ISSN 1749-3889 (print), 1749-3897 (online)
International Journal of Nonlinear Science
Vol.18(2014) No.2,pp.139-144

Local Stability Analysis for a Delayed Differential-algebraic Biological Economic System with Ratio-dependent Functional Response

Lianwang Deng[1, *], Miao Peng[1]
[1] Nonlinear Scientific Research Center, Faculty of Science Jiangsu University, Zhenjiang, Jiangsu, P.R. China, 212013
(Received 9 May 2014, accepted 29 September 2014)

Abstract: This paper is concerned with a delayed differential-algebraic biological economic system. Firstly, the differential-algebraic system is transformed into a differential system by employing the local parameterization method of differential-algebraic system, and then the sufficient conditions for the local stability and the existence of Hopf bifurcation are established by regarding the time delay as the bifurcation parameter. Finally, numerical simulations are given to verify the theoretical analysis.

Keywords: biological economic system; local stability; Hopf bifurcation; time delay; ratio-dependent

1 Introduction

In population dynamics, the functional response of predator to prey density refers to the change in the density of prey attacked per unit time per predator as the prey density changes. Based on the Holling type II functional response [1], Arditi and Ginzburg [2] proposed a ratio-dependent functional response of the form:

$$P\left(\frac{x}{y}\right) = \frac{a_1(x/y)}{m+(x/y)} = \frac{a_1 x}{my+x}, \tag{1}$$

which was more appropriate for describing the relationship between predator and its prey. The theory has been introduced into their researches by many scholars and produced numerous valuable research results [3-5].

In this paper, we incorporate the ratio-dependent functional response above and an economic theory from Gordon [6] into a predator-prey system with non-selective harvesting and time delay [7]. Thus, a delayed differential algebraic predator-prey system with ratio-dependent functional response is considered as following:

$$\begin{aligned}
&\tfrac{dx(t)}{dt} = x(t)\left(r_1 - ax(t-\tau)\right) - \tfrac{a_1 x(t) y(t)}{my(t)+x(t)} - E(t)x(t),\\
&\tfrac{dy(t)}{dt} = \tfrac{a_2 x(t) y(t)}{my(t)+x(t)} - dy(t),\\
&0 = E(t)(px(t)-c) - r,
\end{aligned} \tag{2}$$

where $x(t)$ and $y(t)$ denote prey and predator population densities at time t respectively, $E(t)$ is the harvest effort for prey at time t; $r_1, a, a_1, a_2, m, p, c$ and r are positive constants in which r_1 represents the intrinsic growth rate of the prey, a is the intra-specific competition rate of the prey, a_1 is the capturing rate, $\frac{a_2}{a_1}$ is the conversion rate of the predator, m is the half capturing saturation constant, d is the death rate of predator , p is harvesting reward per unit harvesting effort for unit weight prey, c is the cost per unit harvest effort for prey and r is the economic profit; the delay $\tau \geqslant 0$ represents a gestation period of the prey.

This paper is organized as follows. In Section 2, the local stability of the interior equilibrium point for system (2) is discussed and the existence of Hopf bifurcation at the interior equilibrium is established by analyzing the corresponding characteristic equations. In Section 3, numerical simulations are given to verify the theoretical analysis. Finally, a conclusion is given.

*Corresponding author. *E-mail address*: denglianwang@163.com

IJNS.2014.10.15/832

2 Local stability and Hopf bifurcation

In this section, we will employ the local parameterization method of the differential-algebraic system [8] and Hopf bifurcation Theorem [9] to investigate the local stability of the interior equilibrium point and the existence of Hopf bifurcation for system (2).

