed, densely linear operator. Let $\alpha \in (1,2)$. For some $\phi_0 \in (0,\pi/2]$ for every $\phi < \phi_0$, there exists a constant $C_0 = C_0(\phi) > 0$ such that $\lambda \in \rho(A)$ for each

$$\lambda \in \sum_{0,\alpha\eta} = \{\lambda \in \mathbb{C},\ \lambda \neq 0,\ |\arg(\lambda)| < \alpha\eta\}, \tag{2.9}$$

here $\eta = \phi + \pi/2$ and $\|R(\lambda,A)\| \leq C_0/|\lambda|$ for all $\lambda \in \sum_{0,\alpha\eta}$.

$(P2)$ $f(t) : D(f(t)) \subseteq X \to X$ for $t \geq 0$ is a closed linear operator with $D(A) \subseteq D(f(t))$ and $f(\cdot)x$ is strongly measurable on $(0,\infty)$ for every $x \in D(A)$. For $t > 0$ and $x \in D(A)$, there exists $d(\cdot) \in L^1_{loc}(\mathbb{R}^+)$ such that $\widehat{d}(\lambda)$(Laplace of $d(\cdot)$) exists for $\text{Re}(\lambda) > 0$ and $\|f(t)x\| \leq d(t)\|x\|_1$. Furthermore, the operator valued function $\widehat{f} : \sum_{0,\pi/2} \to L([D(A)],X)$ has an analytical extension which is denoted by $\widehat{f}$ to $\sum_{0,\eta}$ such that $\|\widehat{f}(\lambda)y\| \leq \|\widehat{f}(\lambda)\| \cdot \|y\|_1$ for each $x \in D(A)$ and $\|\widehat{f}(\lambda)\| = O(\frac{1}{|\lambda|})$, $\lambda \to \infty$.

$(P3)$ There exists positive constant C_1 and a subspace $\widehat{\mathcal{D}} \subset D(A)$ dense in $[D(A)]$ such that $A(\widehat{\mathcal{D}}) \subset D(A)$, $\widehat{f}(\lambda)(\widehat{\mathcal{D}}) \subset D(A)$ and $\|A\widehat{f}(\lambda)y\| \leq C_1\|y\|$ for all $y \in \widehat{\mathcal{D}}$ and $\lambda \in \sum_{0,\eta}$.

In the continuation, we have that for $\theta \in (\pi/2, \eta)$ and $r > 0$,

$$\sum_{r,\theta} = \{\lambda \in \mathbb{C} : \ \lambda \neq 0, \ r < |\lambda|, \ \ |\arg(\lambda)| < \theta\},$$

and for $\Gamma_{r,\theta}$

$$\begin{aligned} \Gamma^1_{r,\theta} &= \{te^{i\theta} : \ t \geq r\}, \\ \Gamma^2_{r,\theta} &= \{re^{i\zeta} : \ -\theta \leq \zeta \leq \theta\}, \\ \Gamma^3_{r,\theta} &= \{te^{-i\theta} : \ t \geq r\}, \end{aligned} \tag{2.10}$$

where $\Gamma^i_{r,\theta}, \ i = 1, 2, 3$ are the paths such that $\Gamma_{r,\theta} = \cup_{i=1}^3 \Gamma^i_{r,\theta}$ is oriented counterclockwise. Moreover, we introduce following sets $\rho_\alpha(G_\alpha)$ as

$$\rho_\alpha(G_\alpha) = \{\lambda \in \mathbb{C} : \ G_\alpha(\lambda) = \lambda^{\alpha-1}(\lambda^\alpha I - A - A\widehat{f}(\lambda))^{-1} \in \mathcal{L}(X)\}. \tag{2.11}$$

Now, consider operator family $S_\alpha(t), \ t \geq 0$ defined by

$$S_\alpha(t) = \begin{cases} \frac{1}{2\pi i} \int_{\Gamma_{r,\theta}} e^{\lambda t} G_\alpha(\lambda) d\lambda, & t > 0, \\ I, & t = 0. \end{cases} \tag{2.12}$$

Lemma 1 *[21]Let $\alpha \in (1, 2)$, then we define $R_\alpha(t), \ t \geq 0$ by*

$$R_\alpha(t)x = \int_0^t g_{\alpha-1}(t-s) S_\alpha(s) ds, \ \ t \geq 0, \tag{2.13}$$

where $g_{\alpha-1}(t) = \frac{t^{\alpha-2}}{\Gamma(\alpha-1)}, \ t > 0, \ \alpha - 1 \geq 0$.

Lemma 2 *[21]There exists a positive number r_1 such that $\sum_{r_1,\eta} \subseteq \rho_\alpha(G_\alpha)$ and the map $G_\alpha : \sum_{r_1,\eta} \to L(X)$ is analytic. Furthermore, we have*

$$G_\alpha(\lambda) = \lambda^{\alpha-1} R(\lambda^\alpha, A)[I - \widehat{f}(\lambda) R(\lambda^\alpha, A)]^{-1} \tag{2.14}$$

and there are constants $\widetilde{M}_i$ for $i = 1, 2$ such that

$$\begin{aligned} \|G_\alpha(\lambda)\| &\leq \frac{\widetilde{M}_1}{|\lambda|}, \\ \|AG_\alpha(\lambda)y\| &\leq \frac{\widetilde{M}_2}{|\lambda|}\|y\|_1, \ y \in D(A), \\ \|AG_\alpha(\lambda)\| &\leq \frac{\widetilde{M}_2}{|\lambda|^{1-\alpha}} \end{aligned} \tag{2.15}$$

for each $\lambda \in \sum_{r_1,\eta}$.

Lemma 3 *[20]. Let us assume that conditions $(P1)$-$(P3)$ are satisfied. Then there exists a unique α-resolvent operator for the system (2.6)-(2.7).*

Lemma 4 *[20]. The function $S_\alpha : [0, \infty) \to L(X)$ is strongly continuous and $S_\alpha : (0, \infty) \to L(X)$ is uniformly continuous.*

Lemma 5 *[20]. If the function $S_\alpha(\cdot)$ is exponentially bounded in $L([D(A)])$, then $R_\alpha(\cdot)$ is exponentially bounded in $L([D(A)])$.*

Lemma 6 *[20]. The operator families $S_\alpha(t)$ and $R_\alpha(t)$ are compact for all $t \geq 0$ if $R(\lambda_0^\alpha, A)$ is compact for some $\lambda_0^\alpha \in \rho(A)$*

Theorem 7 *[20],[22]. Let $(P1)$-$(P3)$ hold and $\alpha \in (1,2)$, $\eta \in (0,1)$ such that $\alpha\eta \in (0,1)$. Then there exists number $C_\eta > 0$ such that*

$$\begin{aligned} \|(-A)^\eta S_\alpha(t)\| &\leq C_\eta e^{rt} t^{-\alpha\eta}, \\ \|(-A)^\eta R_\alpha(t)\| &\leq C_\eta e^{rt} t^{\alpha(1-\eta)-1}, \forall\, t > 0. \end{aligned} \tag{2.16}$$

Moreover, if $y \in [D((-A)^\eta)]$, then

$$\begin{aligned} (-A)^\eta S_\alpha(t) y &= S_\alpha(t)(-A)^\eta y, \\ (-A)^\eta R_\alpha(t) y &= R_\alpha(t)(-A)^\eta y, \ \textit{for all} \ \ t > 0. \end{aligned} \tag{2.17}$$

Before expressing and demonstrating the main result, we present the definition of the mild solution to the nonlocal problem (1.1)-(1.2).

