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Korteweg-de Vries Hierarchy from Benjamin-Bona-Mahoney Equation by Multiple Scale Method

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Abstract: We perform a multiple-time scales analysis and compatibility condition to the family of Benjamin-Bona-Mahoney (BBM) equation. We derive Korteweg-de Vries (KdV) flow equation in the bi-Hamiltonian form as an amplitude equation.

Keywords: BBM equation; multiple-time scales method; KdV equation

1 Introduction

We consider the family of Benjamin-Bona-Mahoney (BBM) equation (same accuracy as KdV) in its general form:

$$u_t + \alpha u_x + \beta u u_x - \gamma u_{xxt} + \delta u_{xxx} = 0, \tag{1}$$

where $\alpha, \beta, \gamma, \delta$ are constants specified later and $x \in R$, $t > 0$, u denotes the free surface elevation above the still water level $u = 0$. Benjamin-Bona-Mahoney (BBM) equation (1) incorporates nonlinear and dispersive effects and has been suggested as a model for surface water waves in a uniform channel with flat bottom, cf [1, 2].

The Benjamin-Bona-Mahoney (BBM) equation (1) has been studied thoroughly in the past and the Cauchy problem is known to be well-posed in appropriate Sobolev spaces, at least locally in time. Also the well-posedness of some initial-boundary value problems, including the initial-periodic boundary value problem, can be proved, e.g [1–3].

One may easily check that equation (1) admits exact solitary wave solutions of the form:

$$u(x,t) = 3\frac{c_s - \alpha}{\beta} \sec h\left(\frac{1}{2}\sqrt{\frac{c_s-\alpha}{\gamma c_s + \delta}}\,(x - c_s t)\right), \tag{2}$$

that travel rightwards with a given speed c_s. Further it is well known that (1) possesses two quantities invariant under its evolution dynamics. Assuming either the solution has compact support or $u \to 0$ as $x \to \pm\infty$, one can easily check that quantities

$$I_1(t) = \int_R u(x,t)dx, \qquad I_2(t) = \int_R (u^2(x,t) + \gamma u_x^2(x,t))dx, \tag{3}$$

are conserved in time, i.e. $I_1(t) = I_1(0)$, $I_2(t) = I_2(0)$, $\forall t > 0$. The invariant I_1 reflects the physical property of the mass conservation, while the invariant I_2 can be assimilated to the generalized kinetic energy. Invariants conservation is a fundamental property important not only for theoretical investigations but also for numerics since it allows to validate numerical schemes and to quantify the accuracy of the obtained results.

In mathematics, the Korteweg–de Vries equation (KdV equation for short) (4) is a mathematical model of waves on shallow water surfaces. It is particularly notable as the prototypical example of an exactly solvable model, that is, a non-linear partial differential equation whose solutions can be exactly and precisely specified. The solutions in turn include prototypical examples of solitons. KdV equation (4) can be solved by means of the inverse scattering transform. The mathematical theory behind the KdV equation (4) is rich and interesting, and, in the broad sense, is a topic of active mathematical research.

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$$u_t + u_{xxx} - 6uu_x = 0. \tag{4}$$

The first two terms of this expansion coincide exactly with the complete linear dispersion of the KdV equation (4). For this reason, for sufficiently long wavelengths, the traveling-wave solutions of Eqs. (1) and (4) are expected to be quite similar. Inspite of this, there is deep difference between these two cases, since a polynomial is definitely not equivalent to infinite series. This difference, appearing when higher-order terms of the dispersion relation expansion are considered, might also show up at higher-order approximation in a perturbation theory [4]. As it is well known, the KdV equation (4) governs the first relevant order of an asymptotic perturbation expansion, which describes weakly nonlinear dispersive waves. However, to make sense of it as really governing such waves, the large time behavior of the perturbative series must be analyzed [5]. It has been shown to have solitary wave solutions and to govern a large number of important physical phenomena such as shallow water waves and plasma waves [6].