Firstly, let $g(r,Y)=E(t)(px(t)-c)-r$, where $Y=(x,y,E)^T$. From system (2), we can see that there exists an equilibrium point $Y_0=(x_0,y_0,E_0)^T$ if and only if the equations

$$\begin{aligned}&x(t)\left(r_1-ax(t-\tau)\right)-\tfrac{a_1x(t)y(t)}{my(t)+x(t)}-E(t)x(t)=0,\\&\tfrac{a_2x(t)y(t)}{my(t)+x(t)}-dy(t)=0,\\&E(t)(px(t)-c)-r=0,\end{aligned}\tag{3}$$

hold. Then it is easy to obtain that the system (2) has an equilibrium point $Y_0=(x_0,\frac{a_2-d}{md}x_0,\frac{r}{px_0-c})^T$, where x_0 satisfies the equation

$$k_1x^2+k_2x+k_3=0\tag{4}$$

with $k_1=mpaa_2,k_2=-mcaa_2-p[mr_1a_2-a_1(a_2-d)],k_3=c[mr_1a_2-a_1(a_2-d)]+mra_2$.

From the viewpoint of biology, this paper only concentrates on the interior equilibrium point of the system (2), since the biological significances of the interior equilibrium point indicates that the prey, the predator, and the harvest effort on prey all exist, that is, $x_0>0$,$y_0>0$ and $E_0>0$, which are relevant to our study. Therefore, throughout the paper, we assume that the condition (H_0) $a_2>d,px_0>c,mr_1a_2>a_1(a_2-d),k_2^2>4k_1k_3$ holds.

In order to investigate the local stability of the interior equilibrium point for the system (2), we first use the linear transformation $Y(t)=QN(t)$, where $N(t)=(u(t),v(t),\bar{E}(t))^T$, $Q=\begin{pmatrix}1&0&0\\0&1&0\\-\frac{pE_0}{px_0-c}&0&1\end{pmatrix}$. Then we have $D_Yg(Y_0)Q=(0,0,px_0-c),u(t)=x(t),v(t)=y(t),\bar{E}(t)=\frac{pE_0}{px_0-c}x(t)+E(t)$, for which system (2) yields

$$\begin{aligned}&\tfrac{du(t)}{dt}=u(t)\left(r_1-au(t-\tau)\right)-\tfrac{a_1u(t)v(t)}{mv(t)+u(t)}-\bar{E}(t)u(t)+\tfrac{pE_0u^2(t)}{px_0-c},\\&\tfrac{dv(t)}{dt}=\tfrac{a_2u(t)v(t)}{mv(t)+u(t)}-dv(t),\\&0=(\bar{E}(t)-\tfrac{pE_0u(t)}{px_0-c})(pu(t)-c)-r.\end{aligned}\tag{5}$$

Obviously, system (5) exists an interior equilibrium point $N_0=(u_0,v_0,\bar{E_0})^T$, where $u_0=x_0,v_0=y_0,\bar{E_0}=\frac{pE_0x_0}{px_0-c}+E_0$. Now we derive the formula for determining the properties of the interior equilibrium point of the system (5). Firstly, according to the literature [8], we consider the local parametric of the third equation of system (5), which is defined as follows:

$$N(t)=\tilde{\psi}(Z(t))=N_0+U_0Z(t)+V_0h(Z(t)),g(\tilde{\psi}(Z(t)))=0,\tag{6}$$

where $Z(t)=(y_1(t),y_2(t))^T,h(Z(t))=(h_1(y_1(t),y_2(t)),h_2(y_1(t),y_2(t)),h_3(y_1(t),y_2(t))):R^2\to R$ is a smooth mapping, $U_0=\begin{pmatrix}1&0&0\\0&1&0\end{pmatrix}^T,V_0=\begin{pmatrix}0&0&1\end{pmatrix}^T$. From (6) it is easy to know that $u(t)=u_0+y_1(t),v(t)=v_0+y_2(t),\bar{E}(t)=\bar{E}_0+h_3(y_1(t),y_2(t))$. Thus, we can obtain the parametric system of the system (5) as follows:

