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Existence of Solution for a Second-Order Neutral Differential Equation with State Dependent Delay and Non-instantaneous Impulses

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Abstract: In this paper existence of mild solution of a class of second order partial neutral differential equation with state dependent delay and non-instantaneous impulses is investigated. Hausdorff measure of noncompactness and Darbo Sadovskii fixed point theorem are used to prove the existence. Also, some restrictive conditions such as the compactness assumption on the associated cosine or sine family of operators and the Lipschitz conditions on the nonlinear functions are replaced by simple and natural assumptions. In the last section we also study an example to illustrate the presented result.

Keywords:cosine family; state dependent delay; non-instantaneous impulses; second-order neutral differential equation.

1 Introduction

Neutral differential equations are functional differential equations in which the highest order derivative of the unknown function appear both with and without deviations. Neutral differential equations with unbounded delay appear abundantly as mathematical models in mechanics, electrical engineering, medicine, biology, ecology etc. Hence it is a widely studied topic in several papers and monographs for instance, partial neutral differential equation with unbounded delay arise in the theory of heat conduction of materials with fading memory. For instance, one may see [10],[11],[15],[18],[20],[21],[28-31] and the references cited therein. Second order neutral differential equations model variational problems in calculus of variation and in the study of vibrating masses attached to an electric bar.

Of late, much attention is paid to functional differential equation with state dependent delay. We refer [1-3],[9],[14],[33-35] for details. The literature related to state dependent delay mostly deals with functional differential equations in which the state belongs to a finite dimensional space. As a consequence, the study of partial functional differential equations with state dependent delay is neglected. This is one of the motivations of our paper.

Impulsive differential equations are known for their utility in simulating processes and phenomena subject to short term perturbations during their evolution. Discrete perturbations are negligible to the total duration of the process. We refer [8],[12],[24],[26],[27],[34] regarding discrete impulses. However not instantaneous impulses start abruptly at certain time points and continue their action on specified time intervals. The significance of the study of non-instantaneous impulsive differential equations lies in its diverse fields of applications such as in the theory of stage by stage rocket combustion, maintaining hemodynamical equilibrium etc. One such application is the introduction of insulin in the bloodstream which is abrupt and the consequent absorption which is a gradual process as it remains active for a finite interval of time. Differential equation with non-instantaneous impulses are recently studied by Hernandez et.al [19].

We study the second order partial neutral differential equation with state dependent delay modelled in the form

$$\begin{aligned} \frac{d^2}{dt^2}(x(t)-g(t,x_t)) &= Ax(t)+f(t,x_{\rho(t,x_t)},x'(t)),\ t\in(s_i,t_{i+1}],\ i=0,...,n \\ x_0 &= \phi\in\mathfrak{B}, \\ x'(0) &= \xi\in X, \\ x(t) &= J_i^1(t,x_t),\ t\in(t_i,s_i],\ i=1,2,...,n \\ x'(t) &= J_i^2(t,x_t),\ t\in(t_i,s_i],\ i=1,2,...,n \end{aligned} \tag{1}$$

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where A is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in \mathsf{R}\}$ of bounded linear operators on a Banach space X. The history valued function $x_t : (-\infty, 0] \to X$, $x_t(\theta) = x(t+\theta)$ belongs to some abstract phase space $\mathfrak{B}$ defined axiomatically; $g, f, J_i^1, J_i^2, i = 1, .., n$ are appropriate functions. $0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2, ..., < t_n \leq s_n \leq t_{n+1} = a$ are prefixed numbers.

Recently, second order abstract partial neutral differential equation similar to (1) is extensively studied in [4],[5],[7],[25]. As a matter of fact, in these papers the authors assume severe conditions on the operator family generated by A, which imply that the underlying space X has finite dimension. Thus the equations treated in these works are really ordinary and not partial differential equations. In most of the papers the cosine family generated by the operator A is such that $C(.) \in C([0, a]; \mathcal{L}(X))$ which implies that A is bounded. Hence motivated by this fact and the results in [22] and their various applications we study the existence of mild solution of the partial neutral differential equation of second order with state delay and non-instantaneous impulses.

In this paper we establish the existence of mild solution of the non- instantaneous impulsive partial neutral second order functional differential equation (1), using Hausdorff measure of noncompactness and Darbo Sadovskii fixed point theorem The compactness condition of the operator families generated by A and other restrictive conditions have been omitted. An example is given in the last section to illustrate the result.

2 Preliminaries

In this section some definitions, notations and lemmas that are used throughout this paper are stated. The family $\{C(t) : t \in \mathbb{R}\}$ of operators in $B(X)$ is a strongly continuous cosine family if the following are satisfied:

(a) $C(0) = I$ (I is the identity operator in X);

(b) $C(t+s) + C(t-s) = 2C(t)C(s)$ for all $t, s \in \mathbb{R}$

(c) The map $t \to C(t)x$ is strongly continuous for each $x \in X$.

The one parameter family of operators $\{S(t) : t \in \mathbb{R}\}$ is the sine family associated to the strongly continuous cosine family $\{C(t) : t \in \mathbb{R}\}$ and it is defined as $S(t)x = \int_0^t C(s)xds$, $x \in X$, $t \in \mathbb{R}$.

The operator A is the infinitesimal generator of a strongly continuous cosine family of bounded linear operators $(C(t))_{t\in R}$ and $S(t)$ is the associated sine function. Let $N, \widetilde{N}$ be certain constants such that $\|C(t)\| \leq N$ and $\|S(t)\| \leq \widetilde{N}$ for every $t \in J = [0, a]$. For more details see books by Goldstein[16] and Fattorini[13]. In this work we use the axiomatic definition of phase space $\mathfrak{B}$, introduced by Hale and Kato [10].