Definition 5 *A continuous function $y(\cdot)$ is said to be a mild solution for the nonlocal problem (1.1)-(1.2)*
(i) $y(0) = y_0 + h(y)$, $y'(0) = 0$ and $y_0 \in D(A)$
(ii) $y(\cdot)$ satisfies the following integral equation

$$\begin{aligned} y(t) &= S_\alpha(t)[y_0 + h(y) - F(0, y(h_1(0)), 0)] + F(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))ds) \\ &+ \int_0^t R_\alpha(t-s) A F(s, y(h_1(s)), \int_0^s a_1(s,\xi,y(h_2(\xi)))d\xi) ds \\ &+ \int_0^t \int_0^s f(s-\tau) R_\alpha(t-s) F(\tau, u(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,u(h_2(\xi)))d\xi) d\tau ds \\ &+ \int_0^t R_\alpha(t-s) G(t, y(h_3(s))) ds, \quad t \in [0,T]. \end{aligned} \tag{2.18}$$

3 Existence Result

In this section, we obtain the existence results of the mild for the nonlocal problem (1.1)-(1.2). Now, we make the following assumptions:

$(B1)$ (1) The operator families $S_\alpha(t),\ t > 0$ and $R_\alpha(t),\ t > 0$ are compact and there exist constants $M_1 > 0$ and $M_2 > 0$ such that $\|S_\alpha(t)\|_{L(X)} \leq M_1$ and $\|R_\alpha(t)\|_{L(X)} \leq M_2$ for each $t > 0$ and

$$\|(-A)^\eta R_\alpha(t)\| \leq M_2 t^{\alpha(1-\eta)-1}, \quad t \in (0,T]. \tag{3.1}$$

(2) For each $z \in [D((-A)^{1-\eta})]$, $f(\cdot)z \in C([0,T];X)$ and there is a function $\mathcal{W}(\cdot) \in L^1([0,T];\mathbb{R}^+)$ and a constant M_3 such that

$$\|f(s)R_\alpha(t)\|_{L([D((-A)^\eta)];X)} \leq M_3 \mathcal{W}(s) t^{\alpha\eta-1}, \ \ 0 \leq s < t \leq T. \tag{3.2}$$

$(B2)$ For $0 < \beta < 1$, the function $F : [0,T] \times X_\eta \times X_\eta \to X_{\beta+\eta}$ is a Lipschitz continuous function and there is a constant $L_F > 0$ such that

$$\begin{aligned} \|(-A)^{\beta+\eta} F(t_1, y_1, z_1) - (-A)^{\beta+\eta} F(t_2, y_2, z_2)\| &\leq L_F[|t_1 - t_2| + \|y_1 - y_2\|_\eta \\ &+ \|z_1 - z_2\|_\eta], \end{aligned} \tag{3.3}$$

for each $(t,y,z), (t_1,y_1,z_1),\ (t_2,y_2,z_2) \in [0,T] \times X_\eta \times X_\eta$ with $L_2 = \sup_{t \in J} \|(-A)^{\beta+\eta} F(t,0,0)\|$. Here L_2 is a positive constant.

$(B3)$ (1) The map $a_1 : D \times X_\eta \to X_\eta$, where $D = \{(t,s) \in J \times J : t \geq s\}$ is a continuous mapping and there exists a positive constant L_{a_1} such that

$$\| \int_0^t [a_1(t,s,z_1) - a_1(t,s,z_2)] ds \|_\eta \leq L_{a_1} \|z_1 - z_2\|_\eta, \tag{3.4}$$

for all $z_1, z_2 \in X_\eta$ and $t \in J$ with $L_1 = T \sup_{(t,s)\in D} \|a_1(t,s,0)\|_\eta$.
(2) There exists a positive constant L_a such that

$$\|\int_0^{t_1} a_1(t_1,s,y)ds - \int_0^{t_2} a_1(t_2,s,y)ds\|_\alpha \le L_a|t_1 - t_2|, \ \ t_1, t_2 \in [0,T]. \tag{3.5}$$

$(B4)$ The nonlinear function $G : [0,T] \times X_\eta \to X$ satisfies the following conditions.
(1) The function $G : [0,T] \times X_\eta \to X$ fulfills the Carath$\acute{e}$odory type conditions, that is, $G(\cdot, y)$ is strongly measurable for each $y \in X_\eta$ and $G(t,\cdot)$ is continuous for almost everywhere $t \in [0,T]$.
(2) For $r > 0$, there exist constant $\alpha_1 \in [0,\alpha)$ and function $0 < m_r \in L^{1/\alpha_1}([0,T],\mathbb{R}^+)$ such that

$$\sup_{\|y\|_\eta \le r} \|G(t,y)\| \le m_r(t), \ \forall \ y \in X_\eta \ \text{ and } \ \lim_{r\to+\infty} \inf \frac{\|m_r\|_{L^{1/\alpha_1}}}{r} = \varpi_1 < +\infty, \tag{3.6}$$

where $\varpi_1 > 0$ is a constant.

$(B5)$ The map $h : C([0,T],X_\eta) \to X_\eta$ is continuous and there exists a increasing continuous map $m_h : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\sup_{\|y\|_\eta \le r} \|h(y)\|_\eta \le m_h(r), \ \ \forall\, y \in X_\eta \ \text{ and } \ \lim_{r\to+\infty} \inf \frac{m_h(r)}{r} = \varpi_2 < +\infty, \tag{3.7}$$

where $\varpi_2 > 0$ is a constant. Moreover, there exists a $\theta = \theta(r) > 0$ such that $h(x) = h(y), \forall \ x, y \in C([0,T],X_\eta)$ such that $x(s) = y(s)$, for each $s \in [\theta,T]$.

Theorem 8 *Let us assume that* $(B1)$-$(B5)$ *are satisfied and*

$$\begin{aligned} &M_1^2\varpi_2 + M_1\|A^{-\beta}\|L_F + \|(-A)^{-\beta}\|L_F(1+L_{a_1}) + M_2L_F(1+L_{a_1})\frac{T^{\alpha\beta}}{\alpha\beta} \\ &+M_3L_F(1+L_{a_1})\|\mathcal{W}\|_{L^1}\frac{T^{\alpha\beta}}{\alpha\beta} + M_2T^{\alpha(1-\eta)-\alpha_1}\left(\frac{1-\alpha_1}{\alpha(1-\eta)-\alpha_1}\right)^{1-\alpha_1}\varpi_1 < 1. \end{aligned} \tag{3.8}$$

Then, the nonlocal problem (1.1)-(1.2) has at least one η*-mild solution.*

Proof. We consider the following problem

$$\begin{aligned} {}^cD_t^\alpha[y(t) - F(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))ds)] &= Ay(t) + \int_0^t f(t-s)y(s)ds \\ &+G(t, y(h_3(t))), \ t \in J = [0,T], \ \ 0 < T < \infty, \end{aligned} \tag{3.9}$$

$$y(0) = u_0 + S_\alpha(\theta_n)h(y) \in X, \ \ y'(0) = 0. \tag{3.10}$$

In order to show the existence of solution for problem (1.1)-(1.2), it is sufficient to show that there exists at least one mild solution for the nonlocal problem (3.9)-(3.10). Now, we consider the operator $Q_n : C([0,T],X_\eta) \to C([0,T],X_\eta)$ defined by

$$\begin{aligned} (Q_ny)(t) \;=\; & S_\alpha(t)[y_0 + S_\alpha(\theta_n)h(y) - F(0, y(h_1(0)), 0)] + F(t, y(h_1(t)), \int_0^t a_1(t,s,y(h_2(s)))ds) \\ &+ \int_0^t R_\alpha(t-s)AF(s, y(h_1(s)), \int_0^s a_1(s,\xi,y(h_2(\xi)))d\xi)ds \\ &+ \int_0^t\int_0^s f(s-\tau)R_\alpha(t-s)F(\tau, y(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,y(h_2(\xi)))d\xi)d\tau ds \\ &+ \int_0^t R_\alpha(t-s)G(t, y(h_3(s)))ds. \end{aligned} \tag{3.11}$$

Clearly, the map Q_n is well-defined by using the facts that $F, \ G$ are continuous functions. Next, we show that $Q_n(B_r(0,C([0,T],X_\eta)))$ $\subset B_r(0,C([0,T],X_\eta))$, where $B_r(0,C([0,T],X_\eta)) = \{y \in C([0,T],X_\eta) : \ \|y\|_\eta \le r\}$. We assume that it is not true.

