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Direct Algebraic Method for finding Complex Solutions of Huxley Equation

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Abstract: In this work, an efficient numerical method for the complex solutions of nonlinear partial differential equations based on the direct algebraic approach is proposed, and tested in the case of generalized Burgers–Huxley equation

$$\frac{\partial u}{\partial t}+\alpha u^{\delta}\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=\beta u(1-u^{\delta})(u^{\delta}-\gamma)$$

When$\alpha=0,\delta=1$, is reduced to the Huxley equation. The proposed scheme can be used in a wide class of nonlinear reaction–diffusion equations. These calculations demonstrate that the accuracy of the direct algebraic solutions is quite high even in the case of a small number of grid points. The present method is a very reliable, simple, small computation costs, flexible, and convenient alternative method.

Keywords: Huxley equation, Direct Algebraic Method; Generalized Burgers–Huxley equation; Complex solutions

1 Introduction

In this paper, the direct algebraic method applied to the nonlinear diffusion equation

$$\frac{\partial u}{\partial t}+\alpha u^{\delta}\frac{\partial u}{\partial x}-\frac{\partial^2 u}{\partial x^2}=\beta u(1-u^{\delta})(u^{\delta}-\gamma) \tag{1}$$

where α,β,γ and δ are parameters, $\beta\geq 0,\gamma\succ 0,\gamma\in(0,1)$. Eq. (1) is a generalized Burgers–Huxley equation. When$\alpha=0,\delta=1$, Eq. (1) is reduced to the Huxley equation which describes nerve pulse propagation in nerve fibers and wall motion in liquid crystals [1,2]. When $\beta=0,\delta=1$, Eq. (1) is reduced to the Burgers equation describing the far field of wave propagation in nonlinear dissipative systems [3]. When$\alpha=0,\beta=1,\delta=1$, Eq. (1) becomes the FitzHugh–Nagumo (FN) equation which is a reaction–diffusion equation used in circuit theory, biology and the area of population genetics [4]. And $\delta=1,\alpha\neq 0,\beta\neq 0$, Eq. (1) is turned into the Burgers–Huxley equation. This equation, which shows a prototype model for describing the interaction between reaction mechanisms, convection effects and diffusion transport, was investigated by Satsuma [5]. Various numerical techniques were used in the literature to obtain numerical solutions of the Burgers–Huxley equation. Wang et al. [6] studied the solitary wave solution of the generalized Burgers–Huxley equation while Estevez [7] presented nonclassical symmetries and the singular modified Burgers and Burgers- Huxley equation. Also Estevez and Gordoa [8] applied a complete Painleve test to the generalized Burgers–Huxley equation. In the past few years, various mathematical methods such as spectral methods [9-11], Adomian decomposition method [12–13], homotopy analysis method [14], the tanh-coth method [15], variational iteration method [16-18], Hopf-Cole transformation [19] and polynomial differential quadrature method [20] have been used to solve the equation.
Travelling wave solutions usually can be characterized as solutions invariant with respect to translation in space, and determine the behavior of the solutions of the Cauchy-type problems. From the physical point of view, travelling waves usually describe transition processes. In Mathematical Biology, there is a very famous branch called the Reaction Diffusion system. The Burgers-Huxley equation is one kind of partial differential equations in the Reaction Diffusion systems.

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2 An Analysis of the Method

For a given partial differential equation

$$G(u, u_x, u_t, u_{xx}, u_{tt}, ...), \tag{2}$$

Our method mainly consists of four steps:

Step 1: We seek complex solutions of Eq. (2) as the following form:

$$u = u(\xi), \qquad \xi = ik(x - ct), \tag{3}$$

where k and c are real constants. Under the transformation (3), Eq. (2) becomes an ordinary differential equation

$$N(u, iku', -ikcu', -k^2u'', ...), \tag{4}$$

where $u' = \frac{du}{d\xi}$.