$$\begin{aligned}\tfrac{dy_1(t)}{dt}&=(u_0+y_1(t))\left(r_1-a(u_0+y_1(t-\tau))\right)-\tfrac{a_1(u_0+y_1(t))(v_0+y_2(t))}{m(v_0+y_2(t))+(u_0+y_1(t))}\\&\quad-(\bar{E}_0+h_3(y_1(t),y_2(t)))(u_0+y_1(t))+\tfrac{pE_0(u_0+y_1(t))^2}{px_0-c},\\\tfrac{dy_2(t)}{dt}&=\tfrac{a_2(u_0+y_1(t))(v_0+y_2(t))}{m(v_0+y_2(t))+(u_0+y_1(t))}-d(v_0+y_2(t)),\end{aligned}\tag{7}$$

Noticing that $g(\tilde{\psi}(Z(t)))=0$, so we can get the linearized system of parametric system (7) at $(0,0)$ as follows:

$$\begin{aligned}\tfrac{dy_1(t)}{dt}&=a_{11}y_1(t)+a_{12}y_2(t)+b_{11}y_1(t-\tau),\\\tfrac{dy_2(t)}{dt}&=a_{21}y_1(t)+a_{22}y_2(t),\end{aligned}\tag{8}$$

where $a_{11}=\frac{a_1u_0v_0}{(mv_0+u_0)^2}+\frac{pE_0u_0}{pu_0-c},a_{12}=-\frac{a_1u_0^2}{(mv_0+u_0)^2},a_{21}=\frac{ma_2v_0^2}{(mv_0+u_0)^2},a_{22}=-\frac{ma_2u_0v_0}{(mv_0+u_0)^2},b_{11}=-au_0$. Therefore, the corresponding characteristic equation of system (8) is

$$\lambda^2+m_1\lambda+m_0+(n_1\lambda+n_0)e^{-\lambda\tau}=0,\tag{9}$$

where $m_1=-(a_{11}+a_{22}),m_0=a_{11}a_{22}-a_{12}a_{21},n_1=-b_{11},n_0=a_{22}b_{11}$. Then, we have the following Theorem 1 on stability of the interior equilibrium point Y_0 for system (2) when $\tau=0$.

Theorem 1 *For the system (5), if (H_1) holds, then the interior equilibrium point Y_0 of system (2) with $\tau = 0$ is locally asymptotically stable; if (H_2) holds, then the interior equilibrium point Y_0 of system (2) with $\tau = 0$ is unstable.*

Proof. When $\tau = 0$, the equation (9) becomes

$$\lambda^2 + (m_1 + n_1)\lambda + m_0 + n_0 = 0. \tag{10}$$

A set of necessary and sufficient conditions that two roots of (10) have negative real part is given in the following form:(H_1) $m_1 + n_1 > 0,\ m_0 + n_0 > 0$, that is, $0 < r < \min\{\frac{a(pu_0-c)^2}{p}, \frac{(pu_0-c)^2}{p}[a + \frac{(ma_2-a_1)v_0}{(mv_0+u_0)^2}]\}$.i.e., the interior equilibrium point Y_0 of system (2) with $\tau = 0$ is locally asymptotically stable if (H_1) holds. Similarly, it is obvious that the conditions (H_2) $m_1 + n_1 < 0$ *or* $m_1 + n_1 > 0$ *and* $m_0 + n_0 < 0$ holds, the roots of the equation (10) have always positive real parts. i.e., the interior equilibrium point Y_0 of system (2) with $\tau = 0$ is unstable if (H_2) holds. ■

From Theorem 1, we know that all roots of the equation (9) have negative real parts when $\tau = 0$ if (H_1) holds. In the next, we want to investigate if the real part of some root increases to reach zero and eventually becomes positive as τ varies under the condition (H_1), namely, whether the stability of the interior equilibrium point Y_0 switches when $\tau > 0$.