PC$([0, a], X)$ is the space formed by normalized piecewise continuous function from $[0, a]$ into X. In particular it is the space *PC* formed by all functions $u : [0, a] \to X$ such that u is continuous at $t \neq t_i$, $u(t_i^-) = u(t_i)$ and $u(t_i^+)$ exists for all $i = 1, 2, ..., n$. It is clear that *PC* endowed with the norm $\|x\|_{PC} = sup_{t\in J}\|x(t)\|$ is a Banach space. For any $x \in$ *PC*

$$\widetilde{x}_i(t) = \begin{cases} x(t), & t \in (t_i, t_{i+1}]; \\ x(t_i^+), & t = t_i,\ i = 1, 2, ..., n. \end{cases} \tag{2}$$

So, $\widetilde{x} \in C([t_i, t_{i+1}], X)$.

Definition 1 *[17]:The phase space* $\mathfrak{B}$ *is a linear space of functions mapping* $(-\infty, 0]$ *into X endowed with seminorm* $\|.\|_{\mathfrak{B}}$ *and satisfies the following conditions:*

(A) If $x : (-\infty, \sigma + b] \to X, b > 0$, such that $x_\theta \in \mathfrak{B}$ and $x|_{[\sigma,\sigma+b]} \in C([\sigma, \sigma + b] : X)$, then for every $t \in [\sigma, \sigma + b)$ the following conditions hold :
(i) x_t is in $\mathfrak{B}$,
(ii) $\|x(t)\| \leq H\|x_t\|_{\mathfrak{B}}$,
(iii) $\|x_t\|_{\mathfrak{B}} \leq K(t-\sigma)sup\{\|x(s)\| : \sigma \leq s \leq t\} + M(t-\sigma)\|x_\sigma\|_{\mathfrak{B}}$,
where $H > 0$ is a constant $K, M : [0, \infty) \to [1, \infty)$, K is continuous, M is locally bounded and H, K, M are independent of $x(.)$

(B) The space $\mathfrak{B}$ is complete.

Definition 2 *[6]: The Hausdorff's measure of noncompactness* χ_Y *is defined by*
$\chi_Y(B) = inf\{r > 0, B$ *can be covered by finite no. of balls with radius r*$\}$
for a bounded set B in any Banach space Y.

Lemma 1 *[6]: Let Y be a Banach space and $B, C \subset Y$ be bounded, then the following properties hold:*

(1) *B is pre-compact if and only if $\chi_Y(B) = 0$;*

(2) *$\chi_Y(B) = \chi_Y(\overline{B}) = \chi_Y(convB)$, where $\overline{B}$ and conv B are closure and convex hull of B respectively;*

(3) *$\chi_Y(B) \leq \chi_Y(C)$ when $B \subset C$;*

(4) *$\chi_Y(B + C) \leq \chi_Y(B) + \chi_Y(C)$ where $B + C = \{x + y; x \in B, y \in C\}$;*

(5) *$\chi_Y(B \cup C) = max\{\chi_Y(B), \chi_Y(C)\}$;*

(6) *$\chi_Y(\lambda B) = |\lambda|\chi_Y(B)$ for any $\lambda \in$ R;*

(7) *If the map $Q : D(Q) \subset Y \to Z$ is Lipschitz continuous with constant k then $\chi_Z(Q(B)) \leq k\chi_Y(B)$ for any bounded subset $B \subset D(Q)$, where Z is a Banach space;*

(8) *If $\{W_n\}_{n=1}^{+\infty}$ is a decreasing sequence of bounded closed nonempty subset of Y and $lim_{n\to\infty}\chi_Y(W_n) = 0$, then $\cap_{n=1}^{+\infty}W_n$ is nonempty and compact in Y.*

Definition 3 *[6]: The map $Q : W \subset Y \to Y$ is said to be a $\chi-$ contraction if there exists a positive constant $k < 1$ such that $\chi_Y Q(C) \leq k\chi_Y(C)$ for any bounded closed subset C of W where Y is a Banach space.*

Lemma 2 *(Darbo-Sadovskii)[6]. If $W \subset Y$ is closed and convex and $0 \in W$, the continuous map $Q : W \to W$ is $\chi-$contraction , then the map Q has atleast one fixed point.*

Lemma 3 *[6]:* (1) *If $W \subset PC([a, b]; X)$ is bounded, then $\chi(W(t)) \leq \chi_{PC}(W)$ for any $t \in [a, b]$ where $W(t) = \{u(t) : u \in W\} \subset X$;*
(2) *If W is piecewise equicontinuous on $[a, b]$, then $\chi(W(t))$ is piecewise continuous for $t \in [a, b]$, and*

$$\chi_{PC}(W) = sup\{\chi(W(t)), t \in [a, b]\}$$

(3) *If $W \subset PC([a, b]; X)$ is bounded and piecewise equicontinuous , then $\chi(W(t))$ is piecewise continuous for $t \in [a, b]$ and*

$$\chi(\int_a^t W(s)ds) \leq \int_a^t \chi(W(s))ds \ \ t \in [a, b]$$

Lemma 4 *[6]: If the semigroup $S(t)$ is equicontinuous and $\eta \in L([0, a]; R^+)$, then the set $\{\int_0^t S(t-s)u(s)ds : \|u(s)\| \leq \eta(s) \ for \ a.e \ s \in [0, a]\}$ is equicontinuous for $t \in [0, a]$.*

Lemma 5 *[6] If $W \subset PC^1(J, X)$ is bounded and the elements of W' are equicontinuous on each $J_k(k = 1, 2, ..., m)$ then*

$$\chi_{PC^1}(W) = max\{sup_{t\in J}\chi W(t), sup_{t\in J}\chi(W'(t))\}$$

where χ_{PC^1} denotes the Hausdorff measure of noncompactness in the space $PC^1(J, X)$.