Step 2: We assume that the solution of Eq. (4) is of the form

$$u(\xi) = \sum_{i=0}^{n} a_i F^i(\xi), \tag{5}$$

where $a_i(i = 1, 2, .., n)$ are real constants to be determined later. $F(\xi)$ expresses the solution of the auxiliary ordinary differential equation

$$F'(\xi) = b + F^2(\xi), \tag{6}$$

Eq. (6) admits the following solutions:

$$\begin{aligned} &F(\xi) = \begin{cases} -\sqrt{-b}\tanh(\sqrt{-b}\xi), & b \prec 0 \\ -\sqrt{-b}\coth(\sqrt{-b}\xi), & b \prec 0 \end{cases} \\ &F(\xi) = \begin{cases} \sqrt{b}\tan(\sqrt{b}\xi), & b \succ 0 \\ -\sqrt{b}\cot(\sqrt{b}\xi), & b \succ 0 \end{cases} \\ &F(\xi) = -\frac{1}{\xi}, \qquad b = 0 \end{aligned} \tag{7}$$

Integer n in (5) can be determined by considering homogeneous balance [3] between the nonlinear terms and the highest derivatives of $u(\xi)$in Eq. (4).

Step 3: Substituting (5) into (4) with (6), then the left hand side of Eq. (4) is converted into a polynomial in$F(\xi)$, equating each coefficient of the polynomial to zero yields a set of algebraic equations for a_i, k, c.

Step 4: Solving the algebraic equations obtained in step 3, and substituting the results into (5), then we obtain the exact traveling wave solutions for Eq. (2).

3 Application to the Huxley equation

In this case we study the generalized Burgers–Huxley equation when $\delta = 1, \alpha = 0, \beta = 1$ so

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = \beta u(1 - u)(u - \gamma) \tag{8}$$

We use the wave transformation$u = u(\xi)$, with wave complex variable$\xi = ik(x - ct)$, where k and c are real constants. System (8) takes the form as

$$k^2u'' - ikcu' + \beta u^3 - \beta(1 + \gamma)u^2 + \beta\gamma u = 0 \tag{9}$$

Considering the homogeneous balance between u'' and u^3 in (9), we required that $3m = m+2 \Rightarrow m = 1$. So the solution takes the form

$$u = a_1F + a_0, \tag{10}$$

Substituting (10) into Eq. (9) yields a set of algebraic equations for $a_1, a_0, k,$ and c. These equations are found as

$$\begin{aligned} &2a_1k^2 + \beta a_1^3 = 0 \\ &-ikca_1 + 3\beta a_1^2 a_0 - \beta(1 + \gamma)a_1^2 = 0 \\ &2a_1bk^2 + 3\beta a_1 a_0^2 - 2\beta(1 + \gamma)a_1 a_0 + \beta a_1\gamma = 0 \\ &-ikca_1b + \beta a_0^3 - \beta(1 + \gamma)a_0^2 + \beta a_0\gamma = 0 \end{aligned} \tag{11}$$

With solving relations (11) by Maple package we obtain

$$a_1 = \pm\sqrt{\frac{2}{\beta}}ik, \tag{12}$$

Case1: For $a_1 = -\sqrt{\frac{2}{\beta}}ik,$, we have

$$\begin{array}{l} a_0 = \frac{1}{6}\frac{\mp\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta} \\ c = \pm\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2} \end{array} \tag{13}$$

From (7),(10) and (12-13), we obtain the complex travelling wave solutions of (8) as follows

$$u_1 = -\sqrt{\frac{2}{\beta}}ik\left[\sqrt{-b}\tanh(\sqrt{-b}ik(x\mp\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2}\ t)\right] + \frac{1}{6}\frac{\mp\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta},$$

where $b \prec 0$and k is an arbitrary real constant. And

$$u_2 = -\sqrt{\frac{2}{\beta}}ik\left[\sqrt{-b}\coth(\sqrt{-b}ik(x\mp\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2}\ t)\right] + \frac{1}{6}\frac{\mp\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta},$$

where $b \prec 0$and k is an arbitrary real constant.