In section 2, we present some materials, recursion operators and multiple-time formalısm for KdV equation (4). In section 3, we first give a multiple scales method and then we apply the method to the Benjamin-Bona-Mahoney (BBM) equation with derivation of the KdV flow equations.

Throughout the paper, we make extensive use of Reduce to calculate and simplify our results.

2 Recursion operators and KdV flow Eqs.

A function $v = P[u]$ is called a generalised symmetry of a given evolution equation if it is solves the linearised equation,

$$v_t = K'[u]v, \tag{5}$$

here $P[u]$ is a function of u and its spatial derivatives, and the differential operator $K'[u]$ is the Frẻchet derivative of $K[u]$ which is defined by

$$K'[u]v = \frac{d}{d\varepsilon}K[u + \varepsilon v] \mid_{\varepsilon=0.} \tag{6}$$

The evolution of a differential function of u, $K[u]$, whit respect to any independent variable τ is given by

$$(K[u])_\tau = K'[u]u_\tau. \tag{7}$$

Therefore asking that $v = P[u]$ be a solution of (5) is just the requirement

$$(P[u])_t = (K[u])_t, \tag{8}$$

that the flows (5) and

$$u_\tau = P[u], \tag{9}$$

commute. Note that u is now being thought of as a function of x and both flow times t and τ [7, 8].

The KdV equation (4) has the hereditary recursion operator [9, 10]

$$R[u] = \partial^2 + 4u + 2u_x\partial^{-1}. \tag{10}$$

This maps u_x onto the KdV equation (4) itself. The infinite hierarchy of mutually commuting flows satisfies the relation:

$$u_{t_{2n+1}} = R^n[u]u_x = K_{2n+1}[u], \quad n = 0, 1, \cdots . \tag{11}$$

KdV equation (4) for $n = 1$,

$$u_t = u_{xxxxx} - 10uu_{xxx} - 20u_xu_{xx} + 30u^2u_x. \tag{12}$$

fifth order KdV equation for $n = 2$,

$$\begin{aligned} u_t = &-u_{xxxxxxx} + 14uu_{xxxxx} + 42u_xu_{xxxx} + 70u_{xx}u_{xxx} \\ &-280uu_xu_{xx} - 70u^2u_{xxx} - 70u_x^3 + 14u^3u_x. \end{aligned} \tag{13}$$

seventh order KdV equation for $n = 3$ respectively.

2.1 Multiple-time scales formalism

In this section we are going to apply a multiple-time scales version [11] of the reductive perturbation method to study long waves as governed by the Benjamin-Bona-Mahoney (BBM) equation (1). To study the long-wavelength limit of the Benjamin-Bona-Mahoney (BBM) equation (1), we put

$$k = \varepsilon K, \tag{14}$$

with ε a small parameter. In this limit, the dispersion relation can be expanded as

$$w(k) = \varepsilon K - \varepsilon^3 K^3 + \varepsilon^5 K^5 - \varepsilon^7 K^7 + \cdots . \tag{15}$$

Accordingly, the solution of the corresponding linear Benjamin-Bona-Mahoney (BBM) equation (1) can be written in the form

$$u = \alpha \exp\{i[kx - w(k)t]\} \equiv \alpha \exp\{i[K\varepsilon(x - t) + \varepsilon^3 K^3 t - \varepsilon^5 K^5 t + \varepsilon^7 K^7 t + \cdots]\}, \tag{16}$$

where α is a constant. As given by this solution, we now define a slow space

$$\xi = \varepsilon(x - t), \tag{17}$$

as well as an infinity of properly normalized slow time variables:

$$\tau_3 = \varepsilon^3 t, \;\; \tau_5 = \varepsilon^5 t, \;\; \tau_7 = \varepsilon^7 t, \;\; \text{etc.} \tag{18}$$

Consequently, we have

$$\frac{\partial}{\partial x} = \varepsilon \frac{\partial}{\partial \xi}, \tag{19}$$

and

$$\frac{\partial}{\partial t} = -\varepsilon \frac{\partial}{\partial \xi} + \varepsilon^3 \frac{\partial}{\partial t^3} - \varepsilon^5 \frac{\partial}{\partial t^5} + \varepsilon^7 \frac{\partial}{\partial t^7} - \cdots . \tag{20}$$

It is important to note that the introduction of slow time variables normalized according to the dispersion relation expansion are such that they allow for an automatic elimination of the solitary-wave-related secular-procuding terms appearing in the evolution equations for the higher-order terms of the wave field [4].