3 Main Result

We define the mild solution of the problem (1) as follows.

Definition 4 *A function $x : (-\infty, a] \to X$ is a mild solution of the problem* (1) *if $x_0 = \phi$, $x'(0) = \xi$, $x(.)|_{[0,a]} \in PC^1(X)$, $x(t) = J_i^1(t, x_t)$, $\forall\, t \in (t_i, s_i]$, $i = 1, ..., n$, $x'(t) = J_i^2(t, x_t)$, $t \in (t_i, s_i]$, $i = 1, 2, ..., n$ and*

$$\begin{aligned} x(t) &= C(t)(\phi(0) - g(0, \phi)) + S(t)(\xi - \eta) + g(t, x_t) + \int_0^t AS(t-s)g(s, x_s)ds \\ &+ \int_0^t S(t-s)f(s, x_{\rho(s,x_s)}, x'(s))ds, \ t \in [0, t_1] \\ x(t) &= C(t-s_i)(J_i^1(s_i, x_{s_i}) - g(s_i, x_{s_i})) + S(t-s_i)(J_i^2(s_i, x_{s_i}) - g'(s_i, x_{s_i})) \\ &+ g(t, x_t) + \int_{s_i}^t AS(t-s)g(s, x_s)ds + \int_{s_i}^t S(t-s_i)f(s, x_{\rho(s,x_s)}, x'(s))ds, \\ &\quad for \ \ t \in [s_i, t_{i+1}], \ i = 1, ..., n \end{aligned} \tag{3}$$

where $\frac{d}{dt}g(t,x_t)|_{t=0}=\eta$, where η is independent of x. To prove our result we always assume $\rho: J\times\mathfrak{B}\to(-\infty,a]$ is a continuous function. In this section $y:(-\infty,a]\to X$ is the function defined by $y_0=\phi\ and\ y(t)=C(t)(\phi(0)-g(0,\phi))+S(t)(\xi-\eta)$ on $[0,t_1]$. Clearly $\|y_t\|_{\mathfrak{B}}\le K_a\|y\|_a+M_a\|\phi\|_{\mathfrak{B}}$ where $\|y\|_b=sup_{0\le t\le b}\|y(t)\|$

Let $S(a)$ be the space $S(a)=\{x:(-\infty,a]\to X:\ x_0=0,\ x'(0)=0,\ x|_J\in PC^1\}$ endowed with norm $\|u\|_1=\|u\|_\infty+\|u'\|_\infty$. The following hypotheses are used.

(H_ϕ) The function $t\to\phi_t$ is continuous from $\mathbb{R}(\rho^-)=\{\rho(s,\psi):\rho(s,\psi)\le 0\}$ into $\mathfrak{B}$ and there exists a continuous bounded function $J^\phi:\mathbb{R}(\rho^-)\to(0,\infty)$ such that $\|\phi_t\|_{\mathfrak{B}}\le J^\phi(t)\|\phi\|_{\mathfrak{B}}$ for every $t\in\mathbb{R}(\rho^-)$.

(Hf) The function $f: J\times\mathfrak{B}\to X$ satisfies the following:

(1) For every $x:(-\infty,a]\to X, x_0\in\mathfrak{B}\ and\ x|_J\in PC$, the function $f(.,\psi,x): J\to X$ is strongly measurable for every $\psi\in\mathfrak{B},\ x\in X$ and $f(t,.,.)$ is continuous for a.e. $t\in J$.

(2) There exists an integrable function $\alpha: J\to[0,+\infty)$ and a monotone continuous nondecreasing function $\Omega:[0,+\infty)\to(0,+\infty)$ such that $\|f(t,v,x)\|\le\alpha(t)\Omega(\|v\|_{\mathfrak{B}}+\|x\|)\ \forall t\in J\ and\ v\in\mathfrak{B}$.

(3) There exists an integrable function $\eta: J\to[0,\infty)$ such that
$\chi(S(s)f(t,D_1,D_2))\le\eta_1(t)\sup_{-\infty<\theta\le 0}\chi(D_1(\theta))$ for a.e. $s,t,\in J$,
where $D_1(\theta)=\{v(\theta):v\in D_1\}$.
$\chi(C(s)f(t,D_1,D_2))\le\eta_2(t)\sup_{-\infty<\theta\le 0}\chi(D_2(\theta))$ for a.e. $s,t,\in J$, where $D_2(\theta)=\{v(\theta):v\in D_2\}$

(Hg) The function $g: J\times\mathfrak{B}$ satisfies the following.

(1) $g(t,.):\mathfrak{B}\to X$ is continuous $\forall t\in J$.

(2) If $x:(-\infty,a]\to X$ be such that $x_0=\phi$ and $x|_J\in PC$ then the function $t\to g(t,x_t)$ belongs to PC and $t\to g(t,x_t)$ is strongly measurable function.

