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Stability and Hopf Bifurcation of a Delayed Ratio-dependent Eco-epidemiological Model with Two time Delays and Holling type III Functional Response

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Abstract: In this paper, a delayed ratio-dependent eco-epidemiological model with Holling type III functional response and two time delays is investigated.By regarding the delay as the bifurcation parameter, local stability of each equilibrium is discussed at the endemic equilibrium. By analyzing the corresponding characteristic equations, the conditions for existence of Hopf bifurcation for the system are obtained, respectively. By utilizing normal form method and center manifold theorem, the explicit formulas which determine the stability of bifurcating period solutions are derived. Finally, numerical simulations supporting the theoretical analysis are given.

Keywords: eco-epidemiological system; functional response; time delay; local stability; Hopf bifurcation

1 Introduction

In recent years, great attention has been paid to the eco-epidemiological system, and an increasing number of works have been devoted to the study of the relationships between demographic processes among different populations and diseases([1-8]). There has been a great and continuing interest on eco-epidemiological model with time delay, stage structure and functional response, and more complicated dynamics is illustrated, in terms of stability, bifurcation periodic solutions and so on [9-13].Moreover, Xu[6] proposed a delayed eco-epidemiological system with nonlinear incidence rate for the predator.

By leading several factors to a type III functional response such as predator leaning, prey refuge and the presence of alternative prey, and considering the influence about the feedback and gestation time delays, a delayed radio eco-epidemiological model with Holling III functional response and two delays is considered as follows:

$$\begin{cases} \dot{x}(t) = x(t)[r - a_{11}x(t-\tau_1) - \frac{a_{12}x^2(t)S(t)}{S^2(t)+mx^2(t)}, \\ \dot{S}(t) = \frac{a_{21}x^2(t-\tau_2)S(t-\tau_2)}{S^2(t-\tau_2)+mx^2(t-\tau_2)} - r_1S(t) - \beta S(t)I(t), \\ \dot{I}(t) = \beta S(t)I(t) - r_2I(t), \end{cases} \tag{1}$$

where $x(t)$, $S(t)$ and $I(t)$ represent the prey population, the densities of the susceptible and the infected predator population at time t, respectively. The parameters r, r_1, r_2, a_{11}, a_{12}, a_{21} and β in model (1) are all positive constants and their ecological meaning are interpreted as follows: r denotes the intrinsic growth rate of the prey, and r/a_{11} represents the carrying capacity of prey, a_{12} is the capturing rate of the susceptible predators capturing the prey, a_{21}/a_{12} is the conversion rate of nutrients into the reproduction of the susceptible predators, r_1 is the natural death rate of the susceptible predator, r_2 is the natural and disease-related mortality rate of the infected predator, $\beta > 0$ is called the disease transmission coefficient, τ_1 is the feedback time delay of the prey, and τ_2 is the time delay due to the gestation of the susceptible predator.

The organization of this paper is as follows. In section 2, the local stability of the positive equilibrium and the existence of Hopf bifurcation for system are derived. In section 3, numerical simulations are carried out to illustrate the validity of the main results. Finally, a brief conclusion is given.

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2 Local stability and Hopf bifurcation

In this section, we shall discuss the existence of Hopf bifurcation at the coexistence equilibrium. Further, we assume the equation

$$\beta^2 a_{12}(x^0)^2 > r_1(r_2^2 + m\beta^2(x^0)^2)$$

holds, in which x^0 is a positive real root of the following cubic equation:

$$ma_{11}\beta^2(x^0)^3 - m\beta^2 r(x^0)^2 + (a_{11}r_2 + a_{12}\beta)r_2x^0 + r(r_2)^2 = 0.$$

Then, we will consider the local stability and the existence of Hopf bifurcation at $E^0(x^0, S^0, I^0)$, where $S^0 = \frac{r_2}{\beta}$, $I^0 = \frac{\beta a_{21}x^2}{r_2^2+m\beta^2x^2} - \frac{r_1}{\beta}$.

