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Solving the Duffing Equation with the Homotopy Mapping Method

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Abstract: In this paper, we firstly use the homotopy mapping method to obtain the approximate solutions which is in a concise form. To begin with, we standardize the Duffing equation and build a homotopy mapping. According to the method of variation of constant and Cramers Rule, we get the approximate solution of the equation which is limited to the original conditions. The period and shape of this approximate solution is similar to the accurate solution which is given by Matlab. From this paper, we verify a method to solve nonlinear differential equation, which is efficient, shortcut, accurate.

Keywords: homotopy mapping; method of variation of constant; Cramers rule; approximate solutions

1 Introduction

The rise of nonlinear science was a glorious page in the history of scientific development in twentieth Century. Its influence not only promoted the development of mathematics, mechanics and physics, but also promoted the development of almost all natural science, engineering technology and social science. Therefore, nonlinear science had become a very important new discipline which covered many specialties. Nonlinear science aimed at revealing the general character, basic features and the law of motion of the nonlinear system, in order to achieve the ultimate goal of controlling and utilizing the law.

The nonlinear Duffing equation [1] in single degree freedom system was a kind of vibration equation with nonlinear property , which was widely used in the fields of electrical and mechanical. Therefore, the research of the properties of solutions Duffing equation solutions was attracting many scholars.

Since the homotopy analysis method was intruduced by Liao in 1992 [2], it has been successfully employed to get the approximate analytical solutions for many kinds of nonlinear problems, such as boundary-layer flows over an impermeable stretched plate[3], the KdV-type equations [4–6], solitary-wave solutions of CH equation [7] and so on.

In this paper, we use the homotopy mapping method [2] to study the conservative Duffing equation with exact period, and obtain the approximate solution in the form of polynomials. Using numerical simulation to prove that the method is precise and feasible. From this paper, we find that the homotopy mapping method is efficient and simple in finding a approximate solution of general nonlinear differential equations.

2 Definition of Homopoty method and Duffing equation

2.1 Duffing equation

The form of many single freedom degree conservative system free vibration equations is:

$$f(x) = -\frac{d^2x}{dt^2}. \tag{1}$$

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There $f(x)$ is a nonlinear function, $\frac{d^2x}{dt^2}$ is the acceleration of the system, $f(x)$ is the resilience. Supposing that $x = x_0$ is the balance position of the system, then$f(x_0) = 0$. In addition, it is assumed that$f(x)$ is the analytic function at the point of $x = x_0$, which can be written as a Taylor series:

$$f(x) = k_1(x - x_0) + k_2(x - x_0)^2 + k_3(x - x_0)^3 + \cdots \tag{2}$$

in the formula $k_n = \frac{1}{n!}\frac{d^n f}{dx^n}(x_0)$. Then, we can adapt the formula (1) as below:

$$\frac{d^2x}{dt^2} + k_1(x - x_0) + k_2(x - x_0)^2 + k_3(x^* - x_0^*)^3 + \cdots = 0. \tag{3}$$

Equation (3) describes the system movement beside the balance position. For convenience, let $u = x - x_0$, then change (3) as the following:

$$\frac{d^2u}{dt^2} + k_1u + k_2u^2 + k_3u^3 + \cdots = 0. \tag{4}$$

A special form in Eq. (4) is:

$$\frac{d^2u}{dt^2} + k_1u + k_2u^3 = 0. \tag{5}$$

This special form is what we will research. In Eq. $(5), k_1 > 0$ and k_2 maybe positive or negative. We usually name equation (5) as the Duffing equation. After make nondimensionalization of (5), we get:

$$u'' + u + \varepsilon u^3 = 0. \tag{6}$$

We noticed the ε is dimensionless quantity, which is a measure of the intensity of the nonlinearity. In this paper, we mainly find the approximate solution of the equation (6).

2.2 Homotopy mapping method

Given the nonlinear differential equation $A(u) - f(r) = 0$, A is a general differential operator, $f(r)$ is analytic functions which we already known. In general, the operator A can be decomposed into two parts : linear part L and nonlinear part N , so the equation $A(u) - f(r) = 0$ can be written as $L(u) + N(u) - f(r) = 0$. Now we established the homotopy mapping:

$$H(u,p) = L(u) - L(v) + p[L(v) + N(u) - f(r)], \tag{7}$$

there p is a parameter, v is an auxiliary function, which meets the formula $L(v) + N(v) = 0$. According to formula (7), we get

$$H(u,0) = L(u) - L(v) = 0,$$
$$H(u,1) = A(u) - f(r) = 0.$$

As p from 0 to 1, $H(u,p)$ changes from $L(u) - L(v)$ to $A(u) - f(r)$, this is called Homotopy transformation.