(3) There exists a function a non decreasing function ω_g such that
$\|g(t,\psi)\|_Y\le m_g(t)\Omega_g(\|\psi\|_{\mathfrak{B}})$, for all $(t,\psi)\in J\times\mathfrak{B}$

(4) The set $V(r)=\{AS(\theta)g(s,\psi):\theta,s\in J,\ \psi\in B_r(0,\mathfrak{B})\}$ is precompact in X for all $r>0$.

(5) The set $\{\widetilde{v}_i: v\in V(r,g)\}$ is equicontinuous subset of $C([t_i,t_{i+1}],X)$ for all $i=1,\cdots,n$

(6) $t\to g(t,x_t)$ is C^1 on J and $\frac{d}{dt}g(t,x_t)|_{t=0}=\eta$ where η is independent of x.

(7) The operator $P: S(a)\to C(J,X)$, is a completely continuous operator defined as $P(x)(t)=\frac{d}{dt}g(t,x_t+y_t)$ is such that $\|Px\|\le c_p\|x\|+d_p$. Thus, the set $\{Px(t):x\in S_a, t\in J\}$ is precompact in X.

(HJ) (1) For the maps $J_i^1(t,\phi): J\times\mathfrak{B}\to X$ there exist positive constants c_i^1,c_i^2,d_i^1,d_i^2 such that
$\|J_i^j(t,v)\|\le c_i^j\|v\|_{\mathfrak{B}}+d_i^j,\ \forall\ j=1,2,$

(2) The maps $J_i^1(.,\psi), J_i^2(.,\psi)$ are completely continuous $\forall\ (.,\psi)\in(t_i,s_i]\times\mathfrak{B}\ \ i=1,...,n,$

(HI) $c_p(a+1)+((N+\widetilde{N})c_i^1+(\widetilde{N}+N)c_i^2)K_a+(c_i^1+c_i^2)K_a+max\{\int_0^a\eta_1(s)ds,\int_0^t\eta_2(s)ds\}$
$limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau}\int_{s_j}^{t_k}((\widetilde{N_1}+\widetilde{N_2})m_g(s)+(\widetilde{N}+N)m_f(s))ds<1$

(H1) There exists a Banach space $(Y,\|.\|_Y)$ continuously included in X such that $AS(t)\in\mathcal{L}(Y,X)$, for all $t\in J$ and $AS(.)x\in C(J;X)$ for every $x\in Y$. There exists constants $N_Y,\widetilde{N_1}$ such that $\|y\|\le N_Y\|y\|_Y,\ \forall y\in Y$ and $\|AS(t)\|_{\mathcal{L}(Y,X)}\le\widetilde{N_1},\ \forall t\in J$

(H2) $\mathcal{R}(C(t)-I)$ is closed and dim Ker$(C(t)-I)<\infty,\ \forall\ 0<t\le a$

Lemma 6 *[18]: If* $y:(-\infty,a]\to X$ *is a function such that* $y_0=\phi$ *and* $y|_J\in PC(X)$ *then*
$\|y_{\rho(s,y_s)}\|_{\mathfrak{B}}\le(M_a+\widetilde{J^\phi})\|\phi\|_{\mathfrak{B}}+K_a sup\{\|y(\theta)\|;\theta\in[0,max\{0,s\}]\}$,

$$s\in\mathbb{R}(\rho^-)\cup[0,a]$$

where $\widetilde{J^\phi}=sup_{t\in\mathbb{R}(\rho^-)}J^\phi(t),\ M_a=sup_{t\in J}M(t)\ and\ K_a=max_{t\in J}K(t)$.

Lemma 7 *[23]: Let condition* $(H2)$ *be satisfied and* $B\subset Y$. *If* B *is bounded in* X *and the set* $\{AS(t)y:t\in[0,a],y\in B\}$ *is relatively compact in* X, *then* B *is relatively compact in* X.

Proof. Since for $y \in B$, $C(t)y - y = A\int_0^t S(s)ydy = \int_0^t AS(s)ydy$, it follows from mean value theorem for Bochner integral that

$$C(t)y - y \in t \times \overline{conv(AS(s)y : 0 \leq s \leq t, y \in B)}$$

where $conv$ is the convex hull. Then by hypothesis $(H2)$ the result follows. ■

Lemma 8 *[23]: A set $B \subset PC^1$ is relatively compact in PC^1 if and only if each set $\widetilde{B_i}, i = 1, ..., n$ is relatively compact in $C^1([t_i, t_{i+1}], X)$.*

Theorem 9 *If the hypotheses $(H_\phi), (Hf), (Hg), (HI), (H1)$ are satisfied, then the initial value problem (1) has atleast one mild solution.*

Proof. Let $\Gamma = \Gamma_i^1 + \Gamma_j^2,\ \forall i = 1, \cdots, n \ and \ j = 0, \cdots, n$

$$(\Gamma_i^1 x)(t) = \begin{cases} J_i^1(t, \overline{x}_t), & t \in (t_i, s_i]; i = 1, \cdots, n \\ C(t - s_i)[J_i^1(t, x_{s_i}) - g(s_i, x_{s_i})] & \\ +S(t - s_i)[J_i^2(t, x_{s_i}) - g'(s_i, x_{s_i})], & t \in (s_i, t_{i+1}]; i = 1, \cdots, n \end{cases} \tag{4}$$

$$(\Gamma_j^2 x)(t) = \begin{cases} g(t, \overline{x}_t) + \int_{s_j}^t AS(t - s)g(s, \overline{x}_s)ds & \\ + \int_{s_j}^t S(t - s)f(s, \overline{x}_{\rho(s, \overline{x_s})})ds, & t \in (s_j, t_{j+1}]; j = 0, \cdots, n \\ 0, & t \notin (s_j, t_{j+1}], j = 0, \cdots, n. \end{cases} \tag{5}$$

where $\overline{x}_0 = \phi$ and $\overline{x} = x + y$ on J.