Let $\bar{x}(t) = x(t) - x^0, \bar{S}(t) = S(t) - S^0, \bar{I}(t) = I(t) - I^0$, and still denote $\bar{x}(t), \bar{S}(t), \bar{I}(t)$ by $x(t), S(t), I(t)$. Respectively, using Taylor expansion to expand the system (1) at the positive equilibrium $E^0(x^0, S^0, I^0)$, we have

$$\begin{cases} \dot{x}(t) = a_{11}x(t) + a_{12}S(t) + b_{11}x(t-\tau_1), \\ \dot{S}(t) = a_{22}S(t) + b_{21}x(t-\tau_2) + b_{22}S(t-\tau_2) + a_{23}I(t), \\ \dot{I}(t) = a_{32}S(t), \end{cases} \tag{2}$$

where

$$a_{11} = r - a_{11}x^0 - \frac{2a_{12}x^0(S^0)^3}{((S^0)^2 + m(x^0)^2)^2}, a_{12} = \frac{a_{12}(x^0)^2(S^0)^2 - ma_{12}(x^0)^4}{((s^0)^2 + m(x^0)^2)^2}, b_{11} = -a_{11}x^0, a_{22} = -r_1 - \beta I^0,$$
$$b_{21} = \frac{2a_{21}x^0(S^0)^3}{((s^0)^2 + m(x^0)^2)^2}, a_{23} = -\beta S, b_{22} = \frac{ma_{21}(x^0)_4 - a_{21}(x^0)^2(S^0)^2}{((s^0)^2 + m(x^0)^2)^2}, a_{32} = \beta I.$$

Therefore, the corresponding characteristic equation of system (2) is

$$\lambda^3 + m_2\lambda^2 + m_1\lambda + m_0 + (n_2\lambda^2 + n_1\lambda + n_0)e^{-\lambda\tau_1} + (p_2\lambda^2 + p_1\lambda)e^{-\lambda\tau_2} + (q_1\lambda)e^{-\lambda(\tau_1+\tau_2)} = 0, \tag{3}$$

where $m_0 = a_{11}a_{23}a_{32}, m_1 = -a_{23}a_{32} + a_{11}a_{22}, m_2 = -(a_{11} + a_{22}), n_0 = a_{23}a_{32}b_{11}, n_1 = b_{11}a_{22}, n_2 = -b_{11}, p_1 = b_{22}a_{11} - a_{12}b_{21}, p_2 = -b_{22}, q_1 = b_{11}b_{22}$.
Then, according to the corollary of Ruan and Wei[14], we know the distribution of roots of the transcendental equation (4) . Because system (1) has two time delays, that is, τ_1 and τ_2, we consider the following cases.

Case 1: On substituting $\tau_1 > 0$, and $\tau_2 = 0$, then (3) reduces to

$$\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20} + (n_{22}\lambda^2 + n_{21}\lambda + n_{20})e^{-\lambda\tau_1} = 0, \tag{4}$$

where $m_{20} = m_0, m_{21} = p_1, m_{22} = m_2 + p_2, n_{20} = n_0, n_{21} = n_1 + q_1, n_{22} = n_2$.

For $\omega > 0$, suppose $i\omega$ to be a root of Eq. (4), then, substitute $i\omega$ into (4) and separate real and imaginary parts, one obtains that

$$\begin{cases} n_{21}\omega_1 \sin\omega_1\tau_1 + (n_{20} - n_{22}\omega_1^2)\cos\omega_1\tau_1 = m_{22}\omega_1^2 - m_{20}, \\ n_{21}\omega_1 \cos\omega_1\tau_1 - (n_{20} - n_{22}\omega_1^2)\sin\omega_1\tau_1 = \omega_1^3 - m_{21}\omega_1, \end{cases} \tag{5}$$

which follows that

$$\omega^6 + e_{22}\omega_1^4 + e_{21}\omega_1^2 + e_{20} = 0 \tag{6}$$

where $e_{20} = m_{20}^2 - n_{20}^2, e_{21} = m_{21}^2 - n_{21}^2 - 2m_{20}m_{22} + 2n_{20}n_{22}, e_{20} = m_{22}^2 - n_{22}^2 - 2m_{21}$.