Let $iw(w > 0)$ be a root of (9), separating real and imaginary parts, we get

$$\begin{aligned} n_1 w \sin w\tau + n_0 \cos w\tau &= w^2 - m_0, \\ n_1 w \cos w\tau - n_0 \sin w\tau &= -m_1 w. \end{aligned} \tag{11}$$

Squaring and adding the two equations of (11), which follow that

$$w^4 + (m_1^2 - n_1^2 - 2m_0)w^2 + m_0^2 - n_0^2 = 0. \tag{12}$$

It is easy to know that if the condition (H_3) $m_1^2 - 2m_0 - n_1^2 > 0,\ m_0^2 - n_0^2 > 0$ holds, then (12) has no positive roots. Therefore, Eq. (9) does not have purely imaginary roots. Since the condition (H_1) ensures that all roots of Eq. (10) have negative real parts, by Rouche's theorem [10], we know that the sum of orders of the zeros of $P(\lambda, e^{-\lambda\tau}) = \lambda^2 + m_1\lambda + m_0 + (n_1\lambda + n_0)e^{-\lambda\tau}$ in the open right half plane can change, and only a zero appears on or crosses the imaginary axis. It follows that the roots of Eq. (9) have negative real roots too. This can be summarized as follows:

Theorem 2 *For the system (5), if the conditions (H_1) and (H_3) hold, then all the roots of the equation (9) have negative real roots for all $\tau \geqslant 0$, namely, the interior equilibrium point Y_0 of system (2) is locally asymptotically stable when $\tau \geqslant 0$.*

On the other hand, if (H_1) and (H_4) $m_0^2 - n_0^2 < 0$ hold, then (12) has a unique positive root w_0^2 . Substituting w_0^2 into (10), the corresponding critical value of time delay τ_n is

$$\tau_n = \frac{1}{w_0} \arccos\left\{\frac{(n_0 - n_1 m_1)w_0^2 - n_0 m_0}{n_1^2 w_0^2 + n_0^2}\right\} + \frac{2n\pi}{w_0}, n = 0, 1, 2, \ldots, \tag{13}$$

Again, if the conditions (H_5) hold: $m_1^2 - n_1^2 - 2m_0 < 0,\ m_0^2 - n_0^2 > 0$ and $(m_1^2 - n_1^2 - 2m_0)^2 > 4(m_0^2 - n_0^2)$ hold, then (12) has two positive roots $w_\pm^2$. Substituting $w_\pm^2$ into (11), the corresponding critical value of time delay $\tau_k^\pm$ is

$$\tau_k^\pm = \frac{1}{w_\pm} \arccos\left\{\frac{(n_0 - n_1 m_1)w_\pm^2 - n_0 m_0}{n_1^2 w_\pm^2 + n_0^2}\right\} + \frac{2k\pi}{w_\pm}, n = 0, 1, 2, \ldots, \tag{14}$$

Differentiating (9) with respect to τ, and noticing that λ is a function of τ, we can obtain

$$\left(\frac{d\lambda}{d\tau}\right)^{-1} = -\frac{2\lambda + m_1}{\lambda(\lambda^2 + m_1\lambda + m_0)} + \frac{n_1}{\lambda(n_1\lambda + n_0)} - \frac{\tau}{\lambda}. \tag{15}$$

Hence, a direct calculation shows that

$$sign\left\{\left(\frac{d(Re\lambda)}{d\tau}\right)\right\}_{\lambda=iw} = sign\left\{Re\left(\frac{d\lambda}{d\tau}\right)^{-1}\right\}_{\lambda=iw} = sign\left\{\frac{2w^2 + m_1^2 - n_1^2 - 2m_0}{m_1^2 w^2 + (m_0 - w^2)^2}\right\}. \tag{16}$$

According to the analysis above, we have the following results on stability of the interior equilibrium point Y_0 for system (2) when $\tau > 0$.

