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A Family of Explicit Nonstandard Finite Difference Schemes with Positivity Property for MSEIR Models

M. Mehdizadeh Khalsaraei *, L. Davari Algoo, Sh. Heydari
Faculty of Mathematical Science, University of Maragheh, Maragheh, Iran

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Abstract: A number of important phenomena in epidemiology can be modeled by systems of ordinary differential equations (ODEs). This paper considers a modified system of ODEs which is nonlinear and describes the MSEIR model. Based on the nonstandard discretization method, we propose a family of explicit schemes to solve numerically the MSEIR model. The proposed schemes preserve the positivity property as well as the requirement of conservation law and boundedness. Numerical tests of new schemes are done. A brief discussion of how the work can be extended is also given.

Keywords: ordinary differential equation; positivity; boundedness; nonstandard finite difference

1 Introduction

The mathematical model of MSEIR [1] consists of the following system of differential equations:

$$
\begin{aligned}
\frac{dM}{dt} &= b(N-S)-(\delta+d)M,\\
\frac{dS}{dt} &= bS+\delta M-\beta SI/N-dS,\\
\frac{dE}{dt} &= \beta SI/N-(\epsilon+d)E,\\
\frac{dI}{dt} &= \epsilon E-(\gamma+d)I,\\
\frac{dR}{dt} &= \gamma I-dR,\\
\frac{dN}{dt} &= (b-d)N,
\end{aligned}
\tag{1}
$$

where M, S, E, I and R are infants with passive immunity, susceptibles, exposed individuals, infectives and recovered individuals respectively. The parameters d, q, β, δ, ϵ and γ are positive numbers. In what follows, it is convenient to consider as dependent variables the fractions (divided by N) of the sub-populations in the compartments : m=M/N, s=S/N, e=E/N, i=I/N and r=R/N. This reduces to the following equivalent system for the MSEIR :

*Corresponding author. *E-mail address*: Muhammad.mehdizadeh@gmail.com

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$$
\begin{aligned}
\frac{dm}{dt} &= (d+q)(e+i+r) - \delta m, \\
\frac{ds}{dt} &= \delta m - \beta s i, \\
\frac{de}{dt} &= \beta s i - (\epsilon + d + q)e, \\
\frac{di}{dt} &= \epsilon e - (\gamma + d + q)i, \\
\frac{dr}{dt} &= \gamma i - (d+q)r.
\end{aligned} \tag{2}
$$

Numerical simulation of system (2) can be obtained based on the finite difference approximations, such as Euler, Runge-kutta, or Adams methods [4, 6]. Since the unknown quantities in (2) are necessarily positive, it is important to have numerical schemes that preserve the positivity of the solution[1, 7]. One of the shortcomings of the above mentioned standard finite difference method is that the positivity property of the exact solution usually is not transferred to the numerical solution. Furthermore many problems may affect the stability properties of the standard approach. One way of avoiding this disadvantage is to employ the finite difference schemes that are nonstandard in the sense of Mickens' definition [15].

The present work which is motivated by many successful papers on the matter [1, 3, 5, 10–12, 16, 17], introduces a generic family of the finite-difference schemes to approximate dynamically consistent solution for (2) with respect to positivity.

The rest of the paper is organized as follows: In Section 2, we present preliminaries. In Section 3, we propose the new family of schemes and investigate the positivity and conservation law. In Section 4, we compared the obtained results from new schemes with the results of classical methods. Finally, we end the paper with some conclusions in Section 5.

2 Preliminaries

We now give a brief summery of the nonstandard finite differences (NSFDs) for ordinary differential equations (ODEs), for more details see [8, 9, 15]. Let us consider the systems of ODEs, which can be written in the form

$$\frac{d}{dt}u(t) = F(u(t)), \;\; (t \geq 0), \qquad u(0) = u_0, \tag{3}$$

where u may be a single function or a vector of functions of length k mapping $[t_0, T) \to \mathcal{C}^k$ and the corresponding F a single function or a vector of functions of length k mapping $([t_0, T), \mathcal{C}^k) \to ([t_0, T), \mathcal{C}^k)$. Discretization of the continuous differential equation, or beginning instead with a difference equation, we define $t_n = t_0 + \Delta t n$, where Δt is a positive step size, and say that the discretized version of the function u at time t_n is

$$u_n \approx u(t_n), \tag{4}$$

then the discretized version of Eq. (3) becomes

$$\mathcal{D}_{\Delta t} u_n = \mathcal{F}_n(F, u_n), \tag{5}$$

where $\mathcal{D}_{\Delta t} u_n$ represents the discretized version of $\frac{d}{dt}u(t)$ and $\mathcal{F}_n(F, u_n)$ approximates $F(u(t_n))$ at time t_n.