Let $\omega_1^2 = v_1$, then (6) can be written as

$$v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20} = 0. \tag{7}$$

Denote $f_1(v_1) = v_1^3 + e_{22}v_1^2 + e_{21}v_1 + e_{20}$, since $f_1(0) = e_{20}, \lim_{v_1\to+\infty} f_1(v_1) = +\infty$. We get

$$f_1'(v_1) = 3v_1^2 + 2e_{22}v_1 + e_{21},$$

the following lemma is given.

Lemma 1 *For the transcendental equation (7), we have the following results:*
(1) If (H11): $e_{20} \geq 0$ *and* $\Delta = e_{22}^2 - 3e_{21} \leq 0$*, then (7) has no positive root;*
(2) If (H12): $e_{20} \geq 0$, $\Delta = e_{22}^2 - 3e_{21} > 0$, $v_1^0 = \frac{-e_{21}+\sqrt{\Delta}}{3} > 0$, $f_1(v_1^0) \leq 0$*, then (7) has positive root;*
(3) If (H13): $e_{20} < 0$*, then (7) has positive root.*

Assume that (7) has positive roots. Without loss of generality, we suppose that it has three positive roots, namely, v_{11}, v_{12} and v_{13}.

For $k = 1, 2, 3$, from (6) ,one has three positive roots $\omega_{1k} = \sqrt{v_{1k}}$ and the corresponding critical value of time delay $\tau_{1k}^{(j)}$ is given by

$$\tau_{1k}^{(j)} = \frac{1}{\omega_{1k}} \arccos\{\frac{A_{24}\omega_{1k}^4 + A_{22}\omega_{1k}^2 + A_{20}}{B_{24}\omega_{1k}^4 + B_{22}\omega_{1k}^2 + B_{20}}\} + \frac{2\pi j}{\omega_{1k}}, k = 1, 2, 3; j = 0, 1, 2, \cdots \quad (8)$$

where $A_{20} = -n_{20}m_{20}$, $A_{22} = n_{20}m_{22} + n_{20}m_{20} - n_{21}m_{21}$, $A_{24} = n_{21} - m_{22}n_{22}$, $B_{20} = n_{20}^2$, $B_{22} = n_{21}^2 - 2n_{20}n_{22}$, $B_{24} = n_{22}^2$.

Therefore, $\pm\omega_{1k}$ is a pair of purely imaginary roots of (4) with $\tau_1 = \tau_{1k}^{(j)}$, satisfying $\tau_{10} = \min_{k\in 1,2,3} \tau_{1k}^{(0)}$,$\omega_{10} = \omega_{1k}$.

According to the Hopf Bifurcation Theorem [16], we need to verify the transversality condition. Differentiating (4) with respect to τ_1, and noticing that λ is a function of to τ_1, we obtain

$$(\frac{d\lambda}{d\tau_1})^{-1} = \frac{-(3\lambda^2 + 2m_{22}\lambda + m_{21})}{\lambda(\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20}} + \frac{2n_{22}\lambda + n_{21}}{\lambda(n_{22}\lambda^2 + n_{21}\lambda + n_{20})} - \frac{\tau_1}{\lambda}. \quad (9)$$