3 The Multiple-time scales method

Following Zakharov and Kuznetsov [11, 12] we use a multiple-time scales method to derive the KdV flow equations (4) from the Benjamin-Bona-Mahoney (BBM) equation (1).

We now consider Benjamin-Bona-Mahoney (BBM) equation (1) and we assume the following series expansions for solution:

$$u(x,t) = \sum_{n=0}^{\infty} \varepsilon^{n+1} u_n(\xi_0, \xi_1, \ldots, \xi_n, \tau_0, \tau_1, \ldots, \tau_n), \quad n = 0, 2, 4, \cdots \tag{21}$$

In order to study the long-wave limit of Benjamin-Bona-Mahoney (BBM) equation (1), we will introduce slow space and multiple-time variables based on the long-wave limit of the linear dispersion relation

$$w(k) = \frac{k}{1 + k^2} = k \sum_{n=0}^{\infty} (-1)^n k^{2n}, \tag{22}$$

where k is the wave number. Based on this solution (21), we also define now the slow space and multiple-time variables with respect to the scaling parameter $\varepsilon > 0$ respectively as follows:

$$\xi = \varepsilon(x - t), \quad \tau_3 = \alpha_1 \varepsilon^3 t, \;\; \tau_5 = -\alpha_2 \varepsilon^5 t, \;\; \tau_7 = \alpha_3 \varepsilon^7 t, \;\; \cdots \tag{23}$$

Taking $\alpha = 1$, $\beta = -6$, $\gamma = -1$, and $\delta = 2$ equation (1) and introducing the expansions (19) and (20) into the field equation (1) and setting the coefficients of like powers of ε equal to zero, we obtain the following sets of differential equations:

(i) For the coefficients of ε^2 , we find

$$\alpha_1 u_{0\tau_3} + u_{0\xi\xi\xi} - 6u_0 u_{0\xi} = 0. \tag{24}$$

(ii) For the coefficient of ε^4 , we find

$$u_{2\xi\xi\xi} - 6u_0 u_{2\xi} + \alpha_1 u_{2\tau_3} - \alpha_1 u_{0\xi\xi\tau_3} - 6u_2 u_{0\xi} - \alpha_2 u_{0\tau_5} = 0. \tag{25}$$

(iii) For the coefficients of ε^6 , we find

$$\begin{aligned} u_{4\xi\xi\xi} - 6u_0 u_{4\xi} + \alpha_1 u_{4\tau_3} - 2\alpha_1 u_{2\xi\xi\tau_3} - 6u_2 u_{2\xi} & \\ -\alpha_2 u_{2\tau_5} - 2\alpha_2 u_{0\xi\xi\tau_5} - 6u_4 u_{0\xi} - \alpha_3 u_{0\tau_7} &= 0. \end{aligned} \tag{26}$$

(iv) For the coefficients of ε^8 , we find

$$\begin{aligned} u_{6\xi\xi\xi} - 6u_0 u_{6\xi} + \alpha_1 u_{6\tau_3} - 2\alpha_1 u_{4\xi\xi\tau_3} - 6u_2 u_{4\xi} - \alpha_2 u_{4\tau_5} & \\ +2\alpha_2 u_{2\xi\xi\tau_5} - 6u_4 u_{2\xi} - \alpha_3 u_{2\tau_7} + 2\alpha_3 u_{0\xi\xi\tau_7} - 6u_6 u_{0\xi} - \alpha_4 u_{0\tau_9} &= 0 \end{aligned} \tag{27}$$

$$\vdots$$

and so on.