Let

$$u(t) = \sum_{i=0}^{\infty} u_i p^i = u_0 + u_1p + u_2p^2 + u_3p^3 + \cdots \tag{8}$$

as the answer of $H(u,p) = 0$. When p changes from 0 to 1, we can get the approximate solution of the formula $A(u) - f(r) = 0$ and

$$u(t) = u_0 + u_1 + u_2 + u_3 + u_4 + \cdots \tag{9}$$

3 Solving the Duffing equation with the homotopy mapping method

Firstly, we establish a homotopy mapping for (6)

$$H(u,p) = L(u) - L(v) + p[L(v) + N(u). \tag{10}$$

The linear part of (10) is $L(u) = u'' + u$. The nonlinear part of (10) is $N(u) = \varepsilon u^3$.

To ensure the conciseness of this method, we take

$$u(t) = u_0 + u_1 p + u_2 p^2 + u_3 p^3. \tag{11}$$

Substituting (11) to (10) we get the equation

$$H(u,p) = L(u_0) + pL(u_1) + p^2 L(u_2) + p^3 L(u_3) - L(v_1) + p[L(v_1) + \varepsilon(u_0 + u_1 p + u_2 p^2 + u_3 p^3)^3] = 0. \tag{12}$$

According to linear operator property, we choose

$$v_1 = A\cos t + B\sin t. \tag{13}$$

Substituting (13) to the (12) , using method of undetermined coefficients, we let the coefficients of p^0, p^1, p^2 and p^3 be 0, we get

$$p^0 : L(u_0) = L(v), \tag{14}$$

$$p^1 : L(u_1) + L(v_1) + \varepsilon u_0^3 = 0, \tag{15}$$

$$p^2 : L(u_2) + 3\varepsilon u_0^2 u_1 = 0, \tag{16}$$

$$p^3 : L(u_3) + 3\varepsilon(u_0 u_1^2 + u_0^2 u_2) = 0. \tag{17}$$

We get the solution by using the method of variation of constant and Cramer's Rule

$$u_0 = A\cos t + B\sin t, \tag{18}$$

and

$$u_1 = \sin t\left(\frac{\cos 2t}{2} - \frac{3t}{4} - \frac{\sin 2t}{4} + \frac{sin4t}{16}\right) - \cos\left(\frac{\cos 2t}{2} - \frac{3t}{4} + \frac{\sin 2t}{4} + \frac{sin4t}{16}\right), \tag{19}$$

and

$$u_2 = \frac{\cos t}{32}\left(\frac{99\cos 2t}{128} - \frac{51t}{64} - 3\cos 4t\right) + \frac{\cos 6t}{128} + \frac{39\sin 2t}{128} + \frac{9\sin 4t}{128} + \cdots . \tag{20}$$

Substituting the u_0, u_1, u_2, u_3 into (11) , we can obtain the approximate solutions of the equations.

4 Numerical simulation

We draw the approximate solution by using the Matlab, and compare the numerical solution with computer simulation (see Figs. 1-4.)