Theorem 3 *For the system (5), if (H_1) and (H_4) hold, then there exists a positive number τ_0 such that the interior equilibrium point Y_0 of system (2) is locally asymptotically stable when $\tau \in [0, \tau_0)$ and unstable when $\tau > \tau_0$. Furthermore, system (2) undergoes a Hopf bifurcation at Y_0 when $\tau = \tau_n, n = 0, 1, 2, \ldots$*

Proof. For $\tau = 0$, Y_0 is locally asymptotically stable if (H_1) holds. Hence, by Butler's lemma [11], we know that Y_0 remains stable for $\tau < \tau_0$. We now have to show that

$$\left(\frac{d(Re\lambda)}{d\tau}\right)_{\tau=\tau_0,\lambda=iw_0} > 0. \tag{17}$$

This will signify that there exists at least one eigenvalue with positive real part for $\tau > \tau_0$. From (16) and the condition (H_4), it follows that

$$sign\left\{\left(\frac{d(Re\lambda)}{d\tau}\right)\right\}_{\tau=\tau_0,w=iw_0} = sign\left\{\frac{\sqrt{(m_1^2-n_1^2-2m_0)^2-4(m_0^2-n_0^2)}}{m_1^2w_0^2+(m_0-w_0^2)^2}\right\} > 0. \tag{18}$$

Therefore, the transversability condition holds. According to the Hopf bifurcation Theorem [9], Hopf bifurcation occurs at $w = w_0$,$\tau = \tau_0$ and yields the periodic solution. ■

Theorem 4 *Let $\tau_k^{\pm}$ be defined in (14). For the system (5), if (H_1) and (H_5) hold, then there exists a positive integer j such that there are j switches from stability to instability and to stability. In other words, when $\tau \in [0, \tau_0^+), (\tau_0^-, \tau_1^+), \cdots, (\tau_{j-1}^-, \tau_j^+)$, the interior equilibrium point Y_0 of system (2) is locally asymptotically stable, and when $\tau \in [\tau_0^+, \tau_0^-), (\tau_1^+, \tau_1^-), \cdots, (\tau_j^+, \tau_j^-)$, Y_0 is unstable. Therefore, there are bifurcations at the interior equilibrium point Y_0 of system (2) when $\tau = \tau_j^{\pm}$,$k = 0, 1, 2, \ldots, j$.*

Proof. If conditions (H_1) and (H_5) hold, then to prove the theorem we only need to verify the transversablity conditions. From (16) and the condition (H_5), we can obtain

$$\left(\frac{d(Re\lambda)}{d\tau}\right)_{\tau=\tau_k^+,w=w_+} > 0, \left(\frac{d(Re\lambda)}{d\tau}\right)_{\tau=\tau_k^-,w=w_-} < 0. \tag{19}$$

Hence, the transversablity conditions hold. ■

Remark *Differential-algebraic system (2) is transformed into the equivalent two-dimensional differential system (7) based on the local parameterization method of differential-algebraic system. Therefore, the sufficient conditions for the local stability the interior equilibrium point and the existence of Hopf bifurcation of system (2) are obtained by studying the same properties of system (7).*

3 Numerical simulations

In this section, we will give numerical simulations for system (2) to verify the theoretical analysis in the previous sections. Let $r_1 = a_1 = 3$, $a = c = 1/2$, $a_2 = 2$, $r = 0.74$, $m = p = d = 1$. Then we consider the following specific example of system (2):

$$\begin{array}{l} \frac{dx(t)}{dt} = x(t)(3 - \frac{1}{2}x(t-\tau_1)) - \frac{3x(t)y(t)}{y(t)+x(t)} - E(t)x(t) \\ \frac{dy(t)}{dt} = \frac{2x(t)y(t)}{y(t)+x(t)} - y(t) \\ 0 = E(t)(x(t) - \frac{1}{2}) - 0.74. \end{array} \tag{20}$$

It is easy to verify that the condition (H_0) holds, then we can get the interior equilibrium $Y_0(2.0372, 2.0372, 0.4815)$; and the condition (H_1) holds, which indicates that the interior equilibrium (Y_0) of system (20) is locally asymptotically stable when $\tau = 0$, as is shown in Fig.1.