For the construction of the new schemes we will use the rules 1-2, coming below of the non-standard modeling as in Mickens [13, 15].

Rule 1:

The denominator function for the discrete derivatives must be expressed in terms of more complicated function φ of the step-sizes than those conventionally used. This rule allows the introduction of complex analytic function of Δt in the denominator with the condition that

$$\varphi(\Delta t) = \Delta t + O(\Delta t^2) \;\; as \;\; 0 < \Delta t \to 0. \tag{6}$$

Rule 2:

The non-linear terms must in general be modelled (approximated) non-locally on the computational grid or lattice in many different ways, for instance, the non-linear terms u, u^2 and u^3 can be modelled as follows as in Anguelov and Lubuma [2]:

$$u \approx \frac{2u_{n+1} + u_n}{3}, \tag{7}$$

$$u \approx 3u_{n+1} - 2u_n,$$

$$u^2 \approx au_n^2 + (1-a)u_n u_{n+1}, \ \ a \in R$$

$$u^3 \approx \frac{2u_{n+1}^2 u_n^2}{u_{k+1} + u_k}.$$

As mentioned above the preservation of the qualitative properties of the considered differential equation is of great interest in finite difference methods for solving the differential equations. The major consequence of these result are that such schemes do not allow numerical instabilities to occur[14].

3 New scheme

In this section, we propose our new family of nonstandard finite difference schemes as:

$$\begin{aligned}
&(1-\alpha)\tfrac{m_{n+1}-m_n}{\varphi(\Delta t)} + \alpha\tfrac{m_n - m_{n-1}}{\varphi(\Delta t)} = (d+q)(e_n + i_n + r_n) - \delta m_n,\\
&(1-\alpha)\tfrac{s_{n+1}-s_n}{\varphi(\Delta t)} + \alpha\tfrac{s_n - s_{n-1}}{\varphi(\Delta t)} = \delta m_n - \beta s_{n+1} i_n,\\
&(1-\alpha)\tfrac{e_{n+1}-e_n}{\varphi(\Delta t)} + \alpha\tfrac{e_n - e_{n-1}}{\varphi(\Delta t)} = \beta s_{n+1} i_n - (\epsilon + d + q)e_n,\\
&(1-\alpha)\tfrac{i_{n+1}-i_n}{\varphi(\Delta t)} + \alpha\tfrac{i_n - i_{n-1}}{\varphi(\Delta t)} = \epsilon e_n - (\gamma + d + q)i_n,\\
&(1-\alpha)\tfrac{r_{n+1}-r_n}{\varphi(\Delta t)} + \alpha\tfrac{r_n - r_{n-1}}{\varphi(\Delta t)} = \gamma i_n - (d+q)r_n,
\end{aligned} \tag{8}$$

here α is arbitrary parameter to be determined below based on the positivity property. After simplifying we have:

$$\begin{aligned}
m_{n+1} &= \Big(m_n\big(1-2\alpha-\delta\varphi(\Delta t)\big) + \alpha m_{n-1} + \varphi(\Delta t)(d+q)(e_n+i_n+r_n)\Big)/\big(1-\alpha\big),\\
s_{n+1} &= \big(s_n(1-2\alpha) + a s_{n-1} + \varphi(\Delta t)\delta m_n\big)/\big(1-\alpha+\varphi(\Delta t)\beta i_n\big),\\
e_{n+1} &= \Big(\big(1-2\alpha-\varphi(\Delta t)(\epsilon+d+q)\big)e_n + \alpha e_{n-1} + \varphi(\Delta t)\beta s_{n+1} i_n\Big)/\big(1-\alpha\big),\\
i_{n+1} &= \Big(\big(1-2\alpha-\varphi(\Delta t)(\gamma+d+q)\big)i_n + \alpha i_{n-1} + \varphi(\Delta t)\epsilon e_n\Big)/\big(1-\alpha\big),\\
r_{n+1} &= \Big(\big(1-2\alpha-\varphi(\Delta t)(d+q)\big)r_n + \alpha r_{n-1} + \varphi(\Delta t)\gamma i_n\Big)/\big(1-\alpha\big),
\end{aligned} \tag{9}$$

which is a family of three time levels schemes. By using the underlying positive explicit method from [1].