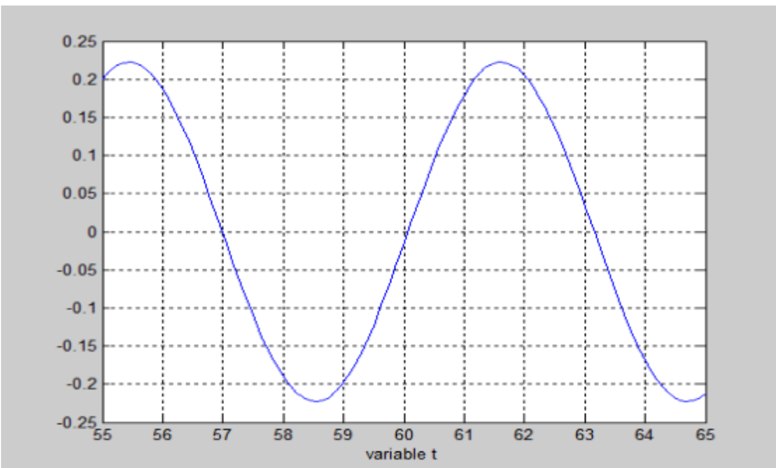

Figure 1: Approximate solution

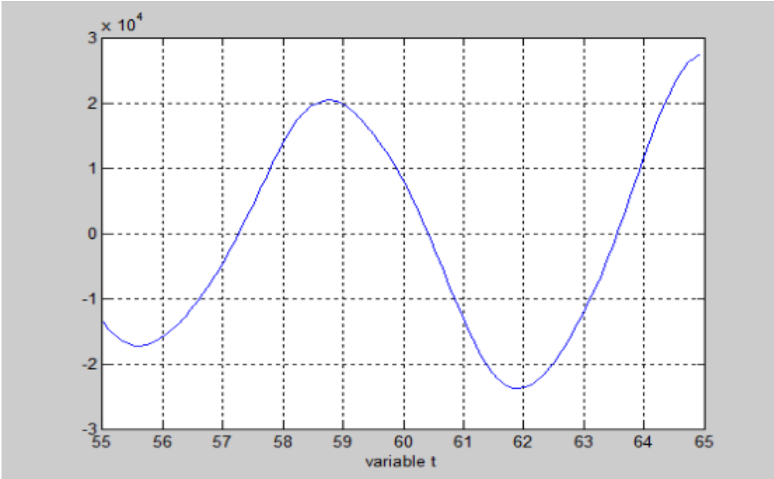

Figure 2: ode45 numerical simulation solution

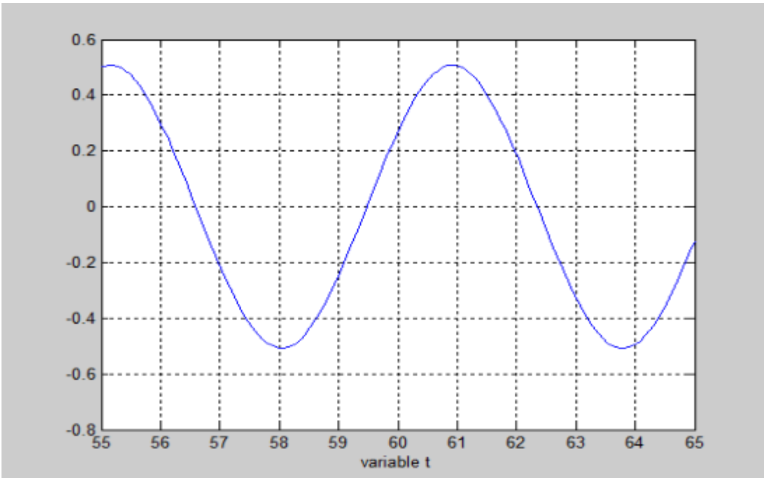

Figure 3: ode45 numerical simulation solution

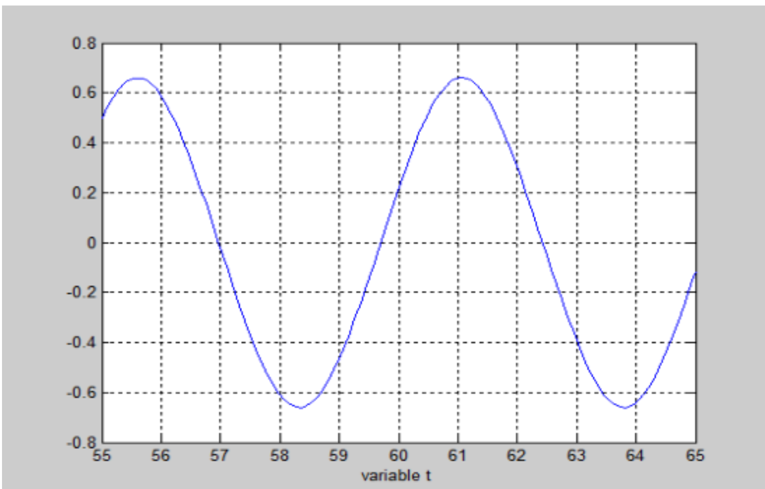

Figure 4: ode45 numerical simulation solution

5 Conclusions

In this paper, we transfer solving nonlinear differential equation to solving linear differential equation. From the above graph, we can conclude that the approximate solution is similar to the numerical simulation solution. Hence, Homotopy mapping method is an efficient and relatively accurate way to solve Duffing equation.

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