$$(\Gamma_i^1 x)'(t) = \begin{cases} J_i^2(t, \overline{x}_t), & t \in (t_i, s_i]; i = 1, \cdots, n \\ AS(t - s_i)[J_i^1(t, x_{s_i}) - g(s_i, x_{s_i})] & \\ +C(t - s_i)[J_i^2(t, x_{s_i}) - g'(s_i, x_{s_i})], & t \in (s_i, t_{i+1}]; i = 1, \cdots, n \end{cases} \tag{6}$$

$$(\Gamma_j^2 x)'(t) = \begin{cases} Px(t) + \int_{s_j}^t AC(t - s)g(s, \overline{x}_s)ds & \\ + \int_{s_j}^t C(t - s)f(s, \overline{x}_{\rho(s, \overline{x_s})})ds, & t \in (s_j, t_{j+1}]; j = 0, \cdots, n \\ 0, & t \notin (s_j, t_{j+1}], j = 0, \cdots, n. \end{cases} \tag{7}$$

It is easy to see that

$$\|\overline{x}_t\|_{\mathfrak{B}} \leq K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a\|x\|_t,$$

where $\|x\|_t = sup_{0 \leq s \leq t}\|x(s)\|$.

$$\|\overline{x}_{\rho(s, \overline{x}_s)}\|_{\mathfrak{B}} \leq k^* := (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a\|y\|_a + K_a\|x\|_a.$$

Thus Γ is well defined and has values in $S(a)$. Also by axioms of phase space, the Lebesgue dominated convergence theorem , and the conditions $(Hf), (Hg)$ it can be shown that Γ is continuous.

Step1 : There exists $k > 0$ such that $\Gamma(B_k) \subset B_k$, where $B_k = \{x \in S(a) : \|x\|_a \leq k\}$. In the following $\hat{k} = K_a k + \|y_s\|_{\mathfrak{B}} = K_a k + K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}}$. In fact, if we assume that the assertion is false, then for $k > 0$ there exist

$x_k \in B_k$ and $t_k \in (s_j, t_j+1]$ for some $j \in \{0, \cdots, n\}$ and $i \in 1, \cdots, n$ such that $k < \|\Gamma x_k(t_k)\|_1$. Then,

$$\begin{aligned}
k &\leq \|\Gamma_j^2 x(t))\| + \|\Gamma_i^1 x(t))\| + \|(\Gamma_j^2 x)'(t)\| + \|(\Gamma_i^1 x)'(t)\| \\
&\leq c_p a\|x\|_1 + c + \widetilde{N_1}\int_{s_j}^{t_k} m_g(s)\Omega\|\overline{x_k}_s\|_{\mathfrak{B}} ds \\
&+ \widetilde{N}\int_{s_j}^{t_k} \alpha(s)\Omega(\|\overline{x_k}_{\rho(s,\overline{x_k}_s)}\|_{\mathfrak{B}} + \|\overline{x_k'}(s)\|) + (Nc_i^1 + \widetilde{N}c_i^2)\|x_{s_i}\| \\
&+ c_p\|x\|_1 + c + \widetilde{N_2}\int_{s_j}^{t_k} m_g(s)\Omega\|\overline{x_k}_s\|_{\mathfrak{B}} ds \\
&+ N\int_{s_j}^{t_k} \alpha(s)\Omega(\|\overline{x_k}_{\rho(s,\overline{x_k}_s)}\|_{\mathfrak{B}} + \|\overline{x_k'}(s)\|) + (\widetilde{N}c_i^1 + Nc_i^2)\|x_{s_i}\| \\
&\leq c_p(a+1)k + c + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a k \\
&+ (\widetilde{N_1} + \widetilde{N_2})\int_{s_j}^{t_k} m_g(s)\Omega(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k)ds \\
&+ (\widetilde{N}+N)\int_{s_j}^{t_k} \alpha(s)ds\Omega(K_a\|y\|_a + (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a k + k)ds \\
&\leq c_p(a+1)k + c + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a k \\
&+ \int_{s_j}^{t_k}((\widetilde{N_1}+\widetilde{N_2})m_g(s) + (\widetilde{N}+N)m_f(s)) \\
&\times \Omega(K_a\|y\|_a + (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a k + k)ds
\end{aligned} \tag{8}$$

Hence

$$\begin{aligned}
1 &< c_p(a+1) + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a \\
&+ limsup_{\tau\to\infty}\frac{\Omega(K_a\|y\|_a + (M_a + \widetilde{J^\phi})\|\phi\|_{\mathfrak{B}} + K_a k + k)}{k} \\
&\times \int_{s_j}^{t_k}((\widetilde{N_1}+\widetilde{N_2})m_g(s) + (\widetilde{N}+N)m_f(s))ds \\
&\leq c_p(a+1) + ((N+\widetilde{N})c_i^1 + (\widetilde{N}+N)c_i^2)K_a \\
&+ limsup_{\tau\to\infty}\frac{\Omega(\tau)}{\tau}\int_{s_j}^{t_k}((\widetilde{N_1}+\widetilde{N_2})m_g(s) + (\widetilde{N}+N)m_f(s))ds
\end{aligned} \tag{9}$$