$$u_3 = -\sqrt{\frac{2}{\beta}}ik\left[\sqrt{b}\tan(\sqrt{b}ik(x\mp\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2}\ t)\right] + \frac{1}{6}\frac{\mp\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta},$$

where $b \succ 0$and k is an arbitrary real constant.

$$u_4 = -\sqrt{\frac{2}{\beta}}ik\left[\sqrt{b}\cot(\sqrt{b}ik(x\mp\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2}\ t)\right] + \frac{1}{6}\frac{\mp\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta},$$

where $b \succ 0$and k is an arbitrary real constant. For $b = 0$

$$u_5 = \sqrt{\frac{2}{\beta}}\frac{1}{x\mp\sqrt{2\beta-2\beta\gamma+2\beta\gamma^2 t}} + \frac{1}{6}(\mp\sqrt{2(1-\gamma+2\gamma^2)}+2+2\gamma),$$

Case2: For $a_1 = \sqrt{\frac{2}{\beta}}ik,$, we have

$$\begin{array}{l} a_0 = \frac{1}{6}\frac{\pm\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta} \\ c = \pm\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2} \end{array} \tag{14}$$

From (7),(10) and (14), we obtain the complex travelling wave solutions of (8) as follows

$$u_1 = -\sqrt{\frac{2}{\beta}}ik\left[\sqrt{-b}\tanh(\sqrt{-b}ik(x\mp\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2}\ t)\right] + \frac{1}{6}\frac{\pm\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta},$$

where $b \prec 0$and k is an arbitrary real constant. And

$$u_2 = -\sqrt{\frac{2}{\beta}}ik\left[\sqrt{-b}\coth(\sqrt{-b}ik(x\mp\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2}\ t)\right] + \frac{1}{6}\frac{\pm\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta},$$

where $b \prec 0$and k is an arbitrary real constant.

$$u_3 = -\sqrt{\frac{2}{\beta}}ik\left[\sqrt{b}\tan(\sqrt{b}ik(x\mp\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2}\ t)\right] + \frac{1}{6}\frac{\pm\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta},$$

where $b \succ 0$and k is an arbitrary real constant.

$$u_4 = -\sqrt{\frac{2}{\beta}}ik\left[\sqrt{b}\cot(\sqrt{b}ik(x\mp\sqrt{-12k^2b+2\beta-2\beta\gamma+2\beta\gamma^2}\ t)\right]+\frac{1}{6}\frac{\pm\sqrt{2\beta(-6k^2b+\beta-\beta\gamma+2\beta\gamma^2)}+2\beta+2\beta\gamma}{\beta},$$

where $b \succ 0$and k is an arbitrary real constant. For $b = 0$

$$u_5 = \sqrt{\frac{2}{\beta}}\frac{1}{x\mp\sqrt{2\beta-2\beta\gamma+2\beta\gamma^2}t} + \frac{1}{6}(\pm\sqrt{2(1-\gamma+2\gamma^2)}+2+2\gamma),$$

4 Conclusion

In this paper, direct algebraic approach is proposed for the generalized Burgers–Huxley equation. Complex solutions of the generalized Burgers–Huxley equation, obtained by computer simulation. Applications of this method are very simple, and also it gives the implicit form of the approximate solutions of the problems. These are the main advantages of the method. Hence, the present method is a very reliable, simple, fast, minimal computation costs, flexible, and convenient alternative method.

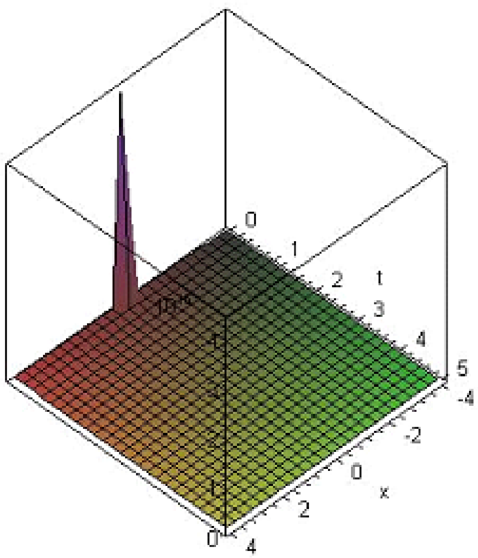

Figure 1: Solutions of Eq. (8), case 2,u_5,at different times ($x = -4..4, t = 0..5$) for$\gamma = 1, k = 1, i = \sqrt{-1}, b = 0$