Then, for each $r > 0$, we have that there exist $y^r \in B_r(0, C([0,T], X_\eta)$ and $t^r \in [0,T]$ such that $\|(Q_n y^r)(t^r)\|_\eta > r$. Thus, from the assumptions $(B2) - (B5)$, we get that

$$\begin{aligned}
r &< \|(Q_n y^r)(t^r)\|_\eta \leq \|S_\alpha(t^r)[y_0 + S_\alpha(\theta_n)h(y^r) - F(0, y^r(h_1(0)), 0)]\|_\eta \\
&\quad + \|F(t^r, y^r(h_1(t^r)), \int_0^{t^r} a_1(t^r, s, y^r(h_2(s)))ds)\|_\eta \\
&\quad + \int_0^{t^r} \|(-A)^{1-\beta} R_\alpha(t^r - s)(-A)^{\beta+\eta} F(s, y^r(h_1(s)), \int_0^s a_1(s, \xi, y^r(h_2(\xi)))d\xi)ds\| \\
&\quad + \int_0^{t^r}\int_0^s \|f(s-\tau)(-A)^{1-\beta} R_\alpha(t^r - s)(-A)^{\beta+\eta} F(\tau, y^r(h_1(\tau)), \int_0^\tau a_1(\tau, \xi, y^r(h_2(\xi)))d\xi)\|d\tau ds \\
&\quad + \int_0^{t^r} \|A^\eta R_\alpha(t^r - s) G(t, y^r(h_3(s)))\|ds, \\
&\leq M_1\|y_0\| + M_1^2 m_h(r) + M_1\|(-A)^{-\beta}\|[L_F(T + r) + L_2] \\
&\quad + \|(-A)^{-\beta}\|[L_F(r + L_{a_1} r + L_1) + L_2] + M_2 \int_0^{t_r} (t^r - s)^{\alpha\beta - 1}[L_F(r + L_{a_1} r + L_1) + L_2]ds \\
&\quad + M_3 \int_0^{t^r}\int_0^s \mathcal{W}(s-\tau)(t^r - s)^{\alpha\beta - 1}[L_F(r + L_{a_1} r + L_1) + L_2]d\tau ds \\
&\quad + M_2 \int_0^{t_r} (t^r - s)^{\alpha(1-\eta)-1} m_r(s)ds, \\
&\leq M_1\|y_0\| + M_1^2 m_h(r) + M_1\|(-A)^{-\beta}\|[L_F(T + r) + L_2] + \|(-A)^{-\beta}\|[L_F(r + L_{a_1} r + L_1) + L_2] \\
&\quad + M_2[L_F(r + L_{a_1} r + L_1) + L_2]\frac{T^{\alpha\beta}}{\alpha\beta} + M_3[L_F(r + L_{a_1} r + L_1) + L_2]\|\mathcal{W}\|_{L^1}\frac{T^{\alpha\beta}}{\alpha\beta} \\
&\quad + M_2 (\int_0^{t_r} (t^r - s)^{\frac{\alpha(1-\eta)-1}{1-\alpha_1}} ds)^{1-\alpha_1} \|m_r\|_{L^{1/\alpha_1}[0,T]}, \\
&\leq M_1\|y_0\| + M_1^2 m_h(r) + M_1\|(-A)^{-\beta}\|[L_F(T + r) + L_2] \\
&\quad + \|(-A)^{-\beta}\|[L_F(r + L_{a_1} r + L_1) + L_2] + M_2[L_F(r + L_{a_1} r + L_1) + L_2]\frac{T^{\alpha\beta}}{\alpha\beta} \\
&\quad + M_3[L_F(r + L_{a_1} r + L_1) + L_2]\|\mathcal{W}\|_{L^1}\frac{T^{\alpha\beta}}{\alpha\beta} + M_2 T^{\alpha(1-\eta)-\alpha_1}(\frac{1-\alpha_1}{\alpha(1-\eta)-\alpha_1})^{1-\alpha_1}\|m_r\|_{L^{1/\alpha_1}[0,T]}.
\end{aligned} \tag{3.12}$$

We divide both side of above inequality by r and take the lower limit as $r \to +\infty$, we get

$$\begin{aligned}
&M_1^2 \varpi_2 + M_1\|A^{-\beta}\|L_F + \|(-A)^{-\beta}\|L_F(1 + L_{a_1}) + M_2 L_F(1 + L_{a_1})\frac{T^{\alpha\beta}}{\alpha\beta} \\
&+ M_3 L_F(1 + L_{a_1})\|\mathcal{W}\|_{L^1}\frac{T^{\alpha\beta}}{\alpha\beta} + M_2 T^{\alpha(1-\eta)-\alpha_1}(\frac{1-\alpha_1}{\alpha(1-\eta)-\alpha_1})^{1-\alpha_1}\varpi_1 > 1.
\end{aligned} \tag{3.13}$$

Thus, we get the contradiction of the inequality (3.8). Hence, there exists $r > 0$ such that $Q_n(B_r(0, C([0,T], X_\eta))) \subset B_r(0, C([0,T], X_\eta))$. Next, we prove that the operator Q_n has a fixed point on $B_r(0, C([0,T], X_\eta))$. Now, we decompose Q_n as $Q_n = Q_n^1 + Q_n^2$ such that

$$\begin{aligned}
(Q_n^1 y)(t) &= S_\alpha(t)[-F(0, y(h_1(0)), 0)] + F(t, y(h_1(t)), \int_0^t a_1(t, s, y(h_2(s)))ds) \\
&\quad + \int_0^t R_\alpha(t-s) A F(s, y(h_1(s)), \int_0^s a_1(s, \xi, y(h_2(\xi)))d\xi)ds \\
&\quad + \int_0^t\int_0^s f(s-\tau) R_\alpha(t-s) F(\tau, y(h_1(\tau)), \int_0^\tau a_1(\tau, \xi, y(h_2(\xi)))d\xi)d\tau ds,
\end{aligned} \tag{3.14}$$

$$(Q_n^2 y)(t) = S_\alpha(t)[y_0 + S_\alpha(\theta_n)h(y)] + \int_0^t R_\alpha(t-s) G(s, y(h_3(s)))ds, \tag{3.15}$$

for each $t \in [0,T]$. To this end, we show that Q_n^1 is a contraction operator and operator Q_n^2 is compact.
For $y_1,\ y_2 \in B_r(0, C([0,T], X_\eta))$ and $t \in [0,T]$, we have
$\|(-A)^\eta(Q_n^1 y_1)(t) - (-A)^\eta(Q_n^1 y_2)(t)\|$

