ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.23(2017) No.2, pp.85-87

Generalized Reflection Function and Periodic Solution of Abel Equation

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(Received 10 August 2013, accepted 5 March 2014)

Abstract: In order to solve the problem associated with Abel equation's periodic solution and stability, this paper proposes a method to find its Poincaré mapping through Abel' s generalized reflection function. The result indicates that based on differential system's generalized reflection function, the condition that enables Abel equation to have the linear generalized reflection function is obtained, the linear generalized reflection function of Abel equation is derived, Poincaré mapping of Abel equation and the condition that enables Abel equation to have periodic solution and stability are obtained.

Keywords: Abel equation; generalized reflective function; periodic solution; stability; Poincaré mapping.

1 Introduction

As is well known, to study the properties of differential system

$$x' = X(t, x) \tag{1}$$

is not easy. When it is $2\omega-$periodic with respect to t, i.e.,$X(t + 2\omega, x) = X(t, x)$($\omega$ is a positive constant),to study the behavior of solutions of (1), we could use, as introduce in [1], the Poincar mapping for many systems which are not integrable in finite terms. In the 1980s, the Russian mathematician, Mironenko [2-3], first established the theory of reflective functions **(RF)**. Since then, a quite new method to study (1) has been found. If $F(t, x)$ is **RF** of system (1), then its Poincaré mapping can be expressed by:$T(x) = F(-\omega, x)$. So now we only need to seek **RF**. The literatures [4-11] are devoted to investigations of qualitative behavior of solutions of differential systems with help of reflective functions. The notion about reflective function has been extended in [7]. In the present section, we introduce the concept of the generalized reflection function **(GRF)**, which will be used throughout the rest of this paper.

Now consider the system (1) with a continuously differentiable right-hand side and with a general solution $\varphi(-t; t_0, x_0)$. For each such system, the **(GRF)** of (1) is defined as $F(t, x) = \varphi(\alpha(t); t, x), (t, x) \in D \subset R \times R^n$. Where $\alpha(t)$ is a continuously differentiable function such that $\alpha(\alpha(t)) = t, \alpha(0) = 0$.Then for any solution $x(t)$ of (1), we have $F(t, x(t)) = x(\alpha(t))$ and $F(\alpha(t), F(t, x)) \equiv F(0, x) \equiv x$.By the definition in [7], a continuously differentiable vector function $F(t, x)$ on$R \times R^n$is called **(GRF)** if and only if it is a solution of the Cauchy problem

$$F_t(t, x) + F_x(t, x)X(t, x) = \alpha'(t)X(\alpha(t), F(t, x)), F(0, x) = x. \tag{2}$$

The (2) is called a basic relation(BR). Besides this, suppose system (1) is $2\omega-$periodic with respect to t, and $F(t, x)$ is its**GRF**,if there exists a number η on R such that $\alpha(\eta) = 2\omega + \eta$, then $T(x) = F(\eta, x) = \varphi(\alpha(\eta); \eta, x)$ is the Poincaré mapping of (1) over the period $[\eta, 2\omega + \eta]$. So, for any solution $x(t)$ of (1) defined on $[\eta, 2\omega + \eta]$, it will be $2\omega-$periodic if and only if $F(\eta, x) = x$. This called a basic lemma.

2 Main results

Now, we consider the Abel equation

$$x' = a_1(t)x + a_2(t)x^2 + a_3(t)x^3 \tag{3},$$

where $a_i(t)(i - 1, 2, 3, t \in R)$ are continuously differentiable functions.

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IJNS.2017.04.15/955

Theorem 1 *$F(t,x) = M(t)x$ is the **GRM** of the Abel equation (3), if and only if*

$$a_2(t)e^{k(t)} = \alpha'(t)a_2(\alpha(t)), a_3(t)e^{2k(t)} = \alpha'(t)a_3(\alpha(t)). \quad (4)$$

*where $k(t) = \int_0^t (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau$. At this time, the **GRM** of the Abel equation (3) is*

$$F(t,x) = e^{-k(t)}x. \quad (5)$$

Proof. Let $F(t,x) = M(t)x$ be the **GRM** of the Abel equation (3). Then, by the basic relation (2), we get

$$M'(t)x + M(t)X(t,x) = \alpha'(t)X(\alpha(t), M(t)x)$$

i.e.,

$$M'(t)x + M(t)[a_1(t)x + a_2(t)x^2 + a_3(t)x^3] = \alpha'(t)[a_1(\alpha(t))M(t)x + a_2(\alpha(t))M^2(t)x^2 + a_3(\alpha(t))M^3(t)x^3]$$
$$M(0)x = x.$$

Equating the coefficients of the same powers, we obtain

$$M'(t) + M(t)a_1(t) = \alpha'(t)a_1(\alpha(t))M(t) \quad (6)$$
$$M(t)a_2(t) = \alpha'(t)a_2(\alpha(t))M^2(t) \quad (7)$$
$$M(t)a_3(t) = \alpha'(t)a_3(\alpha(t))M^3(t) \quad (8)$$
$$M(0) = 1$$

From (6), we get

$$M(t) = exp(-\int_0^t (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau).$$