which is a contradiction to the hypothesis $(H1)$. Similarly, suppose there exists $x_k \in B_k$ and $t_k \in (t_i, s_i]$ for some $i \in \{1, \cdots, n\}$ such that $(\Gamma x_k)(t_k) > k$. Then,

$$\begin{aligned}
k < \|(\Gamma_i^1 x_k)(t_k)\|_1 &= \|J_i^1(t_k, \overline{x_k}_{t_k})\| + \|J_i^2(t_k, \overline{x_k}_{t_k})\| \\
&\leq \{c_i^1\|\overline{x_k}_{t_k}\|_{\mathfrak{B}} + d_i^1\} + \{c_i^2\|\overline{x_k}_{t_k}\|_{\mathfrak{B}} + d_i^2\} \\
&\leq \{c_i^1(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + d_i^1\} \\
&+ \{c_i^2(K_a\|y\|_a + M_a\|\phi\|_{\mathfrak{B}} + K_a k) + d_i^2\}
\end{aligned} \tag{10}$$

Hence,

$$1 < (c_i^1 + c_i^2)K_a \tag{11}$$

which is a contradiction. Hence $\Gamma(B_k) \subset B_k$.
Step 2 : To prove that Γ is a $\chi-$contraction. Let $\Gamma = \Gamma_i^1 + \Gamma_j^2 \ \forall \ i = 1, \cdots, n; j = 0, \cdots, n$ be split into $\Gamma = \Gamma_i^{1a} + \Gamma_i^{1b} + \Gamma_i^{1c} + \{\Gamma_j^{2a} + \Gamma_j^{2b} + \Gamma_j^{2c}\}, \forall\, i = 1, \cdots, n; j = 0, \cdots, n$

$$\Gamma_i^{1a} = C(t-s_i)(-g(s, x_{s_i}) + J_i^1(s_i, x_{s_i}))$$

$$\Gamma_i^{1b} = S(t-s_i)(J_i^2(s_i, x_{s_i}) - g'(s_i, x_{s_i}))$$

$$\Gamma_i^{1c} = J_i^1(t, x_t)$$
$$\Gamma_j^{2a}x(t) = g(s, \overline{x_t})$$
$$\Gamma_j^{2b}x(t) = \int_{s_j}^{t} AS(t-s)g(s, \overline{x_s})ds$$
$$\Gamma_j^{2c}x(t) = \int_{s_j}^{t} S(t-s)f(s, \overline{x}_{\rho(s,\overline{x}_s)})ds$$

The properties of the function g in (Hg) , lemmas (3.2) and lemma (3.3) imply that for all $j = 0, ..., n$, the set of functions $V(k, g)_j = \{t \to [\widetilde{g}(t, x_t + y_t)]_j : x \in B_k\, j = 0, \cdots, n\}$ is relatively compact in $C([s_j, t_{j+1}], X)$. By lemma (2.3)(2) $\chi_{PC}(W) = sup\{\chi(W(t)), t \in J\}$. By lemma (2.1)(1), for any $W \subset \Gamma_j^{2a}(B_k)$

$$\begin{aligned} \chi_{PC^1}(\Gamma_j^{2a}W(t)) &= \chi_{PC^1}(g(t, W_t + y_t)) \\ &= max\{sup_{t\in J}\chi_{PC}g(t, W_t + y_t)), sup_{t\in J}\chi_{PC}g'(t, W_t + y_t)\} \\ &= 0 \end{aligned} \tag{12}$$

By mean value theorem for Bochner integral, we derive

$$\{\Gamma_j^{2b}x(t) : x \in B_k\} \subset t \times \overline{conv(\{AS(h)g(s,\psi) : 0 \le h, s \le t, \|\psi\|_{\mathfrak{B}} \le \hat{k}\})}$$
$$\{(\Gamma_j^{2b}x(t))' : x \in B_k\} \subset t \times \overline{conv(\{AC(h)g(s,\psi) : 0 \le h, s \le t, \|\psi\|_{\mathfrak{B}} \le \hat{k}\})}$$

This implies $\{\Gamma_j^{2b}x(t) : x \in B_k\}$ and $\{(\Gamma_j^{2b}x(t))' : x \in B_k\}$ is relatively compact in X for all $t \in J$. Hence by Lemma (2.1)(1)

$$\begin{aligned} \chi_{PC^1}(\Gamma_j^{2b}W(t)) \quad &= max\{sup_{t\in J}\chi_{PC}(\int_{s_j}^{t} AS(t-s)g(s, W_s + y_s)ds), \\ &sup_{t\in J}\chi_{PC}(\int_{s_j}^{t} AC(t-s)g(s, W_s + y_s)ds)\} = 0 \end{aligned} \tag{13}$$

By lemma (2.4) for any $W \subset \Gamma_j^{2c}(B_k)$, since $S(t)$ is equicontinuous so, W is piecewise equicontinuous . Hence from the fact that $\rho(s, \overline{x_s}) \le s, s \in [0, a]$ and lemma (2.3)(2) and $\chi_{PC}(W) = \sup\{\chi(W(t)), t \in [s_j, t_{j+1}],\ j = 0, \cdots, n\}$ such that for all $j = 0, \cdots, n$.