$$\begin{aligned}
\leq\ & M_1\|(-A)^{-\beta}\|\cdot\|(-A)^{\beta+\eta}[F(0,y_1(h_1(0)),0) - F(0,y_2(h_1(0)),0)\| \\
& +\|(-A)^{-\beta}\|\cdot\|(-A)^{\beta+\eta}F(t,y_1(h_1(t)),\int_0^t a_1(t,s,y_1(h_2(s)))ds) \\
& -(-A)^{\beta+\eta}F(t,y_2(h_1(t)),\int_0^t a_1(t,s,y_2(h_2(s)))ds)\| \\
& +M_2\int_0^t (t-s)^{\alpha\beta-1}\cdot\|(-A)^{\beta+\eta}F(s,y_1(h_1(s)),\int_0^s a_1(s,\xi,y_1(h_2(\xi)))d\xi) \\
& -(-A)^{\beta+\eta}F(s,y_2(h_1(s)),\int_0^s a_1(s,\xi,y_2(h_2(\xi)))d\xi)\|ds + M_3\int_0^t\int_0^s \mathcal{W}(s-\tau)(t-s)^{\alpha\beta-1} \\
& \times\|F(\tau,y_1(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_1(h_2(\xi)))d\xi) - F(\tau,y_2(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_2(h_2(\xi)))d\xi)\|d\tau ds, \\
\leq\ & M_1L_F\|(-A)^{-\beta}\|\|y_1-y_2\|_\eta + \|(-A)^{-\beta}\|L_F(1+L_{a_1})\|y_1-y_2\|_\eta + M_2L_F(1+L_{a_1})\frac{T^{\alpha\beta}}{\alpha\beta} \\
& \times\|y_1-y_2\|_\eta + M_3L_F(1+L_{a_1})\|\mathcal{W}\|_{L^1}\frac{T^{\alpha\beta}}{\alpha\beta}\|y_1-y_2\|_\eta, \\
\leq\ & L_F\{M_1\|(-A)^{-\beta}\| + (1+L_{a_1})[\|(-A)^{-\beta}\| + (M_2 + M_3\|\mathcal{W}\|_{L^1})\frac{T^{\alpha\beta}}{\alpha\beta}]\}\times\|y_1-y_2\|_\eta, \\
\leq\ & \Theta\times\|y_1-y_2\|_\eta, \qquad (3.16)
\end{aligned}$$

where

$$\Theta = L_F\{M_1\|(-A)^{-\beta}\| + (1+L_{a_1})[\|(-A)^{-\beta}\| + (M_2 + M_3\|\mathcal{W}\|_{L^1})\frac{T^{\alpha\beta}}{\alpha\beta}]\} < 1. \qquad (3.17)$$

Taking supremum of t over $[0,T]$ and getting

$$\|(Q_n^1 y_1) - (Q_n^1 y_2)\|_\eta \leq \Theta\|y_1-y_2\|_\eta, \qquad (3.18)$$

which implies that Q_n^1 is a contraction mapping on $B_r(0, C([0,T], X_\eta))$.
Next, we prove that the mapping Q_n^2 is compact. We show the compactness of the mapping Q_n^2 in several steps.
Step 1. The mapping Q_n^2 is continuous on $B_r(0, C([0,T], X_\eta))$.
Let us consider $\{y_m\}_{m=1}^\infty$ be a sequence in $B_r(0, C([0,T], X_\eta))$ such that $\lim_{m\to\infty} y_m = y \in B_r(0, C([0,T], X_\eta))$. Then, we get
$\|(-A)^\eta(Q_n^2 y_m)(t) - (-A)^\eta(Q_n^2 y)(t)\|$

$$\leq\ M_1\|S_\alpha(\theta_n)[h(y_m) - h(y)]\|_\eta + \|\int_0^t R_\alpha(t-s)[G(s,y_m(h_3(s))) - G(s,y(h_3(s)))]ds\|_\eta.$$

Since G and h are continuous mappings, therefore, we have

$$\begin{aligned}
\|G(t,y_m(h_3(t))) - G(t,y(h_3(t)))\| &\to 0, \quad \text{as } m\to\infty, \quad \text{for each } t\in[0,T], \\
\|h(y_m) - h(y)\| &\to 0, \quad \text{as } m\to\infty. \qquad (3.19)
\end{aligned}$$

Thus, from the dominated convergence theorem, we obtain

$$\|(-A)^\eta(Q_n^2 y_m)(t) - (-A)^\eta(Q_n^2 y)(t)\| \to 0, \quad \text{as } m\to\infty. \qquad (3.20)$$

This gives that the mapping Q_n^2 is continuous on $B_r(0, C([0,T], X_\eta))$.
Step 2. The set $\{Q_n^2 y : y \in B_r(0, C([0,T], X_\eta))\}$ is an equicontinuous family of mappings on $[0,T]$.
For $t = 0$, we have $Q_n^2 y(0) = y_0 + S_\alpha(\theta_n)h(y)$. Thus, it is not difficult to verify that $\{S_\alpha(t)[y_0 + S_\alpha(\theta_n)h(y)] : y \in B_r(0, C([0,T], X_\eta))\}$ is equicontinuous in X at $t = 0$ due to the compactness of the operator $S_\alpha(\theta_n),\ t \geq 0, \forall\ n \in \mathbb{N}$

and hypothesis $(B5)$. Furthermore, for $t_2,\ t_1 \in (0, T]$ and $y \in B_r(0, C([0, T], X_\eta))$ with $t_1 < t_2$, we obtain that

$$\begin{aligned}
&\|(-A)^\eta(Q_n^2 y)(t_2) - (-A)^\eta(Q_n^2 y)(t_1)\| \\
\leq\ & \|[S_\alpha(t_2) - S_\alpha(t_1)](y_0 + S_\alpha(\theta_n)h(y))\|_\eta + \|\int_0^{t_1-\epsilon}(-A)^\eta[R_\alpha(t_2 - s) - R_\alpha(t_1 - s)]G(s, y(h_3(s)))ds\| \\
& +\|\int_{t_1-\epsilon}^{t_1}(-A)^\eta[R_\alpha(t_2 - s) - R_\alpha(t_1 - s)]G(s, y(h_3(s)))ds\| \\
& +\|\int_{t_1}^{t_2}(-A)^\eta R_\alpha(t_2 - s)G(s, y(h_3(s)))ds\|, \\
\leq\ & \|[S_\alpha(t_2) - S_\alpha(t_1)](-A)^\eta(y_0 + S_\alpha(\theta_n)h(y))\| + (t_1 - \epsilon)^{1-\alpha(1-\eta)} \\
& \times \int_0^{t_1-\epsilon}\|(-A)^\eta[R_\alpha(t_2 - s) - R_\alpha(t_1 - s)]\|(t_1 - \epsilon - s)^{\alpha(1-\eta)-1}m_r(s)ds \\
& +M_2\int_{t_1-\epsilon}^{t_1}[(t_2 - s)^{\alpha(1-\eta)-1} + (t_1 - s)^{\alpha(1-\eta)-1}]m_r(s)ds + M_2\int_{t_1}^{t_2}(t_2 - s)^{\alpha(1-\eta)-1}m_r(s)ds. \qquad (3.21)
\end{aligned}$$

Thus, we get that the right-hand side of the above inequality tends to zero as $t_2 \to t_1$ with ϵ is sufficiently small by using the facts that $R_\alpha(t),\ t \geq 0$ is compact which implies the continuity of the operator $R_\alpha(t)$ in the uniform operator topology. Hence, Q_n^2 maps $B_r(0, C([0, T], X_\eta))$ into an equicontinuous family of mappings.

Step 3. Q_n^2 maps $B_r(0, C([0, T], X_\eta))$ into a relatively compact in X_η.