which leads to

$$\begin{aligned} Re(\frac{d\lambda}{d\tau_1})^{-1} &= Re(\frac{-(3\lambda^2 + 2m_{22}\lambda + m_{21})}{\lambda(\lambda^3 + m_{22}\lambda^2 + m_{21}\lambda + m_{20}})_{\lambda=i\omega_{10}} + Re(\frac{2n_{22}\lambda + n_{21}}{\lambda(n_{22}\lambda^2 + n_{21}\lambda + m_{20}})_{\lambda=i\omega_{10}} \\ &= \frac{3\omega_{10}^4 + 2(m_{22}^2 - 2m_{21})\omega_{10}^2 + m_{21}^2 - 2m_{20}m_{22}}{(\omega_{10}^3 - m_{21}\omega_{10})^2 + (m_{20} - m_{22}\omega_{10}^2)^2} - \frac{2n_{22}^2\omega_{10}^2 + n_{21}^2 - 2n_{20}n_{22}}{(n_{22}\omega_{10}^2 - n_{20})^2 + n_{21}^2\omega_{10}^2}. \end{aligned}$$

From (5), we can obtain

$$(\omega_{10}^3 - m_{21}\omega_{10})^2 + (m_{20} - m_{22}\omega_{10}^2)^2 + = (n_{22}\omega_{10}^2 - n_{20})^2 + n_{21}^2\omega_{10}^2. \quad (10)$$

Noting that $\{\frac{d(Re\lambda)}{d\tau_1}\}_{\lambda=i\omega_{10}}$ and $\{Re(\frac{d\lambda}{d\tau_1})^{-1}\}_{\lambda=i\omega_{10}}$ have the same sign, then

$$\begin{aligned} sign\{\frac{d(Re\lambda)}{d\tau_1}\}_{\lambda=i\omega_{10}} &= sign\{Re(\frac{d\lambda}{d\tau_1})^{-1}\}_{\lambda=i\omega_{10}} \\ &= \frac{3(\omega_{10}^2)^2 + 2e_{22}\omega_{10}^2 + e_{21}}{n_{21}^2\omega_{10}^2 + (n_{20} - n_{22}\omega_{10}^2)^2} \\ &= \frac{f_1'(\omega_{10}^2)}{n_{21}^2\omega_{10}^2 + (n_{20} - n_{22}\omega_{10}^2)^2} \neq 0. \end{aligned}$$

By the above analysis , we have the following results.

Theorem 2 *When* $\tau_2 = 0$*, system (1) has the following results:*
(1) The positive equilibrium $E^0(x^0, S^0, I^0)$ *is asymptotically stable for all* $\tau_1 \epsilon [0, \tau_{10})$ *hold if (H11) hold.*
(2)The positive equilibrium $E^0(x^0, S^0, I^0)$ *is asymptotically stable is unstable for* $\tau_1 > \tau_{10}$ *if (H12) or (H13) hold.*

Case 2: When $\tau_2 \epsilon [0, \tau_{20})$ and $\tau_1 > 0$, $\tau_1 \neq \tau_2$, we regard τ_2 as the parameter and discuss (4) with τ_1 in its stable interval.
Let $i\omega(\omega > 0)$ be the root of (4), then we have

$$\omega_1^6 + e_{32}\omega_1^4 + e_{31}\omega_1^2 + e_{30} + (c_{34}\omega_1^4 + c_{32}\omega_1^2)\cos\omega_1\tau_2 + (c_{35}\omega_1^5 + c_{33}\omega_1^3 + c_{31}\omega_1)\sin\omega_1\tau_2 = 0, \quad (11)$$

where $e_{30} = m_0^2 - n_0^2$, $e_{31} = m_1^2 - n_1^2 + p_1^2 - q_1^2 + 2n_0n_2 - 2m_0m_2$, $e_{32} = m_2^2 - n_2^2 + p_2^2 - 2m_1$, $c_{31} = 2p_1m_0 - 2n_0q_1$, $c_{32} = 2m_1p_1 - 2n_1q_1 - 2m_0p_2$, $c_{35} = -2p_1$,$c_{33} = 2n_2q_1 - 2p_1m_2 + 2m_1p_1$, $c_{34} = 2m_2p_2 - 2p_2$.

In order to give the main results, we make the following assumption.

(H31): Equation (11) has at least finite positive root is obtained.