Let us denote

$$k(t) = \int_0^t (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau,$$

then

$$M(t) = e^{-k(t)}.$$

Substituting it into (7) and (8), we obtain the identity (4), therefore, the necessary of the present theorem holds. On the other hand, since

$$k(t) = \int_0^t (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau,$$

we differentiating it, then

$$k'(t) = a_1(t) - \alpha'(t)a_1(\alpha(t))$$

So

$$F_t(t,x) + F_x(t,x)X(t,x) = xe^{-k(t)}(-k'(t)) + e^{-k(t)}[a_1(t)x + a_2(t)x^2 + a_3(t)x^3]$$
$$= xe^{-k(t)}\alpha'(t)a_1(\alpha(t)) + e^{-k(t)}[a_2(t)x^2 + a_3(t)x^3].$$

By the identity (4) and a straightforward computation, we get

$$F_t(t,x) + F_x(t,x)X(t,x) = \alpha'(t)X(\alpha(t), F(t,x)).$$

Therefore, identity (2) is valid. It shows that $F(t,x) = xe^{-k(t)}$ is the **GRM** of the Abel equation (3). Then the sufficiency of the present theorem holds. ■

Theorem 2 *Suppose that the identity (4) is satisfied, $a_i(t)$ is continuous function and $a_i(t+2\omega) = a_i(t)(i = 1,2,3; t \in R)$. Moreover, all the solutions of the Abel equation (3) are meaningful in $[-\omega, \omega]$, then*

(i) When $\int_0^\omega (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau < 0$, the Abel equation (3) has just one $2\omega-$periodic stability solution;

(ii) When $\int_0^\omega (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau > 0$, the Abel equation (3) has just one $2\omega-$periodic instability solution;

(iii) When $\int_0^\omega (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau = 0$, all the solutions of the Abel equation (3) over the period $[-\varpi, \varpi]$ are $2\omega-$periodic.

Proof. By Theorem 1, the Poincaré mapping of the Abel equation (3) is

$$T(x) = F(\alpha(\omega), x) = xe^{-k(\alpha(\omega))}$$

Here

$$k(\omega) = \int_0^{\omega} (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau,$$

$$k(\alpha(\omega)) = \int_0^{\alpha(\omega)} (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau. \tag{9}$$

Let $\tau = \alpha(u)$, then $d\tau = \alpha'(u)du$. Besides this,$\alpha(0) = 0, \alpha(\alpha(t)) = t, \alpha'(\alpha(t))\alpha'(t) = 1, \alpha'(\alpha(t)) = \frac{1}{\alpha(t)}$, so $\alpha'(\tau) = \alpha'(\alpha(u)) = \frac{1}{\alpha'(u)}, \alpha(\tau) = \alpha(\alpha(u)) = u$. Substituting them into (9) we get

$$k(\alpha(\omega)) = \int_0^{\omega} (a_1(\alpha(u)) - \frac{1}{\alpha'(u)} a_1(u))\alpha'(u)\, du = -k(\omega),$$

$$T(x) = F(\alpha(\omega), x) = xe^{-k(\alpha(\omega))} = xe^{k(\omega)} = x.$$

when $k(\omega) \neq 0$, i.e.,$\int_0^{\omega} (a_1(\tau) - \alpha'(\tau)a_1(\alpha(\tau)))\, d\tau \neq 0$, the Abel equation (3) has just one solution, otherwise, it will has an infinite number of solution. By the basic lemma introduced in Introduction and the stability theorems about periodic solution, we can easily deduce the conclusions of this theorem. ■

3 Conclusions

By generalized reflective function periodic solution and stability of Abel equation are obtained. Because it is simple and feasible to find Poincaré mapping through the generalized reflection function,so, the obtained results on the Abel equation's periodic solution and stability are also of significance to the study of the other differentia systems.The KM-BBM and LGH equations are solved analytically and 1-soliton solutions are obtained. This paper will study a generalized form of the NLSE that will be useful in various areas of physical sciences and engineering. This method can also be applied to other kinds of nonlinear partial differential equations

References

[1] V. I. Arnol'd. Ordinary Differential Equation. *Science Press, Moscow*, 1971.

[2] V. I. MIRONENKO. On the method that allows one to determine the initial data of periodic solution of differential systems and to compare the mappings for a period. *Differ. Equ*, 14(11)(1980):1985-1994.

[3] V. I. MIRONENKO. Reflecting function and periodic solution of the differential system. *University Press,Minsk*, 1986:12-45.

[4] Zh. X. Zhou. On the reflective function of polynomial differential system triangles. *J. Math. Anal. Appl*, 278(1)(2003):18-26.

[5] J. Zhou. Differential systems with the same reflecting functions, Applied mathematics and computation. *Appl. Math. Comput*, 218(7)(2011):3144-3148.

[6] Zh. X. Zhou. On the qualitative behavior of periodic solutions of periodic differential systems. *J. Comput. Appl. Math*, 218(7)(2011):3144-3148.

[7] Ch. Z. Sun Properties and application of generalized reflection function. *Math. Pract. Theo*, 40(10)(2010): 222-228..

[8] J. Zhou. Generalized reflective function and periodic solution of differential systems with small parameter. *Appl. Math. Comput*, 218(13)(2012):7204-7211.

[9] E.V.Musafirov. Differential systems, the mapping over period for which is represented by product of three exponential matrixes. *J. Math. Anal. Appl*, 329(1)(2007):647-654.