To this end, let us consider ε be a real number which satisfies $0 < \varepsilon < t$. For $y \in B_r(0, C([0, T], X_\eta))$, we consider the operator

$$Q_n^{2,\varepsilon}y(t) = S_\alpha(t)[y_0 + S_\alpha(\theta_n)h(y)] + \int_0^{t-\varepsilon}R_\alpha(t - s)G(s, y(h_3(s)))ds. \qquad (3.22)$$

Since $S_\alpha(t),\ R_\alpha(t),\ t \geq 0$ and $S_\alpha(\theta_n), \theta_n > 0$ for all $n \in \mathbb{N}$ are compact, thus, we deduce that the set $U_\varepsilon(t) = \{(Q_n^{2,\varepsilon}y)(t) :\ y \in B_k\}$ is relatively compact in X_η with for every ε, $0 < \varepsilon < t$. Moreover, for $y \in B_r(0, C([0, T], X_\eta))$ and assumption $(B1)$, $(B4)$, we get

$$\begin{aligned}
\|(-A)^\eta(Q_n^2 y)(t) - (-A)^\eta(Q_n^{2,\varepsilon}y)(t)\| &\leq \|\int_{t-\varepsilon}^{t}(-A)^\eta R_\alpha(t - s)G(s, y(h_3(s)))ds\| \\
&\leq M_2\int_{t-\varepsilon}^{t}(t - s)^{\alpha(1-\eta)-1}m_r(s)ds, \\
&\leq M_2\varepsilon^{\alpha(1-\eta)-\alpha_1}\left(\frac{1 - \alpha_1}{\alpha(1 - \eta) - \alpha_1}\right)^{1-\alpha_1}\|m_r\|_{L^{1/\alpha_1}[0,T]}.
\end{aligned}$$

Clearly, the right hand side of the above inequality tends to zero as $\varepsilon \to 0$. Thus, there exist relatively compact sets arbitrarily close to the set $U(t) = \{(Q_n^2 y)(t) :\ y \in B_r(0, C([0, T], X_\eta))\}$ for $t \in (0, T]$. Therefore, the set $\{(Q_n^2 y)(t) : y \in B_r(0, C([0, T], X_\eta))\}$ is relatively compact in X_η for each $t \in (0, T]$. For $t = 0$, since $S_\alpha(\theta_n),\ n > 0$ is compact. Then, we deduce that the set $\{(Q_n^2 y)(0) = y_0 + S_\alpha(\theta_n)h(y) : y \in B_r(0, C([0, T], X_\eta))\}$ is relatively compact in X_η by using assumption $(B5)$. Hence, we conclude that the set $\{(Q_n^2 y)(t) : B_r(0, C([0, T], X_\eta))\}$ is relatively compact in X_η for $t \in [0, T]$.

As the consequence of the above steps and Arzela-Ascoli theorem, we deduce that operator Q_n^2 is a compact. Therefore, we get that $Q_n = Q_n^1 + Q_n^2$ is a condensing mapping on $B_r(0, C([0, T], X_\eta))$ and by the Sadovskii's fixed, there exists a fixed point $y_n \in B_r(0, C([0, T], X_\eta))$ for the mapping Q_n. Thus, the nonlocal problem (3.9)-(3.10) has a mild solution in $B_r(0, C([0, T], X_\eta))$. Then, for any $t \in [0, T]$, we have

$$\begin{aligned}
y_n(t) &= S_\alpha(t)[y_0 + S_\alpha(\theta_n)h(y_n) - F(0, y_n(h_1(0)), 0)] + F(t, y_n(h_1(t)), \int_0^t a_1(t, s, y_n(h_2(s)))ds) \\
&\quad + \int_0^t R_\alpha(t - s)AF(s, y_n(h_1(s)), \int_0^s a_1(s, \xi, y_n(h_2(\xi)))d\xi)ds \\
&\quad + \int_0^t\int_0^s f(s - \tau)R_\alpha(t - s)F(\tau, y_n(h_1(\tau)), \int_0^\tau a_1(\tau, \xi, y_n(h_2(\xi)))d\xi)d\tau ds \\
&\quad + \int_0^t R_\alpha(t - s)G(t, y_n(h_3(s)))ds. \qquad (3.23)
\end{aligned}$$

Next, we show that the set $\{y_n:\ n\in\mathbb{N}\}\subset B_r(0,C([0,T],X_\eta))$ is relatively compact. Now, we present the decomposition of y_n as $y_n=y_n^1+y_n^2$ defined by

$$\begin{aligned}
y_n^1(t) &= -S_\alpha(t)F(0,y_n(h_1(0)),0)+F(t,y_n(h_1(t)),\int_0^t a_1(t,s,y_n(h_2(s)))ds)\\
&\quad+\int_0^t R_\alpha(t-s)AF(s,y_n(h_1(s)),\int_0^s a_1(s,\xi,y_n(h_2(\xi)))d\xi)ds\\
&\quad+\int_0^t\int_0^s f(s-\tau)R_\alpha(t-s)F(\tau,y_n(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_n(h_2(\xi)))d\xi)d\tau ds\\
&\quad+\int_0^t R_\alpha(t-s)G(s,y_n(h_3(s)))ds, \qquad t\in[0,T], \qquad (3.24)\\
y_n^2(t) &= S_\alpha(t)[y_0+S_\alpha(\theta_n)h(y_n)], \qquad t\in[0,T]. \qquad (3.25)
\end{aligned}$$

Next, we show that $\{y_n^1:\ n\in\mathbb{N}\}$ is relatively compact in $B_r(0,C([0,T],X_\eta))$.
Claim 1. $\{y_n^1: n\in\mathbb{N}\}$ is equicontinuous on $[0,T]$.
Let $\varepsilon>0$. For $y_n\in B_r(0,C([0,T],X_\eta))$, there exists a positive constant $\gamma>0$ such that for each $t\in[0,T]$, $\zeta\in(0,\gamma)$ and $t+\zeta\le T$, we have

$$\begin{aligned}
&\|(-A)^\eta y_n^1(t+\zeta)-(-A)^\eta y_n^1(t)\|\\
&\le \|(-A)^{-\beta}\|\cdot\|[S_\alpha(t+\zeta)-S_\alpha(t)](-A)^{\beta+\eta}F(0,y_n(h_1(0)),0)\|+\|(-A)^{-\beta}\|\\
&\quad\|(-A)^{\beta+\eta}[F(t+\zeta,y_n(h_1(t+\zeta)),\int_0^{t+\zeta}a_1(t+\zeta,s,y_n(h_2(s)))ds)\\
&\quad-F(t,y_n(h_1(t)),\int_0^t a_1(t,s,y_n(h_2(s)))ds)]\|\\
&\quad+\|\int_t^{t+\zeta}(-A)^{1-\beta}R_\alpha(t+\zeta-s)\\
&\quad\times(-A)^{\beta+\eta}F(s,y_n(h_1(s)),\int_0^s a_1(s,\xi,y_n(h_2(\xi)))d\xi)ds\|+\|\int_0^t(-A)^{1-\beta}[R_\alpha(t+\zeta-s)-R_\alpha(t-s)]\\
&\quad\times(-A)^{\beta+\eta}F(s,y_n(h_1(s)),\int_0^s a_1(s,\xi,y_n(h_2(\xi)))d\xi)ds\|\\
&\quad+\|\int_t^{t+\zeta}\int_0^s f(s-\tau)R_\alpha(t+\zeta-s)F(\tau,y_n(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_n(h_2(\xi)))d\xi)d\tau ds\|\\
&\quad+\|\int_0^t\int_0^s f(s-\tau)[R_\alpha(t+\zeta-s)-R_\alpha(t-s)](-A)^\eta F(\tau,y_n(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_n(h_2(\xi)))d\xi)d\tau ds\|\\
&\quad+\|\int_t^{t+\zeta}(-A)^\eta R_\alpha(t+\zeta-s)G(s,y_n(h_3(s)))ds\|+\|\int_0^t(-A)^\eta[R_\alpha(t+\zeta-s)-R_\alpha(t-s)]\\
&\quad\times G(s,y_n(h_3(s)))ds\|,\\
&\le \|(-A)^{-\beta}\|\cdot\|[S_\alpha(t+\zeta)-S_\alpha(t)](-A)^{\beta+\eta}F(0,y_n(h_1(0)),0)\|\\
&\quad+\|(-A)^{-\beta}\|L_F[\zeta+\|y_n(h_1(t+\zeta))-y_n(h_1(t))\|_\eta+L_a\zeta]+M_2N_F\int_t^{t+\zeta}(t+\zeta-s)^{\alpha\beta-1}ds\\
&\quad+N_F\int_0^t\|R_\alpha(t+\zeta-s)-R_\alpha(t-s)](-A)^{1-\beta}\|ds+M_3N_F\|\mathcal{W}\|_{L^1}\int_t^{t+\zeta}(t+\zeta-s)^{\alpha\beta-1}ds\\
&\quad+N_F\int_0^t\int_0^s\|f(s-\tau)[R_\alpha(t+\zeta-s)-R_\alpha(t-s)](-A)^{-\beta}\|d\tau ds\\
&\quad+M_2\int_t^{t+\zeta}(t+\zeta-s)^{\alpha(1-\eta)-1}m_r(s)ds\\
&\quad+M_2T^{1-\alpha(1-\eta)}\int_0^t\|[R_\alpha(t+\zeta-s)-R_\alpha(t-s)](-A)^\eta\|(t-s)^{\alpha(1-\eta)-1}m_r(s)ds. \qquad (3.26)
\end{aligned}$$

