ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.24(2017) No.1, pp.3-8

Cauchy Problem for Sobolev Type Fractional Integro-differential Equation with Nonlocal Conditions

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(Received 24 February 2015, accepted 23 January 2017)

Abstract: By using fractional calculus, uniformly continuous semigroups and the Schauder fixed point theorem, we prove the existence of mild solution of Sobolev type fractional integro-differential equation with nonlocal conditions in Banach spaces. An example in fractional integro-partial differential equation is provided to illustrate the results.

Keywords: Fractional calculus; Sobolev type fractional integro-differential equation; nonlocal condition; uniformly continuous semigroup; mild solution; Schauder fixed point theorem

1 Introduction

Sobolev type equation appears in a variety of physical problems such as flow of fluid through fissures in rocks, thermodynamics and propagation of long waves of small amplitude (see [1-3]). The first result on Sobolev type equation was obtained by Hilbert space methods (see [4]). Subsequently, several authors studied and discussed similar type of problems subject to local or nonlocal conditions. Very strong and complete results are known concerning existence, uniqueness and properties of solutions. Brill [5] and Showalter [6] established the existence of solutions of semilinear evolution equations of Sobolev type in Banach spaces. Balachandran and Karunanithi [7] studied the regularity of solutions of Sobolev type semilinear integrodifferential equations in Banach spaces. Agarwal and Bahuguna [8] studied the existence solutions of Sobolev type partial neutral differential equations. Balachandran and Park [9] investigated integrodifferential equation of Sobolev type with nonlocal condition and proved the existence of mild and strong solutions using semigroup theory and Schauder's fixed point theorem. Recently, differential equations with fractional order have emerged as a new branch of applied mathematics which have been used for many mathematical models in science and engineering. The subject of fractional differential equations is gaining much importance and attention. For details, see [10-17] and the references therein. In this paper, we prove the existence of mild solution of Sobolev type fractional integro-differential equation with nonlocal condition of the following form

$$(1.1)\begin{cases} {}^cD^\alpha(Bu(t)) + Au(t) = f\left(t, u(t), \int_0^t a(t,s)k(s,u(s))ds\right), & t \in I := [0,b], \\ u(0) + g(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p)) = u_0, \end{cases}$$

where ${}^cD^\alpha$ is the Caputo fractional derivative of order $0 < \alpha < 1$, B and A are linear operators with domains contained in a Banach space X and ranges contained in a Banach space Y, the nonlinear operators $f : I \times X \times X \to Y$, $k : I \times X \to X$, $g : I^p \times X^p \to X$ and the function $a : I \times I \to R$ will be given later. Here $u_0 \in D(B)$.

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IJNS.2017.08.15/964

2 Preliminaries

In this section, we introduce preliminary facts which are used throughout this paper.

Definition 1 *(see [18-20]). The fractional integral of order $\alpha > 0$ with the lower limit zero for a function f can be defined as*

$$I^{\alpha}f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t \frac{f(s)}{(t-s)^{1-\alpha}}ds, \ \ t > 0$$

provided the right-hand side is pointwise defined on $[0,\infty)$, where Γ is the Gamma function.

Definition 2 *(see [18-20]). The Caputo derivative of order α with the lower limit zero for a function f can be written as*

$$^{c}D^{\alpha}f(t) = \frac{1}{\Gamma(n-\alpha)}\int_0^t \frac{f^{(n)}(s)}{(t-s)^{\alpha+1-n}}ds = I^{n-\alpha}f^{(n)}(t), \ \ t > 0, \ 0 \le n-1 < \alpha < n.$$

If f is an abstract function with values in X, then the integrals appearing in the above definitions are taken in Bochner's sense.