Suppose that (H31) holds, we denote the positive roots of (11) as $\omega_1^{(i)}(i = 1,2,3,4,5,6)$. For every $\omega_1^{(i)}(i = 1,2,3,4,5,6)$, the corresponding critical value of time delay $\tau_{1i}^{(j)}$ is

$$\tau_{1i}^{(j)} = \arccos\{\frac{E_{31}E_{34} + E_{32}E_{33}}{E_{31}^2 + E_{32}^2} + 2\pi j\}. \tag{12}$$

Let $\tau'_{10} = \min\{\tau_{1i}^{(0)} | i = 1,2,3,4,5,6; j = 0,1,2,\cdots\}$, ω'_{10} is the corresponding root of (11).

In order to verify the transversality condition,we differentiate the two sides of (3) with respect to τ_1.

Taking the derivative of λ with refer to τ_1 in (3) and substituting $\lambda = i\omega'_{10}$, then we have

$$\{(Re\frac{d\lambda}{d\tau_1})^{-1}\}_{\lambda=i\omega_{10}} = Re(\frac{A' + B'i}{C' + D'i}) = \frac{A'C' + B'D'}{(C')^2 + (D')^2},$$

where

$$\begin{aligned}
A' =& -3\omega'_{10} + 2n_2\omega'_{10}\sin\omega'_{1_0}\tau'_{10} + (n_1 + n_2(\omega'_{10})^2\tau'_{10} - n_0\tau'_{10})\cos\omega'_{10}\tau'_{10} + \sin\omega'_{10}\tau_2(p_1\omega'_{10}\tau_2 + 2p_2\omega'_{1_0} - (q_1 \\
&+ q_0\tau_2)\sin\omega'_{10}\tau'_{10} - q_1\tau_2\omega'_{10}\cos\omega'_{10}\tau'_{10}) + \cos\omega'_{10}\tau_2((p_1 - p_0\tau_2 + p_2\tau_2(\omega'_{10})^2) + (q_1 - \tau_2 q_0))\cos\omega'_{10}\tau'_{10} \\
&- q_1\tau_2\omega'_{10}\sin\omega'_{10}\tau'_{10}),\\
B' =& 2m_2\omega'_{10} - n_1\sin\omega'_{10}\tau'_{10} + 2n_2\omega'_{10}\cos\omega'_{10}\tau'_{10} + \sin\omega'_{10}\tau_2(-p_1 + p_0\tau_2 - p_2\tau_2(\omega'_{10})^2 + q_1\tau_2\omega'_{10}\sin\omega'_{10}\tau'_{10} \\
&+ (-q_0 + \tau_2 q_0)\cos\omega'_{10}\tau'_{10}) + \cos\omega'_{10}\tau_2(2p_2\omega'_{10} - p_1\omega'_{10}\tau_2 + (-q_0 + \tau_2 q_0)\sin\omega'_{10}\tau'_{10} - q_1\tau_2\cos\omega'_{10}\tau'_{10}),\\
C' =& (n_0\omega'_{10} - n_2(\omega'_{10})^3)\sin\omega'_{10}\tau'_{10} - n_1(\omega'_{10})^2\cos\omega'_{10}\tau'_{10} - \sin\omega'_{10}\tau_2(-q_1(\omega'_{10})^2\sin\omega'_{10}\tau'_{10} + q_0\omega'_{10}\cos\omega'_{10}\tau'_{10}) \\
&+ \cos\omega'_{10}\tau_2(-q_1(\omega'_{10})^2\cos\omega'_{10}\tau'_{1_0} + q_0\omega'_{10}\sin\omega'_{10}\tau'_{10}),\\
D' =& n_0\omega'_{10}\cos\omega'_{10}\tau'_{10} - n_2(\omega'_{10})^3\cos\omega'_{10}\tau'_{10} + n_1(\omega'_{10})^2\sin\omega'_{10}\tau'_{10} + \cos\omega'_{10}\tau_2(q_1(\omega'_{10})^2\sin\omega'_{10}\tau'_{10} + q_0\omega'_{10} \\
&\cos\omega'_{10}\tau'_{10}) + \sin\omega'_{10}\tau_2(-q_0\omega'_{10}\sin\omega'_{10}\tau'_{1_0} + q_1(\omega'_{10})^2\cos\omega'_{10}\tau'_{10}).
\end{aligned}$$

Obviously, if the following condition holds:

$$(H32): A'C' + B'D' \neq 0.$$

Then $\{\frac{d(Re\lambda)}{d\tau_1}\}_{\lambda=i\omega_{10}} \neq 0$ is given. Through the above analysis, the following results exist.