Then, for all $t \in (0, T]$, using the compact operator property, we obtain

$$\|(-A)^{-\beta}\| \cdot \|[S_\alpha(t+\zeta) - S_\alpha(t)](-A)^{\beta+\eta}F(0, y_n(h_1(0)), 0)\| < \frac{\varepsilon}{8}, \tag{3.27}$$

$$\|(-A)^{-\beta}\|L_F[\zeta + \|y_n(h_1(t+\zeta)) - y_n(h_1(t))\|_\eta + L_a\zeta] < \frac{\varepsilon}{8}, \tag{3.28}$$

$$M_2 N_F \int_t^{t+\zeta} (t+\zeta-s)^{\alpha\beta-1} ds < \frac{\varepsilon}{8}, \tag{3.29}$$

$$N_F \int_0^t \|R_\alpha(t+\zeta-s) - R_\alpha(t-s)](-A)^{1-\beta}\| ds < \frac{\varepsilon}{8}, \tag{3.30}$$

$$M_3 N_F \|\mathcal{W}\|_{L^1} \int_t^{t+\zeta} (t+\zeta-s)^{\alpha\beta-1} ds < \frac{\varepsilon}{8}, \tag{3.31}$$

$$N_F \int_0^t \int_0^s \|f(s-\tau)[R_\alpha(t+\zeta-s) - R_\alpha(t-s)](-A)^{-\beta}\| d\tau ds < \frac{\varepsilon}{8}, \tag{3.32}$$

$$M_2 \int_t^{t+\zeta} (t+\zeta-s)^{\alpha(1-\eta)-1} m_r(s) ds < \frac{\varepsilon}{8}, \tag{3.33}$$

$$M_2 T^{1-\alpha(1-\eta)} \int_0^t \|[R_\alpha(t+\zeta-s) - R_\alpha(t-s)](-A)^\eta\|(t-s)^{\alpha(1-\eta)-1} m_r(s) ds < \frac{\varepsilon}{8}. \tag{3.34}$$

From the inequalities (3.26)-(3.34), we get

$$\|(-A)^\eta y_n^1(t+\zeta) - (-A)^\eta y_n^1(t)\| < \varepsilon. \tag{3.35}$$

Thus, we conclude that $\{y_n^1 : n \in \mathbb{N}\}$ is equicontinuous for $t \in [0, T]$. It is clear that $\{y_n^1(0) : n \in \mathbb{N}\}$ is equicontinuous.
Claim 2. $\{y_n^1 : n \in \mathbb{N}\}$ is relatively compact in X_η.
Let $\varepsilon > 0$. For $t \in [0, T]$ and $y_n \in B_k$, there exists $\eta' > 0$ with $\eta' < \beta + \eta$ such that

$$\begin{aligned} y_n^1(t) &= -S_\alpha(t)F(0, y_n(h_1(0)), 0) + (-A)^{-\eta'}(-A)^{\eta'} F(t, y_n(h_1(t)), \int_0^t a_1(t, s, y_n(h_2(s)))ds) \\ &+ \int_0^t R_\alpha(t-s)AF(s, y_n(h_1(s)), \int_0^s a_1(s, \xi, y_n(h_2(\xi)))d\xi)ds \\ &+ \int_0^t \int_0^s f(s-\tau)R_\alpha(t-s)F(\tau, y_n(h_1(\tau)), \int_0^\tau a_1(\tau, \xi, y_n(h_2(\xi)))d\xi)d\tau ds \\ &+ \int_0^t R_\alpha(t-s)G(s, y_n(h_3(s)))ds = \sum_{j=1}^{5} H_j, \qquad t \in [0, T]. \end{aligned} \tag{3.36}$$

(3.37)

From the conditions $(B2)$ and $(B3)$, it follows that $(-A)^{\eta'}F(t,y_n(h_1(t)),\int_0^t a_1(t,s,y_n(h_2(s)))ds)$ is bounded in X. Since $S_\alpha(t),\ t\geq 0$ and $(-A)^{-\eta'}$ are compact, therefore, we get that H_1 and H_2 are compact in X_η. Also, we have

$$\begin{aligned}
\|(-A)^{\eta'}H_3\| &= \|\int_0^t(-A)^{1+\eta'}R_\alpha(t-s)F(s,y_n(h_1(s)),\int_0^s a_1(s,\xi,y_n(h_2(\xi)))d\xi)ds\|,\\
&\leq \int_0^t\|(-A)^{1+\eta'-\beta-\eta}R_\alpha(t-s)(-A)^{\beta+\eta}F(s,y_n(h_1(s)),\int_0^s a_1(s,\xi,y_n(h_2(\xi)))d\xi)\|ds,\\
&\leq M_2N_F\int_0^t(t-s)^{\alpha(\beta+\eta-\eta')-1}ds,\\
&\leq M_2N_F\frac{T^{\alpha(\beta+\eta-\eta')}}{\alpha(\beta+\eta-\eta')},\\
\|(-A)^{\eta'}H_4\| &= \|\int_0^t\int_0^s(-A)^{\eta'}f(s-\tau)R_\alpha(t-s)F(\tau,y_n(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_n(h_2(\xi)))d\xi)d\tau ds\|,\\
&\leq \int_0^t\int_0^s\|(-A)^{\eta'-\eta}f(s-\tau)R_\alpha(t-s)(-A)^{\eta}F(\tau,y_n(h_1(\tau)),\int_0^\tau a_1(\tau,\xi,y_n(h_2(\xi)))d\xi)\|d\tau ds,\\
&\leq M_3N_F\|\mathcal{W}\|_{L^1}\int_0^t(t-s)^{\alpha(\beta+\eta-\eta')-1}ds = M_3N_F\|\mathcal{W}\|_{L^1}\frac{T^{\alpha(\beta+\eta-\eta')}}{\alpha(\beta+\eta-\eta')},
\end{aligned}$$

and

$$\begin{aligned}
\|(-A)^{\eta'}H_5\| &= \|\int_0^t(-A)^{\eta'}R_\alpha(t-s)G(s,y_n(h_3(s)))ds\|,\\
&\leq M_2\int_0^t(t-s)^{\alpha(1-\eta')-1}m_r(s)ds,\\
&\leq M_2T^{\alpha(1-\eta')-\alpha_1}\left(\frac{1-\alpha_1}{\alpha(1-\eta')-\alpha_1}\right)^{1-\alpha_1}\|m_r\|_{L^{1/\alpha_1}[0,T]}.
\end{aligned}$$