$$\begin{aligned}
m_{n+1} &= m_n\big(1-\delta\varphi(\Delta t)\big)+\varphi(\Delta t)(d+q)(e_n+i_n+r_n),\\
s_{n+1} &= \big(s_n+\varphi(\Delta t)\delta m_n\big)/\big(1+\varphi(\Delta t)\beta i_n\big),\\
e_{n+1} &= \big(1-\varphi(\Delta t)(\epsilon+d+q)\big)e_n+\varphi(\Delta t)\beta s_{n+1}i_n,\\
i_{n+1} &= \big(1-\varphi(\Delta t)(\gamma+d+q)\big)i_n+\varphi(\Delta t)\epsilon e_n,\\
r_{n+1} &= \big(1-\varphi(\Delta t)(d+q)\big)r_n+\varphi(\Delta t)\gamma i_n,
\end{aligned} \tag{10}$$

We can solve system (2) for the first time step to initiate three time levels scheme.

Theorem 1 *Sufficient conditions for the new family of three time levels schemes* (8) *to be positive are*

$$0<\varphi(\Delta t)\le\frac{1}{(\gamma+d+q)},\quad 0\le\alpha\le\frac{1-\varphi(\Delta t)(\gamma+d+q)}{2}.$$

Proof. Assume that m^{k-1}, s^{k-1}, e^{k-1}, i^{k-1}, r^{k-1} are nonnegative real numbers, since for the positivity of (10) we have to put

$$1-\delta\varphi(\Delta t)\ge 0,$$

$$1-(\epsilon+d+q)\varphi(\Delta t)\ge 0,$$

$$1-(\gamma+d+q)\varphi(\Delta t)\ge 0,$$

$$1-(d+q)\varphi(\Delta t)\ge 0,$$

from which

$$0<\varphi(\Delta t)\le\frac{1}{(\gamma+d+q)},$$

then in (9) m^k, s^k, e^k, i^k, r^k are positive, so for the positivity of scheme it is enough to show that

$$(1-2\alpha-\delta\varphi(\Delta t))\ge 0,$$

$$\alpha\ge 0,$$

$$1-\alpha\ge 0,$$

$$(1-2\alpha-\varphi(\Delta t)(\epsilon+d+q))\ge 0,$$

$$(1-2\alpha-\varphi(\Delta t)(\gamma+d+q))\ge 0,$$

$$(1-2\alpha-\varphi(\Delta t)(d+q))\ge 0,$$

which leads to

$$0\le\alpha\le\frac{1-\varphi(\Delta t)(\gamma+d+q)}{2},$$

and this completes the proof. ■

Proposition 1 *The scheme* (8) *preserves the conservation law.*

Proof. With collectors the right-hand side of (8) we have:

$$\frac{d}{dt}(m+s+e+i+r) = (d+q)(e_n+i_n+r_n) - \delta m_n - \beta s_{n+1}i_n + \delta m_n + \beta s_{n+1}i_n - (\epsilon+d+q)e_n + \epsilon e_n - (\gamma+d+q)i_n + \gamma i_n - (d+q)r_n = 0,$$

and also

$$(1-\alpha)m_{n+1} + (2\alpha-1)m_n - \alpha m_{n-1} + (1-\alpha)s_{n+1} + (2\alpha-1)s_n - \alpha s_{n-1} + (1-\alpha)e_{n+1} + (2\alpha-1)e_n - \alpha e_{n-1} + (1-\alpha)i_{n+1} + (2\alpha-1)i_n - \alpha i_{n-1} + (1-\alpha)r_{n+1} + (2\alpha-1)r_n - \alpha r_{n-1} = 0,$$

since

$$m_{n-1} + s_{n-1} + e_{n-1} + i_{n-1} + r_{n-1} = 1,$$

following [1]

$$m_n + s_n + e_n + i_n + r_n = 1,$$

therefor

$$(1-\alpha)\big(m_{n+1} + s_{n+1} + e_{n+1} + i_{n+1} + r_{n+1}\big) = (1-\alpha),$$
$$m_{n+1} + s_{n+1} + e_{n+1} + i_{n+1} + r_{n+1} = 1,$$

and this concludes the conservation law. ■

Proposition 2 *The scheme* (8) *is boundedness.*

Proof. Since the new scheme has the positivity property, then all variables are nonnegative real numbers and according to the conservation law, sum of these variable equal to one, these follow that the method is boundedness. ■