$$\begin{aligned} &\chi_{PC^1}(\Gamma_j^{2c}W(t)) = \chi_{PC^1}(\int_{s_j}^{t} S(t-s)f(s, W_{\rho(s,\overline{x}_s)} + y_s, W'(s) + y'(s))ds \\ &= max\{sup_{t\in J}\chi_{PC}(\int_{s_j}^{t} S(t-s)f(s, W_{\rho(s,\overline{x}_s)} + y_s, W'(s) + y'(s))ds, \\ &sup_{t\in J}\chi_{PC}(\int_{s_j}^{t} C(t-s)f(s, W_{\rho(s,\overline{x}_s)} + y_s, W'(s) + y'(s))\}ds \\ &\le max\{sup_{t\in J}\int_{s_j}^{t} \eta_1(s) sup_{-\infty<\theta\le 0}\chi(W(\rho(s,\overline{x}_s) + \theta) + y(s+\theta))ds, \\ &sup_{t\in J}\int_{s_j}^{t} \eta_2(s) sup_{-\infty<\theta\le 0}\chi(W'(s+\theta) + y'(s+\theta))ds\} \\ &\le max\{sup_{t\in J}\int_{s_i}^{t} \eta_1(s) sup_{-\infty<\theta\le 0}\chi(W(s+\theta) + y(s+\theta))ds, \\ &sup_{t\in J}\int_{s_i}^{t} \eta_2(s) sup_{-\infty<\theta\le 0}\chi(W'(s+\theta) + y'(s+\theta))ds\} \\ &\le max\{sup_{t\in J}\int_{s_i}^{t} \eta_1(s) sup_{0<\tau\le s}\chi W(\tau)ds, \\ &sup_{t\in J}\int_{s_i}^{t} \eta_2(s) sup_{0<\tau\le s}\chi W(\tau)ds\} \\ &\le max\{\int_0^a \eta_1(s)ds, \int_0^a \eta_2(s)ds\}\chi_{PC^1}(W) \end{aligned} \tag{14}$$

Hence

$$\begin{aligned}\chi_{PC^1}(\Gamma_j^{2c}W) &= \sup_{t\in J}\{\chi_{PC^1}(\Gamma_j^{2c}W(t)),\ t\in[s_j,t_{j+1}],\ j=0,\cdots,n\}\\ &\leq \chi_{PC^1}(W)max\{\int_0^a\eta_1(s)ds,\int_0^a\eta_2(s)ds\} \end{aligned}\tag{15}$$

For arbitrary $x_1, x_2 \in B_k$ and $t \in (s_i, t_{i+1}]\ \forall\ i = 1, \cdots, n$

$$\begin{aligned}&\chi_{PC^1}(\Gamma_i^{1a}x)(t) = max\{sup\chi_{PC}(\Gamma_i^{1a}x)(t), sup\chi_{PC}(\Gamma_i^{1b}x)'(t)\}\\ &\leq max\{sup\chi_{PC}\overline{conv(\{C(\theta)[J_i^1(s,\psi)-g(s,\psi)]:0\leq\theta,s\leq t,\|\psi\|\leq k\})},\\ &sup\chi_{PC}\overline{conv(\{S(\theta)(J_i^1(s,\psi)-g(s,\psi)):0\leq\theta,s\leq t,\|\psi\|\leq k\})}\}\\ &=0\end{aligned}\tag{16}$$

Since

$$\begin{aligned}C(t-s_i)[J_i^1(s_i,x_{s_i}) &- g(s_i,x_{s_i})]\\ &\in \overline{conv(\{C(\theta)[J_i^1(s,\psi)-g(s,\psi)]:0\leq\theta,s\leq t,\|\psi\|\leq k\})}.\end{aligned}$$

and

$$\begin{aligned}S(t-s_i)[J_i^1(s_i,x_{s_i}) &- g(s_i,x_{s_i})]\\ &\in \overline{conv(\{C(\theta)[J_i^1(s,\psi)-g(s,\psi)]:0\leq\theta,s\leq t,\|\psi\|\leq k\})}.\end{aligned}$$

is relatively compact.

Similarly $\chi(\Gamma_i^{1b}) = 0,\ \forall i = 1, \cdots, n$ and $\chi(\Gamma_i^{1c}) = 0\ \forall i = 1, \cdots, n$

For each bounded set $W \in PC^1(J; X)$ we have,

$$\begin{aligned}\chi_{PC^1}(\Gamma W) &\leq \chi_{PC^1}(\Gamma_i^{1a}W+\Gamma_i^{1b}W+\Gamma_i^{1c}W)+\chi_{PC^1}(\Gamma_j^{2a}W+\Gamma_j^{2b}W+\Gamma_j^{2c}W)\\ &\leq (0+0+0+max\{\int_0^a\eta_1(s)ds,\int_0^t\eta_2(s)ds\}\chi_{PC^1}(W))\end{aligned}$$

Therefore, Γ is a $\chi-$contraction. So, by Darbo-Sadovskii fixed point theorem we conclude that Γ has a fixed point in $S(a)$. Thence, $z = x + y$ is a mild solution of (1). ■

4 Example

In this section we discuss a partial differential equation applying the abstract results of this paper. In this application, $\mathfrak{B}$ is the phase space $C_0 \times L^2(h, X)$ see ([19]).
Consider the second order neutral differential equation

$$\begin{aligned}\frac{\partial^2}{\partial t^2}(x(t,\sigma) &- \int_{-\infty}^t\int_0^\pi n(t-s,\upsilon,\sigma)x(s,\upsilon)d\upsilon ds)\\ =\frac{\partial^2x(t,\sigma)}{\partial\sigma^2} &+ \int_{-\infty}^t m(t-s)x(s-\rho_1(t)\rho_2(\|x(t)\|,\sigma,\zeta)ds\ \ t\in[0,a],\sigma\in[0,\pi],\\ x(t,0) &= x(t,\pi)=0,\ t\in[0,a],\\ x(s,\sigma) &= \phi(s,\sigma)\ -\infty\leq s\leq 0, 0\leq\sigma\leq\pi,\\ \frac{\partial}{\partial t}x(0,\sigma) &= \xi(\sigma),\ 0\leq\sigma\leq\pi,\\ x(t)(\sigma) &= \int_\infty^{t_i}n_i^1(t_i-s)x(s,\sigma)ds,\ t\in(t_i,s_i],\ i=1,...,n\end{aligned}\tag{17}$$