3.1 The derivation of KdV flow equations

We now use (24) and take $\alpha_1 = 1$, we obtain

$$u_{0\tau_3} + u_{0\xi\xi\xi} - 6u_0 u_{0\xi} = 0, \tag{28}$$

the well known KdV equation (4) with $u = u_0$. It is known that taking in the equation (28) for travelling wave solution in the form $u_0(x,t) = \phi(x+ct)$:

$$u_0(x,t) = 2a^2 \sec h^2 a(x + 4a^2 t + \theta), \tag{29}$$

where θ is the phase [13]. We take u_0 to be the solitary-wave solution of the KdV equation (28),

$$u_0 = -2k^2 \sec h^2 (k\xi - 4k^3\tau_3). \tag{30}$$

where chosing the phase $\theta_3 = k(\xi - 4k^2\tau_3)$.

We now take in equation (25) $\alpha_2 = 1$, we obtain the equation

$$u_{0\tau_5} = u_{2\xi\xi\xi} - 6u_0 u_{2\xi} + u_{2\tau_3} - u_{0\xi\xi\tau_3} - 6u_2 u_{0\xi}, \tag{31}$$

and in this equation chosing $u_2 = 4k^2 u_0$,

$$\theta_5 = k(\xi - 4k^2\tau_3 + 16k^4\tau_5),$$

the solitary wave solution

$$u_0 = -2k^2 \sec h^2 (k\xi - 4k^3\tau_3 + 16k^5\tau_5)$$

and use to imposing the natural (in the multiple-time formalism) compatibility condition

$$(u_{0\tau_3})_{\tau_5} = (u_{0\tau_5})_{\tau_3},$$

we obtain the evolution of u_0 in the time τ_5 derive the fifth-order KdV equation with $u = u_0$

$$u_{0\tau_5} = u_{0\xi\xi\xi\xi\xi} - 10u_0 u_{0\xi\xi\xi} - 20u_{0\xi} u_{0\xi\xi} + 30u_0^2 u_{0\xi}. \tag{32}$$

We take in equation (26) $\alpha_3 = 1$, we obtain the equation

$$u_{0\tau_7} = u_{4\xi\xi\xi} - 6u_0 u_{4\xi} + u_{4\tau_3} - u_{2\xi\xi\tau_3} - 6u_2 u_{2\xi} - u_{2\tau_5} - u_{0\xi\xi\tau_5} - 6u_4 u_{0\xi} \tag{33}$$

and in this equation chosing $u_4 = (4k^2u_0)^2$,

$$\theta_7 = k(\xi - 4k^2\tau_3 + 16k^4\tau_5 - 64k^6\tau_7),$$

the solitary wave solution

$$u_0 = -2k^2 \sec h^2(k\xi - 4k^3\tau_3 + 16k^5\tau_5 - 64k^6\tau_7)$$

and using to compatibility condition

$$(u_{0\tau_3})_{\tau_7} = (u_{0\tau_7})_{\tau_3},$$

we obtain the evolution of u_0 in the time τ_7 derive the seventh-order KdV equation

$$\begin{aligned} u_{0\tau_7} = -u_{0\xi\xi\xi\xi\xi\xi\xi} + 14u_0u_{0\xi\xi\xi\xi\xi} + 42u_{0\xi}u_{0\xi\xi\xi\xi} + 70u_{0\xi\xi}u_{0\xi\xi\xi} \\ -280u_0u_{0\xi}u_{0\xi\xi} - 70u_0^2u_{0\xi\xi\xi} - 70u_{0\xi}^3 + 14u_0^3u_{0\xi}. \end{aligned}$$

Extending this procedure, we obtain all equations of the KdV hierarchy.

4 Conclusions

We have used a multiple-time scales method to provide a new derivation of the KdV flow equations from the Benjamin-Bona-Mahoney (BBM) equation and then represent their solitary wave solution. The equations for the coefficients at each order in epsilon, contain no secular terms in our derivation of KdV flow equations (4). Therefore no freedom is left in choosing coefficients at each order in epsilon and the expansion is uniquely determined. Thus there exists a relation between the Benjamin-Bona-Mahoney (BBM) equation (1) with the KdV flow equations (4).

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