4 Numerical simulations

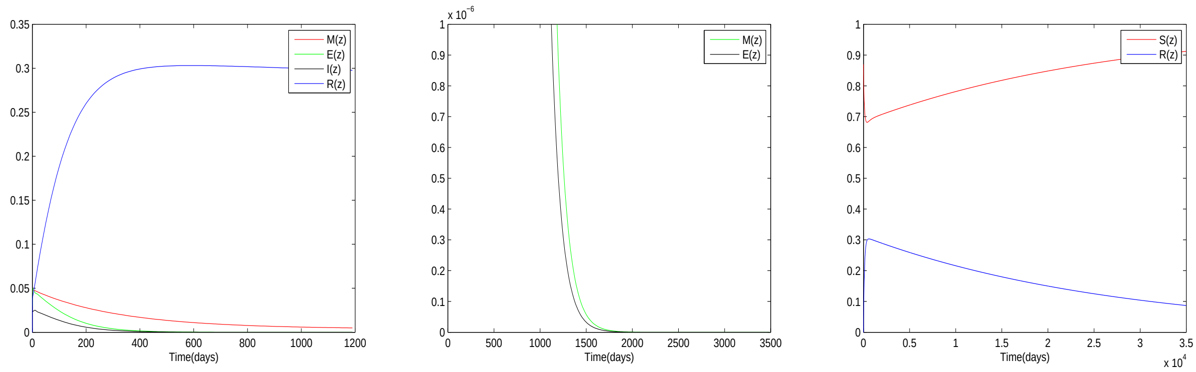
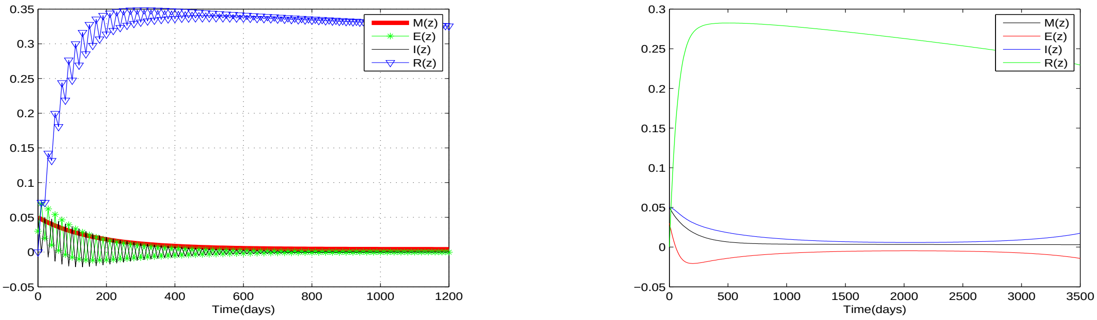


Figure 1: Numerical results of the new scheme with $h = 10$,$m(0) = 0.05$,$s(0) = 0.87$,$e(0) = 0.03$,$i(0) = 0.05$, $r(0) = 0$ and α=0.02.

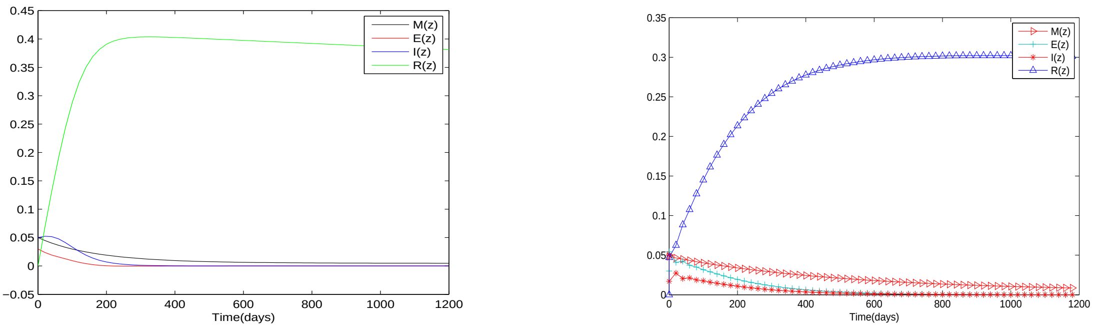
In this section we perform numerical experiments to demonstrate the performance of the new scheme. The numerical simulations are illustrated in Figure 1-4 with $d = 1/(40 * 365)$, $q = 0$, $\beta = 0.14$, $\delta = 1/180$, $\epsilon = 1/14$, $\gamma = 1/7$, and