References

[1] A.C. Scott, Neurophysics, *Wiley, New York*, 1977.
[2] X.Y. Wang, Nerve propagation and wall in liquid crystals, *Phys Lett A*, 112(1985):402-406.
[3] G.B. Whitham, Linear and nonlinear waves, *Wiley, New York*, 1974.
[4] D.G. Aronson, H.F. Weinberger, Multidimensional nonlinear diffusion arising in population genetics, *Adv Math* , 30(1978):33-76.
[5] Satsuma, Exact solutions of Burgers equation with reaction termsTopics in soliton theory and exactly solvable non-linear equations, *World Sci. Pub, Singapore*, (1986):255-262.
[6] X.Y. Wang, Z.S. Zhu, Y.K. Lu, Solitary wave solutions of the generalised Burgers–Huxley equation, *J Phys A Math General*, 23(1990):271-274.

[7] P.G. Estevez, Non-classical symmetries and the singular manifold method: the Burgers and the Burgers– Huxley equations, *J Phys A*, 27(1994): 2113-2127.

[8] P.G. Estevez, P.R. Gordoa, Painleve analysis of the generalized Burgers–Huxley equation, *J Phys A*, 23(1990):4831-4837.

[9] M.T. Darvishi, S. Kheybari, F. Khani, Spectral collocation method and Darvishi's preconditionings to solve the generalized Burgers–Huxley equation,*Commun Nonlinear Sci Numer Simul*, 13(2008):2091-2103.

[10] M. Javidi, A numerical solution of the generalized Burgers–Huxley equation by spectral collocation method, *Appl Math Comput* 178(2)(2006): 338-344.

[11] M. Javidi, A. Golbabai, A new domain decomposition algorithm for generalized Burger's- Huxley equation based on Chebyshev polynomials and preconditioning, *Chaos Solitons Fractals*, 39(2009):849-857.

[12] I. Hashim, M.S.M. Noorani, M.R. Said Al-Hadidi, Solving the generalized Burgers–Huxley equation using the Adomian decomposition method, *Math Comput Model*, 43(11-12)(2006):1404-1411.

[13] I. Hashim, M.S.M. Noorani, B. Batiha, A note on the Adomian decomposition method for the generalize Huxley equation, *Appl Math Comput*, 181(2006):1439-1445.

[14] H.N.A. Ismail, K. Raslan, A.A. Abd Rabboh, Adomian decomposition method for Burger's–Huxley and Burger's–Fisher equations, *Appl Math Comput* , 159(2004):291-301.

[15] A. Molabahramia, F. Khani, The homotopy analysis method to solve the Burgers–Huxley equation, *Nonlinear Anal Real World Appl*, 10(2009):589-600.

[16] A.M. Wazwaz, Analytic study on Burgers, Fisher, Huxley equations and combined forms of these equations, Appl Math Comput 195 (2) (2008): 754-761.

[17] B. Batiha, M.S.M. Noorani, I. Hashim, "Numerical simulation of the generalized Huxley equation by He's variational iteration method", *Appl Math Comput* , 186(2)(2007):1322-1325.

[18] B. Batiha, M.S.M. Noorani, I. Hashim, Application of variational iteration method to the generalized Burgers–Huxley equation, *Chaos Solitons Fractals* , 36(2008):660-663.

[19] O.Yu Efimova, N.A. Kudryashov, Exact solutions of the Burgers–Huxley equation, *J Appl Math Mech* , 68(2004):413-420.

[20] M. Sari, G. Guraslan, "Numerical solutions of the generalized Burgers–Huxley equation by a differential quadrature method. *Hindawi Pub. Corp. Mathematical Problems in Engineering*, 1155(10)(2009)370765.