Thus, we conclude that $\{(-A)^{\eta'}H_j, y_n\in B_k,\ j=3,4,5\}$ are bounded in X. Hence, $H_j,\ j=3,4,5$ are relatively compact in X_η.
As the consequence of the above steps, we deduce that $\{y_n^1 : n\in\mathbb{N}\}$ is relatively compact in X_η.
Next, we are going to show that $\{y_n^2 : n\in\mathbb{N}\}$ is relatively compact in X.
claim 1. the set $\{y_n^2 :\ n\in\mathbb{N}\}$ is equicontinuous on $(0,T]$.
Let $t\in(0,T]$ and $\varepsilon>0$. For $t_1,\ t_2\in(0,T]$ and $y_n\in B_r$ with $t_2>t_1$, we have

$$\begin{aligned}
\|(-A)^\eta[y_n^2(t_2)-y_n^2(t_1)]\| &= \|[S_\alpha(t_2)-S_\alpha(t_1)](-A)^\eta(y_0+S_\alpha(\theta_n)h(y_n))\|\\
&\leq \|[S_\alpha(t_2)-S_\alpha(t_1)]\|\times\|(-A)^\eta(y_0+S_\alpha(\theta_n)h(y_n))\|. \qquad (3.38)
\end{aligned}$$

By the strongly continuity of the operator $S_\alpha(t)$ and $\varepsilon>0$, we can choose $0<\vartheta<T-t$ such that

$$\|[S_\alpha(t_2)-S_\alpha(t_1)]y\|<\varepsilon,\ \ y\in B_r, \qquad (3.39)$$

with $|t_2-t_1|<\vartheta$. Thus, from (3.38) and (3.39) and condition $(B5)$, we get that the right hand side of the inequality (3.38) tends to zero as $t_2\to t_1$ with ε sufficiently small. Hence, the set $\{y_n^2 :\ n\in\mathbb{N}\}$ is equicontinuous on $(0,T]$.
Claim 2. The set $\{y_n^2(t) :\ n\in\mathbb{N}\}$ is relatively compact in X_η for each $t\in(0,T]$.
By the compactness of the operator $S_\alpha(\theta_n)$ and boundedness of the set $\{S_\alpha(\theta_n)h(y) : y\in B_r(0,C([0,T],X_\eta))\}$, we deduce that $\{y_n^2\}$ is relatively compact in X for all $t\in(0,T]$.
Thus, for proving the precompactness of the set $\{y_n : n\in\mathbb{N}\}$ in $C([0,T],X_\eta)$, we only need to show the precompactness of the set $\{y_n(t) : n\in\mathbb{N}\}$ at $t=0$. Clearly, $\{y_n^1(0) : n\in\mathbb{N}\}$ is precompact in X. So, it remain to show that the set $\{y_n^2(t) : n\in\mathbb{N}\}$ is relatively compact and equicontinuous in X at $t=0$. For $y_n\in B_r(0,C([0,T],X_\eta)),\ n\geq 1$, we set

$$y_n^*(t)=\begin{cases} y_n(t), & t\in[\theta,T],\\ y_n(\theta), & t\in[0,\theta].\end{cases} \qquad (3.40)$$

It is easy to verify that the set $\{y_n^* :\ n \in \mathbb{N}\}$ is relatively compact in $C([0,T];X)$. Thus, without loss of generality, let us assume that

$$y_n^* \to y^* \in C([0,T],X), \text{ as } n \to \infty. \tag{3.41}$$

By the condition $(B5)$, we have

$$h(y_n) = h(y_n^*) \to h(y^*). \tag{3.42}$$

Therefore, from the strongly continuity of $S_\alpha(t)$ and continuity of h, we get
$\|(-A)^\eta[y_n^*(0) - (y_0 + h(y^*))]\|$

$$\begin{aligned} &= \|(-A)^\eta[S_\alpha(\theta_n)h(y_n^*) - h(y^*)]\|, \\ &\leq \|(-A)^\eta[S_\alpha(\theta_n)h(y_n^*) - S_\alpha(\theta_n)h(y^*)]\| + \|(-A)^\eta[S_\alpha(\theta_n)h(y^*) - h(y^*)]\|, \\ &\leq M_1\|(-A)^\eta(h(y_n^*) - h(y^*))\| + \|(S_\alpha(\theta_n) - I)(-A)^\eta h(y^*)\|, \\ &\to 0 \text{ as } n \to \infty. \end{aligned} \tag{3.43}$$

This gives that $\{S_\alpha(\theta_n)[y_0 + h(y_n^*)] : 1 \leq n \in \mathbb{N}\}$ is precompact in X at $t = 0$. Since $S_\alpha(\theta_n),\ \theta_n > 0$ is compact and the set $\{y_n^2(0) : n \in \mathbb{N}\}$ is precompact set in X_η , thus, by Arzela-Ascoli theorem, we conclude that $\{y_n^2(0) : n \in \mathbb{N}\}$ is relatively compact in X. Therefore, we get that the set $\{y_n = y_n^1 + y_n^2 : n \in \mathbb{N}\}$ is relatively compact in $C([0,T] : X)$ by using facts that $\{y_n^1 : n \in \mathbb{N}\}$ and $\{y_n^2 : n \in \mathbb{N}\}$ are relatively compact. Hence, without loss of generality, we may assume that $y_n^* \to y^*$ as $n \to \infty$ in $C([0,T];X)$. Letting $n \to \infty$ in inequality (3.23) and getting

$$\begin{aligned} y^*(t) &= S_\alpha(t)[y_0 + h(y^*) - F(0, y^*(h_1(0)), 0)] + F(t, y^*(h_1(t)), \int_0^t a_1(t,s,y^*(h_2(s)))ds) \\ &\quad + \int_0^t R_\alpha(t-s)AF(s, y^*(h_1(s)), \int_0^s a_1(s,\xi,y^*(h_2(\xi)))d\xi)ds \\ &\quad + \int_0^t \int_0^s f(s-\tau)R_\alpha(t-s)F(\tau, y^*(h_1(\tau)), \int_0^\tau a_1(\tau,\xi,y^*(h_2(\xi)))d\xi)d\tau ds \\ &\quad + \int_0^t R_\alpha(t-s)G(t, y^*(h_3(s)))ds, \quad t \in [0,T]. \end{aligned} \tag{3.44}$$

It implies that $y^* \in C([0,T],X)$ is a η-mild solution of the nonlocal problem (1.1). This finishes the proof of the theorem. ■

4 Example

To generalization of theory, consider the following fractional integro-differential equation with nonlocal initial conditions

$$\begin{aligned} D_t^\alpha[w(t,x) - \int_0^\pi a(t,s,\vartheta)w(\sin t,\vartheta)d\theta] &= \frac{\partial^2}{\partial x^2}w(t,x) + \int_0^t (t-s)^\xi e^{-\zeta(t-s)}\frac{\partial^2}{\partial x^2}w(s,x)ds \\ &\quad + \mathcal{G}(t, \frac{\partial}{\partial x}w(t\sin t, x)),\ 0 \leq t \leq T < \infty, \end{aligned} \tag{4.1}$$

$$w(t,0) = w(t,\pi) = 0, \quad 0 \leq t \leq T, \tag{4.2}$$

$$w(0,x) = u_0(x) + \sum_{i=1}^{q} \sqrt[3]{w(s_j,x)},\ 0 < s_1 < \cdots < s_q < T,\ \ 0 \leq x \leq \pi, \tag{4.3}$$

where D_t^α means fractional derivative in Caputo sense of order α, $\alpha \in (1,2)$, c_j are given real numbers and $0 < s_1 < s_2 < \cdots < s_q < T$. The nonlinear function $a : [0,T] \times [0,T] \times [0,\pi] \to \mathbb{R}$ is continuous mapping.
Take $X = L^2([0,\pi])$ with the norm $\|\cdot\|$. We now define the operator $A : X \to X$ by $Au = u''$. The domain of A is given by

$$D(A) = \{u \in X : u,\ u' \text{ are absolutely continuous } u'' \in H \text{ with } u(0) = u(\pi) = 0\}. \tag{4.4}$$

Then, we have
(1) $Au = \sum_{n=1}^\infty n^2(u,u_n)u_n,\ \ u \in D(A)$, where $u_n(x) = \sqrt{\frac{2}{\pi}\sin(nx)},\ n = 1,\cdots$, is the orthogonal set of eigenvectors of A.