In order to prove our main theorem we assume certain conditions on the operators A and B.
Let X and Y be Banach spaces with norm $|.|$ and $\|.\|$ respectively.
The operators $A : D(A) \subset X \to Y$ and $B : D(B) \subset X \to Y$ satisfy the following hypotheses:
(H_1) A and B are closed linear operators,
(H_2) $D(B) \subset D(A)$ and B is bijective,
(H_3) $B^{-1} : Y \to D(B)$ is compact.
The hypotheses H_1, H_2 and the closed graph theorem imply the boundedness of the linear operator $AB^{-1} : Y \to Y$.
From the above fact, $-AB^{-1}$ generates a uniformly continuous semigroup $\{T(t),\ t \ge 0\}$ in X.
(H_4) $-AB^{-1}$ generates a uniformly continuous semigroup $\{T(t),\ t \ge 0\}$, so $\max_{t\in J}\|T(t)\|$ is finite, which means that there exists $M > 0$ such that $\max_{t\in I}\|T(t)\| \le M$.
(H_5) $T(t)$ is a compact operator for every $t > 0$,
Let $B_r = \{x \in X : |x| \le r\}$ where r is arbitrary.
We will need the following assumptions.
(H_6) $a : I \times I \to R$ and $k : I \times B_r \to X$ are continuous,
(H_7) $f : I \times B_r \times B_r \to Y$ is continuous in t on I and there exists a constant $L > 0$ such that $\|f(t,u,v)\| \le L$ for $t \in I$ and $u,\ v \in B_r$,
(H_8) $g : I^p \times B_r^p \to D(B) \subset X$, Bg is continuous and there exists a constant $G > 0$ such that $\|Bg(t_1,t_2,...,t_p,u(t_1),u(t_2),...,u(t_p))\| \le G$ for $t_i \in I$ and $u(t_i) \in B_r$,
(H_9) the set $\{u(0) : u \in C(I,X),\ \|u\| \le r,\ u(0) + g(t_1,t_2,...,t_p,u(t_1),u(t_2),...,u(t_p)) = u_0\}$ where $r = M_0M\left[\|Bu_0\| + G + \frac{Lb^{\alpha}}{\Gamma(1+\alpha)}\right]$, is precompact in X, $M_0 = \|B^{-1}\|$.
According to [21, 22], a mild solution of equation (1.1) can be represented by

$$(2.1)\quad u(t) = B^{-1}S_{\alpha}(t)Bu_0 - B^{-1}S_{\alpha}(t)Bg(t_1,t_2,...,t_p,u(t_1),u(t_2),...,u(t_p))$$
$$+\int_0^t (t-s)^{\alpha-1}B^{-1}T_{\alpha}(t-s)f\left(s,u(s),\int_0^s a(s,\tau)k(\tau,u(\tau))d\tau\right)ds, \ \ t \in I,$$

where

$$S_{\alpha}(t)x = \int_0^{\infty}\xi_{\alpha}(\theta)T(t^{\alpha}\theta)xd\theta, \qquad T_{\alpha}(t)x = \alpha\int_0^{\infty}\theta\xi_{\alpha}(\theta)T(t^{\alpha}\theta)xd\theta$$

with $\xi_{\alpha}(\theta)$ is a probability density function defined on $(0,\infty)$, that is $\xi_{\alpha}(\theta) \ge 0$, $\theta \in (0,\infty)$ and $\int_0^{\infty}\xi_{\alpha}(\theta)d\theta = 1$.

Remark 1 *Notice that $\int_0^{\infty}\theta\xi_{\alpha}(\theta)d\theta = \frac{1}{\Gamma(1+\alpha)}$ (see [22, 23]).*

Definition 3 *By a mild solution of the problem* (1.1)*, we mean that the function* $u \in C(I, X)$ *which satisfies*

$$u(t) = B^{-1}S_\alpha(t)Bu_0 - B^{-1}S_\alpha(t)Bg(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p))$$
$$+ \int_0^t (t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau)ds, \ \ t \in I.$$

Lemma 2 *(see [22]). The operators* $S_\alpha(t)$ *and* $T_\alpha(t)$ *have the following properties:*
(I) *for any fixed* $x \in X$, $\| S_\alpha(t)x \| \leq M \| x \|$, $\| T_\alpha(t)x \| \leq \frac{\alpha M}{\Gamma(\alpha+1)} \| x \|$;
(II) $\{S_\alpha(t), \ t \geq 0\}$ *and* $\{T_\alpha(t), \ t \geq 0\}$ *are strongly continuous;*
(III) *For every* $t > 0$, $S_\alpha(t)$ *and* $T_\alpha(t)$ *are compact operators.*

3 Main Result

Theorem 3 *If the assumptions* (H_4)-(H_9) *are satisfied, then the problem* (1.1) *has a mild solution on* I.

Proof. Let $E \subset C(I, X)$ and $E_0 = \{u \in E : u(t) \in B_r, \ t \in I\}$. Clearly, E_0 is a bounded closed convex subset of E. We define a mapping $F : E_0 \to E_0$ by

$$(Fu)(t) = B^{-1}S_\alpha(t)[Bu_0 - Bg(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p))]$$
$$+ \int_0^t (t-s)^{\alpha-1}B^{-1}T_\alpha(t-s)f\left(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau\right)ds, \ \ t \in I.$$