where $\phi \in H^1([0, \pi]),\ \xi \in X,\ 0 = t_0 = s_0 < t_1 \leq s_1 \leq t_2, ..., t_n \leq s_n \leq t_{n+1} = a$ Here, $X = L^2.([0, \pi])$, $\mathfrak{B} = PC_0 \times L^2(\rho, X)$, $A \subset D(A) \subset X \to X$ is the map defined by $Af = f''$ with domain $D(A) = \{f \in X : f'' \in X,\ f(0) = f(\pi) = 0\}$. It is well known that A is the infinitesimal generator of a strongly continuous cosine function $(C(t))_{t\in R}$ on X. Also, A has a discrete spectrum, the eigenvalues are $-n^2\ \ n \in \mathbb{N}$; with corresponding eigenvectors $z_n(\theta) = (\frac{2}{\pi})^{\frac{1}{2}} sin(n\theta)$ and the following properties hold

(C1) $A\phi = -\sum_{n=1}^{\infty} n^2 < \phi, z_n > z_n$ where $\phi \in D(A)$

(C2) $C(t)\phi = \sum_{n=1}^{\infty} cos(nt) < \phi, z_n > z_n$ and $S(t)\phi = \sum_{n=1}^{\infty} \frac{sin(nt)}{n} < \phi, z_n > z_n$, for $\phi \in X$.

By defining maps $\rho, g, f : [0, a] \times \mathfrak{B} \times X \to X$ by

$$\rho(t, \psi) := \rho_1(t)\rho_2(\|\psi(0)\|),$$

$$g(\psi)(\sigma) := \int_{-\infty}^{0} \int_{0}^{\pi} n(s, \upsilon, \sigma)\psi(s, \upsilon)d\upsilon ds,$$

$$f(\psi)(\sigma) := \int_{-\infty}^{0} m(s)\psi(s, \sigma, \zeta)ds$$

the system (17) can be transformed into system (1) Assume that the functions $\rho_i : \mathbb{R} \to [0, \infty),\ m : \mathbb{R} \to \mathbb{R}$ are piecewise continuous.

(1) The functions $n(s, \upsilon, \sigma), \frac{\partial n(s,\upsilon,\sigma)}{\partial \sigma}$ are measurable, $n(s, \upsilon, \pi) = n(s, \eta, 0) = 0$ and

$$L_g := max\{(\int_{0}^{\pi} \int_{-\infty}^{0} \int_{0}^{\pi} \frac{1}{h(s)} (\frac{\partial^i n(s, \eta, \sigma)}{\partial \sigma^i})^2 d\eta ds d\sigma)^{1/2} :\ i = 0, 1\} < \infty$$

$$\widetilde{L}_g := (\int_{0}^{\pi} \int_{-\infty}^{0} \int_{0}^{\pi} \frac{1}{h(s)} (\frac{\partial^i n(s, \eta, \sigma)}{\partial \sigma^i})^2 d\eta ds d\sigma) < \infty$$

(2) The function $f : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is continuous and there is continuous function $\int_{-\infty}^{0} \frac{\mu(s)^2}{q(s)} ds < \infty$. and $\|f(t, \sigma)\| \leq \mu(s)(\|\sigma\| + \|\zeta\|)$

(3) The functions $n_i^j \in C([0, \infty); \mathbb{R})$ and $L_i^j := (\int_{-\infty}^{0} \frac{(n_i^j(s))^2}{q(s)} ds)^{1/2} < \infty,\ \forall\, i = 1, 2, ..., n,\ j = 1, 2$

So, $g(t, .), J_i, (i = 1, ..., n), f$ are bounded linear operators. Also by using (1) we can prove that g is $D(A)-$valued. Thus we take $Y = D(A)$. Therefore if $\iota : Y \to X$ is the inclusion then $t \to AS(t)$ is uniformly continuous into $L(Y, X)$ and $\|AS(t)\|_{L(Y,X)} \leq 1$ for $t \in [0, a]$ Suppose $u(t)(\sigma) = x(t, \sigma)$ such that $x_0 = \phi$ and continuous on $[0, t_1)$ then the right derivative

$$\begin{aligned} \frac{d}{dt} g(u_t)|_{t=0}(\sigma) &= \int_{-\infty}^{0} \int_{0}^{\pi} \frac{\partial}{\partial s} n(s, \upsilon, \sigma)\phi(s, \upsilon)d\upsilon ds + \int_{0}^{\pi} n(0, \upsilon, \sigma)\psi(0, \upsilon)d\upsilon \\ &= \eta(\sigma) \end{aligned} \tag{18}$$

exists and is independent of x. Hence by assumptions $(a) - (c)$ and theorem (3.2) it is ensured that mild solution to the problem (17) exists.

5 Conclusions

Hence we prove the existence of mild solution using Darbo-Sadovskii theorem and measure of noncompactness replacing Lipschitz conditions and compactness of cosine family by natural assumptions. Our abstract approach permits application to partial differential equations with instantaneous impulsive term involving nonlinear expression also.

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