Theorem 3 *For system (1), while $\tau_1 > 0$, $\tau_2\epsilon[0,\tau_{20})$ and $\tau_1 \neq \tau_2$, we suppose that the conditions (H31) and (H32) hold, then the positive equilibrium $E^0(x^0, S^0, I^0)$ is asymptotically stable for all $\tau_1\epsilon[0,\tau'_{10})$ and unstable for $\tau_1 > \tau'_{10}$. Furthermore, the system (1) undergoes a Hopf bifurcation at the positive equilibrium $E^0(x^0, S^0, I^0)$ when $\tau_1 = \tau'_{10}$.*

3 Numerical simulations

In this section, by illustrating the analytical results and the corresponding waveform, we use Matlab 7.0 to describe some numerical simulations and draw phase plots of system (1).

Let $r = 0.98, a_{11} = 0.11, a_{12} = 1.4, m = 0.08, a_{21} = 1, a_{22} = 0.9, r_1 = 0.11, \beta = 0.5, r_2 = 0.4$. Then, we have the following particular example of system (1):

$$\begin{cases}\dot{x}(t) = x(t)[0.96 - 0.11x(t-\tau_1) - \frac{1.4x^2(t)S(t)}{S^2(t)+0.08x^2(t)},\\ \dot{S}(t) = \frac{0.9x^2(t-\tau_2)S(t-\tau_2)}{S^2(t-\tau_2)+0.08x^2(t-\tau_2)} - 0.11S(t) - 0.5S(t)I(t),\\ \dot{I}(t) = 0.5S(t)I(t) - 0.4I(t).\end{cases} \tag{13}$$

It is not difficult to verify that the (H11) holds, we can get the positive equilibrium $E^0(0.8, 0.5319, 0.5485)$.

For $\tau_1 > 0$, $\tau_2 = 0$, there is $\omega_{10} = 0.2457$, $\tau_{10} = 6.77$. From Theorem 2, it is easy to know that the positive equilibrium E^0 is asymptotically stable when $\tau_1\epsilon[0,\tau_{10})$. When the time delay τ_1 passes through the critical value τ_{10}, the positive equilibrium E^0 will lose its stability and a Hopf bifurcation occurs. And a family of periodic solutions