Obviously F is continuous, since all the functions involved in the definition of the operator are continuous. Further, from our assumptions, we have

$$\|(Fu)(t)\| \leq \|B^{-1}\|\|S_\alpha(t)\|[\|Bu_0\| + \|Bg(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p))\|]$$
$$+ \int_0^t (t-s)^{\alpha-1}\|B^{-1}\|\|T_\alpha(t-s)\|\|f\left(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau\right)\|ds$$
$$\leq M_0M[\|Bu_0\| + G + \frac{Lb^\alpha}{\Gamma(1+\alpha)}] = r,$$

and therefore F maps E_0 into E_0. Moreover, F maps E_0 into a precompact subset of E_0. To prove this, we first show that the set $E_0(t) = \{(Fu)(t) : u \in E_0\}$ is precompact in X, for every fixed $t \in I$. It is clear for $t = 0$ from (H_9).
Let $t > 0$ be fixed. For all ϵ, $0 < \epsilon < t$ and for all $\delta > 0$ take

$$(F_{\epsilon,\delta}u)(t) = \int_\delta^\infty \xi_\alpha(\theta)B^{-1}T(t^\alpha\theta)[Bu_0 - Bg(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p))]d\theta$$
$$+ \alpha \int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)B^{-1}T((t-s)^\alpha\theta)f\left(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau\right)d\theta ds$$
$$= T(\epsilon^\alpha\delta)\int_\delta^\infty \xi_\alpha(\theta)B^{-1}T(t^\alpha\theta - \epsilon^\alpha\delta)[Bu_0 - Bg(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p))]d\theta$$
$$+ \alpha T(\epsilon^\alpha\delta)\int_0^{t-\epsilon}\int_\delta^\infty \theta(t-s)^{\alpha-1}\xi_\alpha(\theta)B^{-1}T((t-s)^\alpha\theta\epsilon^\alpha\delta)f\left(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau\right)d\theta ds$$

Since the operator B^{-1} is compact in X for $t > 0$ and $T(\epsilon^\alpha\delta)$, $\epsilon^\alpha\delta > 0$ is a compact operator, the set $E_{\epsilon,\delta}(t) = \{(F_{\epsilon,\delta}u)(t) : u \in E_0\}$ is precompact in X for every ϵ, $0 < \epsilon < t$ and for all $\delta > 0$. Furthermore, for $u \in E_0$ we have

$$\|(Fu)(t) - (F_{\epsilon,\delta}u)(t)\| \leq \| \int_0^\delta \xi_\alpha(\theta)B^{-1}T(t^\alpha\theta)[Bu_0 - Bg(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p))]d\theta\|$$
$$+ \alpha\| \int_{t-\epsilon}^t\int_\delta^\infty \theta\xi_\alpha(\theta)(t-s)^{\alpha-1}B^{-1}T((t-s)^\alpha\theta)f\left(s, u(s), \int_0^s a(s,\tau)k(\tau, u(\tau))d\tau\right)d\theta ds\|$$

$$+\alpha\|\int_{t-\epsilon}^{t}\int_{0}^{\delta}\theta\xi_\alpha(\theta)(t-s)^{\alpha-1}B^{-1}T((t-s)^\alpha\theta)f\left(s,u(s),\int_0^s a(s,\tau)k(\tau,u(\tau))d\tau\right)d\theta ds\|$$

$$\leq M_0M[\|Bu_0\|+G]\int_0^\delta \xi_\alpha(\theta)d\theta+\frac{M_0ML\epsilon^\alpha}{\Gamma(1+\alpha)}+M_0ML\epsilon^\alpha\int_0^\delta\theta\xi_\alpha(\theta)d\theta.$$

We see that $\|(Fu)(t)-(F_{\epsilon,\delta}u)(t)\|\to 0$ as $\epsilon\to 0^+$ and $\delta\to 0^+$. Therefore, there are precompact sets arbitrary close to the set $E_0(t)$ and so $E_0(t)$ is precompact in X.
Now we shall show that $F(E_0)=Z=\{Fu: u\in E_0\}$ is an equicontinuous family of functions. For $0<s<t$, we have

$$\begin{aligned}\|(Fu)(t)-(Fu)(s)\| &\leq \|B^{-1}\|\|S_\alpha(t)-S_\alpha(s)\|\|Bu_0\|\\ &+\|B^{-1}\|\|S_\alpha(t)-S_\alpha(s)\|\|Bg(t_1,t_2,...,t_p,u(t_1),u(t_2),...,u(t_p))\|\\ &+\int_0^t\|B^{-1}\|\|(t-\eta)^{\alpha-1}T_\alpha(t-\eta)-(s-\eta)^{\alpha-1}S_\alpha(s-\eta)\|\|f\left(\eta,u(\eta),\int_0^\eta a(\eta,\tau)k(\tau,u(\tau))d\tau\right)\|d\eta\\ &+\int_s^t(s-\eta)^{\alpha-1}\|B^{-1}\|\|T_\alpha(s-\eta)\|\|f\left(\eta,u(\eta),\int_0^\eta a(\eta,\tau)k(\tau,u(\tau))d\tau\right)\|d\eta\\ &\leq (M_0\|Bu_0\|+M_0G)\|S_\alpha(t)-S(s)\|+M_0L\int_0^t\|(t-\eta)^{\alpha-1}T_\alpha(t-\eta)-(s-\eta)^{\alpha-1}T_\alpha(s-\eta)\|d\eta\\ &+M_0L\int_s^t(s-\eta)^{\alpha-1}\|T_\alpha(s-\eta)\|d\eta.\end{aligned}$$