$$\varphi(\Delta t) = \frac{1-e^{-Q\Delta t}}{Q}, Q = \max\{\delta, \epsilon+d+q, \gamma+d+q\},$$



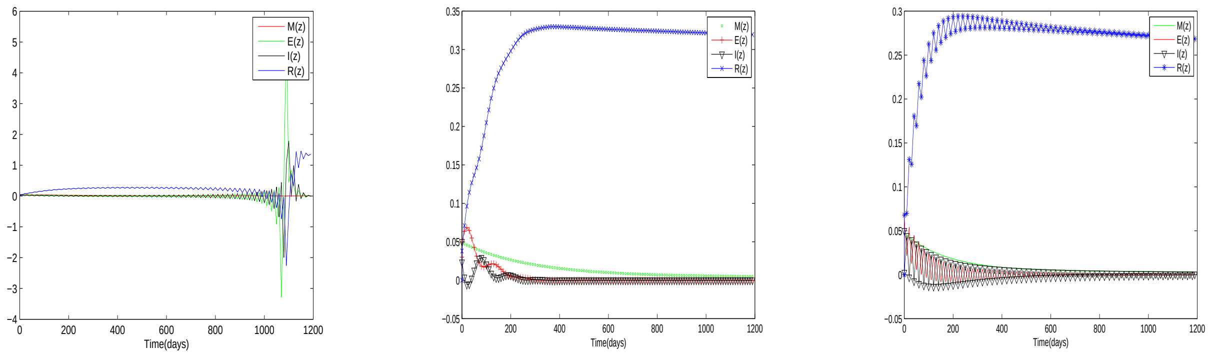
(a) Euler method (b) $RK2$

Figure 2: Numerical results with $h = 10$,$m(0) = 0.05$,$s(0) = 0.87$,$e(0) = 0.03$,$i(0) = 0.05$ and $r(0) = 0$.



(a) $RK4$ (b) The new scheme with α=0.02

Figure 3: Numerical results with $h = 20$,$m(0) = 0.05$,$s(0) = 0.87$,$e(0) = 0.03$,$i(0) = 0.05$ and $r(0) = 0$.



(a) $\alpha = 0.24$ (b) $\alpha = -2.5$ (c) $\varphi(\Delta t) > \frac{1}{(\gamma+d+q)}$

Figure 4: Numerical results of the new scheme with $h = 10$,$m(0) = 0.05$,$s(0) = 0.87$,$e(0) = 0.03$,$i(0) = 0.05$ and $r(0) = 0$.

the parameters used in these simulations have been taken from [1].

As it can be seen in Figure 1 the new scheme preserves the positivity property with $\alpha = 0.02$, while the Euler method and the second-order Runge-Kutta method (RK2) give negative values when applied to the MSEIR model, see Figure 2. Similar behavior is observed with classical fourth order Runge-Kutta method (RK4), see Figure 3.

If one of the sufficient conditions in Theorem 1 is violated, then the numerical solutions may exhibit spurious oscillations and the new scheme produces negative values, see Figure 4.

5 Discussion and conclusions

In this article, we applied the new NSFD scheme to the MSEIR model by the renormalization of denominator of the discrete derivatives and the nonlocal approximation of nonlinear terms. We compared the numerical results of the proposed scheme with the standard ones. We show that the proposed scheme is boundedness and positivity preserving, while the approximations obtained by the Euler, RK2 and RK4 methods produce negative values. Also the new scheme preserves the conservation law. A future work can be investigation of the stability requirement and the necessity of the condition in Theorem 1.

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