(2) For every $u \in X$,

$$(-A)^{-1/2}u = \sum_{n=1}^{\infty} \frac{1}{n}(u, u_n)u_n.$$

(3) $(-A)^{1/2}u = \sum_{n=1}^{\infty} n(u, u_n)u_n$ on the space

$$D((-A)^{1/2}) = \{u(\cdot) \in X : \sum_{n=1}^{\infty} n(u, u_n)u_n \in X\} \text{ and } \|(-A)^{-1/2}\| = 1.$$

Thus, it is well known that the operator A is the infinitesimal generator of a strongly continuous, compact, analytic semigroup $T(t)$ and A is sectorial of type and $(P1)$ is fulfilled. The operator $f(t) : X \supset D(A) \to X,\ t \geq 0$, $f(t)x = t^{\xi}e^{-\zeta t}x''$ for $x \in D(A)$. Moreover, it is not difficult to show that the hypothesis $(P2)$ and $(P3)$ are fulfilled with $t^{\xi}e^{-\zeta t}$ and $D(A) = C_0^{\infty}([0, \pi])$, here $C_0^{\infty}([0, \pi])$ is the space of infinitely differentiable functions such that vanish at $x = 0$ and $x = \pi$.
Let us consider the functions $F : [0, T] \times X \to X$, $G : [0, T] \times X \to X$

$$F(t, w)(\cdot) = \int_0^{\pi} a(t, \cdot, \vartheta)w(\vartheta)d\vartheta, \tag{4.5}$$

$$G(t, w)(\cdot) = \mathcal{G}(t, w'(\cdot)), \tag{4.6}$$

$$h(w)(\cdot) = \sum_{j=1}^{p} \sqrt[3]{(w(s_j)(\cdot))},\ 0 < s_1 < \cdots < s_q < T,\ \ 0 \leq x \leq \pi. \tag{4.7}$$

Take $h_1(t) = h_3(t) = \sin(t)$. Thus, the system (4.1)-(4.3) can be written in the form of system (1.1)-(1.2). Furthermore, $AF : [0, T] \times X_{1/2} \to X_{1/2}$ (we choose $\beta = 1/2$) $G : [0, T] \times X_{1/2} \to X$. Hence, there exists a mild solution for (4.1)-(4.3) under appropriate functions G, F and h satisfying suitable conditions to verify the assumptions of Theorem 8.

Acknowledgment

The authors would like to thank the referee for valuable comments and suggestions.

References

[1] K. S. Miller, B. Ross. An Introduction to the Fractional Calculus and Fractional Differential Equations, *John Wiley and Sons*, Inc., New York, 1993.

[2] S. G. Samko, A. A. Kilbas, O. I. Marichev. Fractional Integrals and Derivatives: Theory and Applications, *Gordon and Breach Science Publisher*, Yverdon, 1993.

[3] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo. Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[4] I. Podlubny. Fractional Differential Equations, Mathematics in Science and Engineering, vol. 198. *Academic Press*, San Diego 1999.

[5] L. Byszewski. Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, *J. Math. Anal. Appl.*, 162(1991): 497-505.

[6] L. Byszewski, V. Lakshmikantham. Theorem about the existence and uniqueness of a solution of a nonlocal abstract Cauchy problem in a Banach space, *Applied Analysis*, 40(1990): 11-19.

[7] Fang Li, J. Liang, H.-K. Xu. Existence of mild solutions for fractional integrodifferential equations of Sobolev type with nonlocal conditions, *J. Math. Anal. Appl.*, 391(2012): 510-525.

[8] X.-B. Shu, Q. Wang. The existence and uniqueness of mild solutions for fractional differential equations with nonlocal conditions of order $1 < \alpha < 2$, *Comput. Math. Appl.*, 64(2012): 2100-2110.

[9] Fang Li, Gaston M. N' Guérékata. An existence result for neutral delay integrodifferential equations with fractional order and nonlocal conditions, *Abst. Appl. Anal.*, 2011(2011):1-20.

[10] H. M. Ahmed, Fractional neutral evolution equations with nonlocal conditions, *Adv. Differ. Equ.*, 2013 (2013), pp-17.

[11] S. Liang, R. Mei. Existence of mild solutions for fractional impulsive neutral evolution equations with nonlocal conditions, *Adv. Diff. Equ.*, 2014 (2014):1-16.

[12] A. Chadha and D. N. Pandey. Existence and approximation of solution to neutral fractional differential equation with nonlocal conditions, *Comp. Math. Appl.*, 69 (2015): 893-908.

[13] R. S. Jain, M. B. Dhakne. On mild solutions of nonlocal semilinear impulsive functional integro-differential equations, *Appl. Math. E-notes*, 13 (2013): 109-119.

[14] R.-N. Wang, Q.-M. Xiang, Y. Zhou. Asymptotically periodic solutions to nonlocal cauchy problems governed by compact evolution families, *J. Math. Sci.*, 197 (2014): 13-28

[15] K. Karthikeyan, A. Anguraj, K. Malar, J. J. Trujillo. Existence of mild and classical solutions for nonlocal impulsive integrodifferential equations in Banach spaces with measure of noncompactness, *Int. J. Diff. Equ.*, 2014 (2014), pp-10.

[16] P. Chen, Y. Li. Nonlocal Cauchy problem for fractional stochastic evolution equations in Hilbert spaces, *Collect. Math.*, 2014 (2014):1-14.

[17] S. Ji. Approximate controllability of semilinear nonlocal fractional differential systems via an approximating method, *Appl. Math. Comp.*, 236 (2014): 43-53.

[18] A. Debbouche, J. J. Nieto. Sobolev type fractional abstract evolution equations with nonlocal conditions and optimal multi-controls, *Appl. Math. Comp.*, 245 (2014): 74-85.

[19] A. Chadha, D. N. Pandey. Existence of a mild solution for an impulsive neutral fractional integro-differential equation with nonlocal conditions, *J. Fract. Cal. Appl.*, 6 (2015): 5-20.

[20] R. P. Agarwal, J. P. C. Santos, C. Cuevas. Analytic resolvent operator and existence results for fractional order evolutionary integral equations, *J. Abstr. Differ. Equ. Appl.*, 2 (2012): 26-47.

[21] B. de Andrade, J.P.C. Santos. Existence of solutions for a fractional neutral integro-differential equation with unbounded delay, *Elect. J. Diff. Equ.*, 2012 (2012):1-13.

[22] B. de Andrade, J. P. C. Santos. Existence of solutions for a fractional neutral integro-differential equation with unbounded delay, *Elect. J. Diff. Equ.*, 2012 (2012): 1-13.

[23] Z. Yan, X. Jia. Impulsive problems for fractional partial neutral functional integro-differential inclusions with infinite delay and analytic resolvent operators, *Mediterr. J. Math.*, 2013 (2013):1-36.

[24] Zuomao Yan. Existence of solutions for nonlocal impulsive partial functional integrodifferential equations via fractional operators, *J. Comp. Appl. Math.*, 235 (2011): 2252-2262.

[25] X. Fu. Existence of solutions for non-autonomous functional evolution equations with nonlocal conditions, *Elect. J. Diff. Equ.*, 2012 (2012):1-15.

[26] A. Chadha, D. N. Pandey. Approximations of solutions for a Sobolev type fractional order differential equation, *Nonlinear Dyn. Sys. The.*, 14 (2014): 11-29.

[27] A. Pazy. Semi-groups of Linear operator and Applications of Partial Differential Equations, Springer verlag, 1983.

[28] K. Yosida. Functional Analysis, 6th edn. Springer, Berlin, 1980.

[29] A. Granas, J. Dugundji. Fixed Point Theory, Springer-Verlag, New York, 2003.