For $\tau > 0$, by calculation, it is easy to verify that the condition (H_4) holds and we can obtain $w_0 = 0.4975$ and $\tau_0 = 0.2430$. From Theorem 3, we know that, the interior equilibrium Y_0 is locally asymptotically stable when $\tau \in [0, \tau_0)$, as is illustrated in Fig.2. Once the time delay τ passes through the critical value τ_0, the interior equilibrium point Y_0 will lose its stability and a Hopf bifurcation occurs around the interior equilibrium point Y_0. The corresponding waveform and the phase trajectory is drawn as shown in Fig.3.

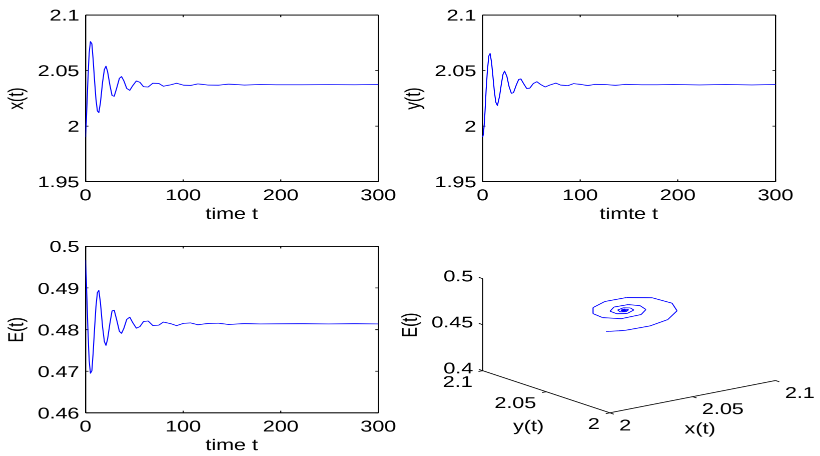

Figure 1: Numerical simulation shows that Y_0 is locally asymptotically stable for $\tau = 0$.

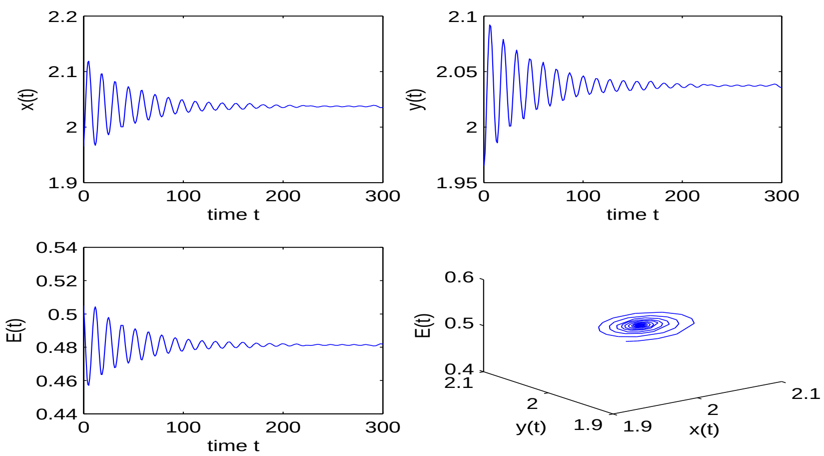

Figure 2: Numerical simulation shows that Y_0 is locally asymptotically stable for $\tau = 0.16 < \tau_0$.