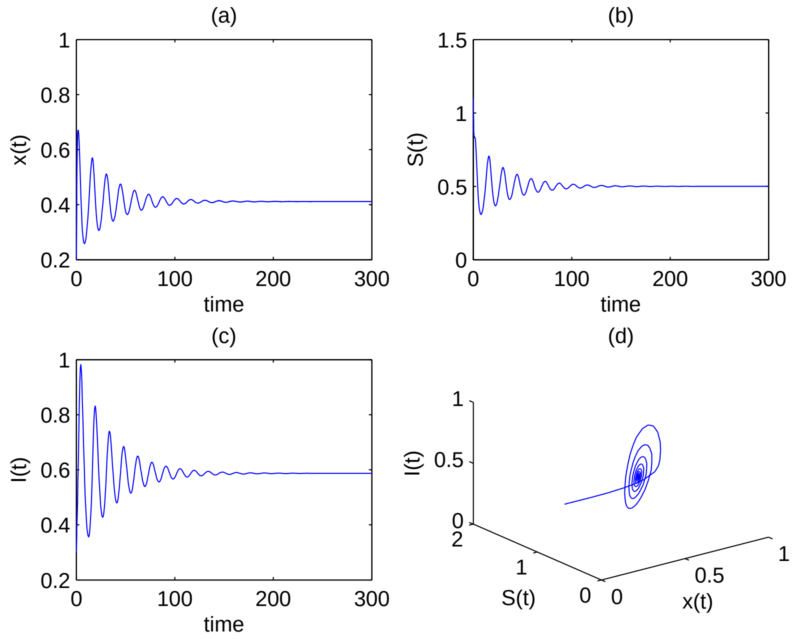

Figure 1: The positive equilibrium E^0 of system (1) is asymptotically stable when $\tau_2 = 0$, $\tau_1 = 2 < \tau_{10} = 6.77$. (a) time series $x(t)$; (b) time series $S(t)$; (c) time series $I(t)$; (d)trajectory.

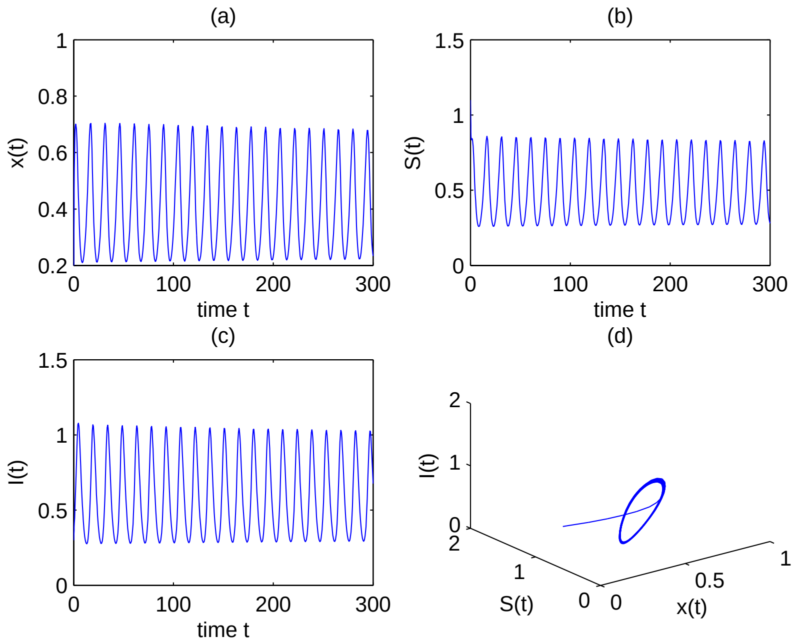

Figure 2: The positive equilibrium E^0 undergoes a Hopf bifurcation when $\tau_2 = 0$, $\tau_1 = 12 > \tau_{10} = 6.77$.(a) time series $x(t)$; (b) time series $S(t)$; (c) time series $I(t)$; (d) trajectory.