The right hand side of the above inequality is independent of $u\in E_0$ and tends to zero as $s\to t$ as a consequence of the continuity of $T(t)$ in the uniform operator topology for $t>0$. It is also clear that Z is bounded in E. Thus by Arzela - Ascoli's theorem, Z is precompact. Hence by the Schauder's fixed point theorem, F has a fixed point in E_0 and any fixed point of F is a mild solution of (1.1) on I such that $u(t)\in X$ for $t\in I$. ■

4 Example

Consider the following fractional integro-partial differential equation

$$(4.1)\begin{cases}\frac{\partial^\alpha}{\partial t^\alpha}\left(u(t,x)-\frac{\partial^2}{\partial x^2}u(t,x)\right)-\frac{\partial^2}{\partial x^2}u(t,x)=\frac{u(t,x)}{1+t^2}+u(t,x)\int_0^t\frac{e^{-u(s,x)}}{(1+t^2)(1+s)}ds,\ t\in I:=[0,1],\ x\in[0,1]\\ u(t,0)=u(t,1)=0,\ t\in J,\\ u(0,x)+\sum_{i=1}^p c_iu(t_i,x)=u_0(x),\ \ x\in[0,1],\end{cases}$$

where $\alpha\in(0,1)$, $0<t_1<t_2<...<t_p<1$, p is a positive integer and $u_0(x)\in X$.
Take $X=Y=L^2[0,1]$ and define the operators $A:D(A)\subset X\to Y$ and $B:D(B)\subset X\to Y$ by $Aw=-w''$ and $Bw=w-w''$, where each domain $D(A)$ and $D(B)$ is given by $\{w\in X: w,\ w'$ are absolutely continuous, $w''\in X,\ w(0)=w(\pi)=0\}$.
Then A and B can be written respectively as [24]

$$Aw=\sum_{n=1}^{\infty}n^2(w,w_n)w_n,\quad w\in D(A),$$

$$Bw=\sum_{n=1}^{\infty}(1+n^2)(w,w_n)w_n,\ \ w\in D(B),$$

where $w_n(x)=\sqrt{2/\pi}\sin nx,\ \ n=1,2,...,$ is the orthonormal set of eigenvectors of A and (w,w_n) is the L^2 inner product. Moreover for $w\in X$, we get

$$B^{-1}w=\sum_{n=1}^{\infty}\frac{1}{1+n^2}(w,w_n)w_n,$$

$$-AB^{-1}w = \sum_{n=1}^{\infty} \frac{-n^2}{1+n^2}(w, w_n)w_n,$$

$$T(t)w = \sum_{n=1}^{\infty} e^{\frac{-n^2}{(1+n^2)}t}(w, w_n)w_n.$$

It is easy to see that $-AB^{-1}$ generates a uniformly continuous semigroup $T(t),\ t \geq 0$ and so $\max_{t\in I} \|T(t)\|$ is finite.
Put

$$u(t) = u(t, \cdot),\ \ f(t, u(t), \int_0^t a(t, s)k(s, u(s))ds) = \frac{u(t, x)}{1+t^2} + u(t, x)\int_0^t \frac{e^{-u(s,x)}}{(1+t^2)(1+s)}ds,$$

$$g(t_1, t_2, ..., t_p, u(t_1), u(t_2), ..., u(t_p)) = \sum_{i=1}^{p} c_i u(t_i, x).$$

with this choice of $A,\ B,\ f,\ a$ and k, the equation (4.1) can be written in the abstract formulation of the system (1.1). Thus all the conditions of Theorem 3.1 are satisfied.
Hence, the equation (4.1) has a mild solution on I.
According to [25] the mild solution of (4.1) is given by

$$u(t, x) = \int_0^{\infty}\int_0^x B^{-1}T(t^{\alpha}\theta, x, y)\xi_{\alpha}(\theta)Bu(0, y)dyd\theta$$

$$+ \alpha\int_0^t\int_0^{\infty}\int_0^x \theta(t-s)^{\alpha-1}\xi_{\alpha}(\theta)B^{-1}T((t-s)^{\alpha}\theta, x, y)[\frac{u(s, y)}{1+s^2} + u(s, y)\int_0^s \frac{e^{-(\tau,y)}}{(1+s^2)(1+\tau)}d\tau]dyd\theta ds.$$

5 Conclusions

In this paper, we have presented, by using fractional calculus and Schauder fixed point theorem, the existence of mild solution of fractional the existence of mild solution of Sobolev type fractional integro-differential equation with nonlocal condition in Banach spaces. We provided example to illustrate our results.

Acknowledgments

I am very grateful to the referees and the Editor in Chief for their careful reading and helpful comments.

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