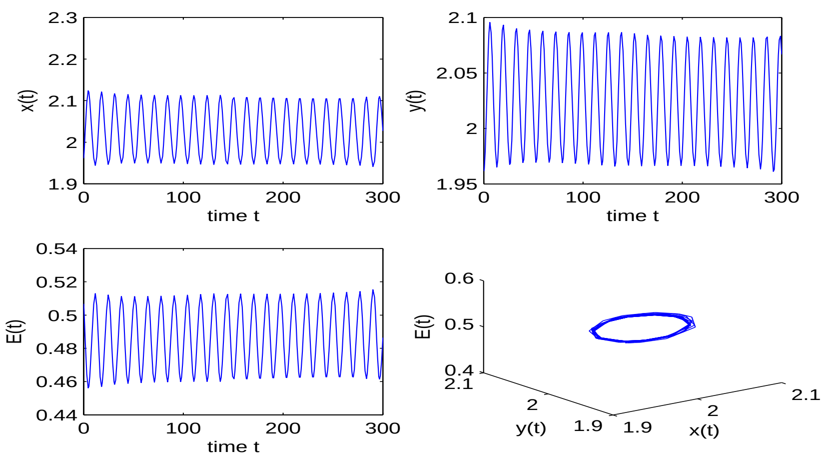

Figure 3: Numerical simulation shows that Hopf bifurcation occurs from Y_0 for $\tau_0 = 0.2430$.

4 Conclusions

In this paper, a differential-algebraic biological economic system which the model incorporate prey harvesting and time delay into a ratio-dependent predator-prey system is investigated. We conclude that, comparing with the system without time delay, the interior equilibrium point of system with time delay will lose its stability gradually as time delay increases and a periodic solution bifurcates from the equilibrium when time delay cross through the critical value. Finally, numerical simulations are given to verify the theoretical analysis. From these waveforms and the phase trajectories above, it is shown that these results are in accord with the theoretical analysis.

From both biologic and economic perspectives, the sustainable development of the predator-prey system will be very important. So with the purpose of maintaining the sustainable development of the biological resources in practice and application, we can predict the dynamical behavior of system by choosing parameters, especially the economic profit and time delay.

Acknowledgments

This work was supported by the Students' Research Foundation of Jiangsu University (Y13A130).

References

[1] C. S. Holling. The functional response of predator to prey density and its role in mimicry and population regulation. *Mem. Ent. Sec. Can.*, 45(1965):1-60.

[2] R. Arditi, L. R. Ginzburg. Coupling in predator-prey dynamics: ratio-dependence. *J. Theor. Biol.*, 139(1989):311-326.

[3] R. Xu, Q. T. Gan, Z. Ma. Stability and bifurcation analysis on a ratio-dependent predatorłprey model with time delay. *J. Comput. Appl. Math.*, 230(2009):187-203.

[4] C. Y. Ji, D. Q. Jiang, X. Y. Li. Qualitative analysis of a stochastic ratio-dependent predator-prey system. *J. Comput. Appl. Math.*, 235(2011):1326-1341.

[5] L. W. Deng, X. D. Wang, M. Peng. Hopf bifurcation analysis for a ratio-dependent predator-prey system with two delays and stage structure for the predator. *Appl. Math. Comput.*, 231(2014):214-230.

[6] H. S. Gordon. The economic theory of a common property resource: the Fishery. *J. Polit. Econ.*, 62(2)(1954):124-142.

[7] T. K. Kar, U. K. Pahari. Non-selective harvesting in prey-predator models with delay. *Commun. Nonlinear S-ci.Numer. Simulat.*, 11(2006):499-509.

[8] B. S. Chen, X. X. Liao, Y. Q. Liu. Normal forms and bifurcations for the differential-algebraic biological economic system. *Appl. Math. Comput.*, 217(1)(2010):330-338(in Chinese).

[9] B. D. Hassard, N. D. Kazarinoff, Y. H. Wan. Theory and Application of Hopf Bifurcation. Cambridge University Press, Cambrideg. 1981.

[10] S. G. Ruan, J. J. Wei. On the zeros of transcendental functionals with applications to stability of delay differential equations with two delays. *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.*, 10(2003):863-874.

[11] H. I. Freedman, V. Sree Hari Rao. The trade-off between mutual interference and time lags in predator-prey systems. *Bull. Math. Biol*, 45(1983):991-1003.