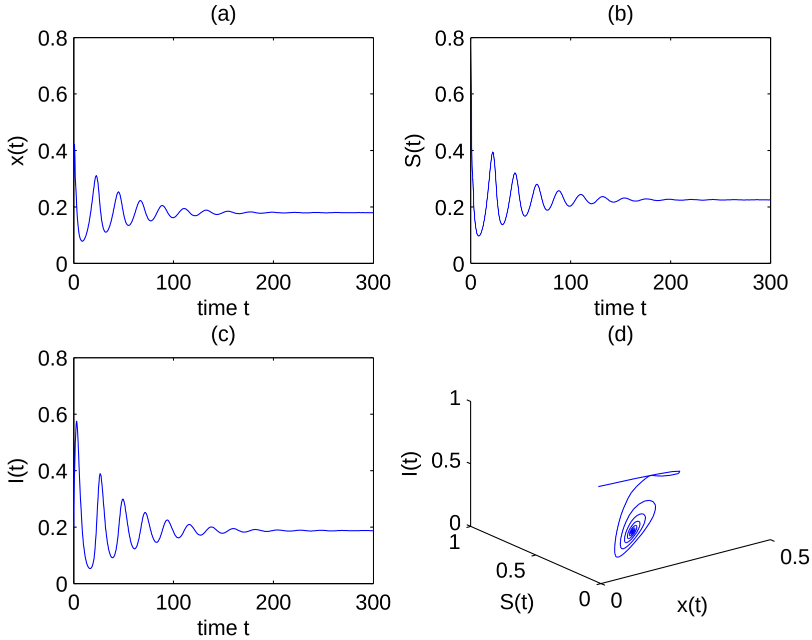

Figure 3: The positive equilibrium E^0 is asymptotically stable when $\tau_2' = 2$, $\tau_1 = 1.1 < \tau_{10}' = 2.4708$.(a) time series $x(t)$; (b)time series $S(t)$; (c) time series $I(t)$; (d)trajectory.

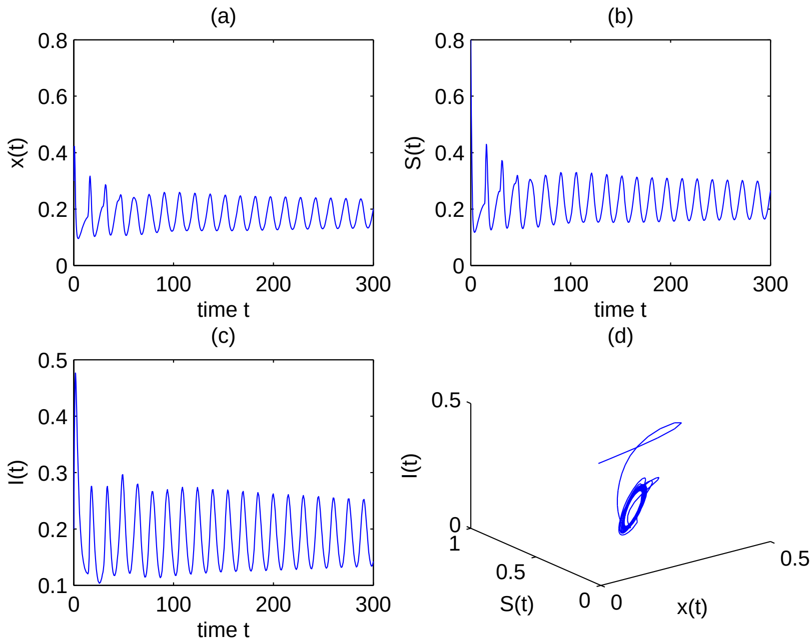

Figure 4: The positive equilibrium E^0 undergoes a Hopf bifurcation when $\tau_2' = 2$, $\tau_1 = 2.6 > \tau_{10}' = 2.4708$.(a) time series $x(t)$; (b) time series $S(t)$; (c) time series $I(t)$; (d) trajectory.

bifurcate from the positive equilibrium E^0. The corresponding waveform and the phase plots are depicted in Figure 1 and Figure 2.

For $\tau_1 > 0$, $\tau_2' = 2\epsilon[0, \tau_{20})$, we have $\omega_{10}' = 0.1269$, $\tau_{10}' = 2.4708$. According to Theorem 3, E^0 is asymptotically stable when $\tau_1 \epsilon [0, \tau_{10}')$ and unstable when $\tau_1 > \tau_{10}'$ .The corresponding waveform and the phase plots are depicted in Figure 3 and Figure 4.

4 Conclusions

In this paper, we have focused our study on incorporating Holling type III ratio-dependent functional response and two time delays into the predator-prey system with a transmissible disease spreading among the predator population.

By analyzing the corresponding characteristic equation, the local stability and the existence of Hopf bifurcation at the coexistence equilibrium is investigated. By using the normal form theory and center manifold theorem, the explicit formulas which determine stability of the bifurcating periodic solution is derived. The numerical results which the Hopf bifurcation is supercritical and the bifurcation periodic solutions are stable are in accord with the theoretical analysis.

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