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Turing Computability of the Solution Operator of the Initial Value Problem for Fifth-order Camassa-Holm Equation

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Abstract: The computability of the solution operator of the Cauchy problem for the Fifth-order Camassa-Holm equation is studied in this paper. Firstly, a nonlinear map $K_R : H^s \to C\left(R; H^s\left(R\right)\right)$ is defined from the initial value φ to the solution u. Then we used the relevant knowledge of type-2 theory of effectivity, functional analysis and Sobolev space to prove that when $s > \left(6\sqrt{10} - 17\right)/4$, the solution operator of the Cauchy problem the Fifth-order Camassa-Holm equation is computable. The conclusion enriches the theories of computability.

Keywords: Fifth-order Camassa-Holm equation; Cauchy problem; Turing Computability; Type-2 theory of effectivity; I-method

1 Introduction

Computability theory is one of the theoretical foundations of computer science, it can be calculated accurately distinguish between what is and what is not computable through the establishment of a mathematical model of the computer (such as abstract computer). Calculation process is the implementation process of the algorithm. One of the most important topics of computability theory is to make this concept accurate. There are many ways to make the concept of algorithms precise, one of which is defined by an abstract computer.The algorithm is regarded as an abstract computer program, we call those functions computable functions whose algorithm exists. The late 90s of last century, Klaus Weihrauch and others improved the original Turing machine model, established the second type of Turing machine, by using the type-II model, they established an effective theory of computability theory and other theories related, K. Weihrauch and N.Zhong studied the computability of generalized functions, KdV equation and Schrödinger equation [1,2]. Among them, KdV equation is a very important equation both in mathematics or practice, it has a huge and far-reaching impact on the development of nonlinear science.

During the study process of the KdV equation, people promoted it with varying degrees, and got its general form:

$$\partial_t u + \partial_x^{2j+1} + f(u, \partial_x u, ..., \partial_x^{2j} u) = 0 \ . \tag{1}$$

e.g. Generalized KdV equation

$$\partial_t u + \partial_x^3 u + \tfrac{1}{k+1}\partial_x u^{k+1} = 0 \ . \tag{2}$$

Where $k \in Z_+$, when $k = 1$ the equation above is called the Classical KdV equation; when $k = 2$,it is called Modified KdV equation.

Many equations in physical models can be attributed to the class broader than (1)(2), just like the situation that nonlinear term f contains non-local items, e.g. Ostrovsky equation and Fifth-order Camassa-Holm equation.This article will study the computability of the solution operator of the Cauchy problem for Fifth-order Camassa-Holm equation.

Fifth-order Camassa-Holm equation forms as follows:

$$\partial_t u - \partial_x^2 \partial_t u + \partial_x^3 u - \partial_x^5 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0. \tag{3}$$

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Equation (3) is the high order correction of the following equation

$$\partial_t u - \partial_x^2 \partial_t u + 2k\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0, \tag{4}$$

where k is a constant associated with the critical velocity in shallow water. Specifically, equation (3) plus fifth-order item $-\frac{2}{3}k\partial_x^5 v$, then do the replacement $v = u - 1$, we can get the equation as follows

$$\partial_t u - \partial_x^2 \partial_t u + (2k-3)\,\partial_x u + \partial_x^3 u - \frac{2}{3}k\partial_x^5 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0. \tag{5}$$

Let $k = \frac{2}{3}$, equation (5) is equivalent to equation (3).

When $k = 0$, equation (3) acted with Bessel Potential operator $\left(1 - \partial_x^2\right)^{-1}$, then the Non-local form goes to

$$\partial_t u + \partial_x^3 u + \frac{1}{2}\partial_x u^2 + \left(1 - \partial_x^2\right)^{-1}\partial_x\left[u^2 + \frac{1}{2}\left(\partial_x u\right)^2\right] = 0. \tag{6}$$

Remove the second term of equation (6) , we can get the Camassa-Holm equation; remove the forth term of equation (6), we can get the KdV equation.

Camassa and Holm obtained the shallow water wave equation4 by using the Hamiltonian method when they were studying the movement of waves in shallow water [3]. Conetantin proved the existence of global weak solutions of Camassa-Holm equation (4) in Sobolev space [4-8]. They also proved that when $s > \frac{2}{3}$, there existed local strong solutions of the Camassa-Holm equation in $H^s(R)$ and $H^s(T)$, where T represents the one-dimensional point in the real axis. Himonas and Misiolek proved when $s < \frac{2}{3}$, the local strong solution of the Cauchy problem of Camassa-Holm equation dose not exist [9]. This is one reason to consider higher order correction for the Camassa-Holm equation, another reason is the study of equation (3) is helpful to the study of equation (4).

2 Preliminaries

In order to state conveniently, we firstly give the following definition.

Definition 1 *For $\forall s, b \in R$, the space $X_{s,b}$ of Bourgain type is defined by the completion of the Schwartz space $S(R^2)$ with respect to the norm: $\|u\|_{X_{s,b}} = \left\|\langle\sigma\rangle^b \langle\xi\rangle^s \hat{u}(\xi,\tau)\right\|_{L_\xi^2 L_\tau^2}$. For $\delta > 0$, define space$X_{s,b}^\delta$ the restriction in the space-time space $T \times [0,\delta]$, given the following norm $\|u\|_{X_{S,B}^\delta} = \inf\left\{\|U\|_{X_{S,B}} : U \in X_{s,b}, U|_{T\times[0,\delta]} = u\right\}$.*

Suppose $s < 1, N \gg 1$ fixed. Define the Fourier multiplier operator

$$I : H^s(T) \to H^1(T),$$

$$f \to If,$$

where $\hat{If}(n) = m(n)\hat{f}(n)$, multiplier $m(\xi)$ is a symmetrical, smooth function of radial decreasing, and it satisfies

$$m(\xi) = \begin{cases} 1, & |\xi| \le N; \\ \left(\frac{|\xi|}{N}\right)^{s-1}, & |\xi| \ge 2N. \end{cases}$$

According to reference [10], we can get the following inequalities:

$$\|u\|_{H^s} \le \|Iu\|_{H^1} \le CN^{1-s}\|u\|_{H^s};$$

$$\|u\|_{X_{s,b}} \le \|Iu\|_{X_{1,b}} \le CN^{1-s}\|u\|_{X_{s,b}}.$$

3 Main result

We are going to study the computability of Cauchy problem for the Fifth-order Camassa-Holm equation given bellow:

$$\begin{cases} \partial_t u - \partial_x^2 \partial_t u + \partial_x^3 u - \partial_x^5 u + 3u\partial_x u - 2\partial_x u \partial_x^2 u - u\partial_x^3 u = 0, x \in R, t > 0, \\ u(x,0) = u_0(x) \in H^s(R). \end{cases} \tag{7}$$

Firstly, a nonlinear map $K_R : H^s \to C(R; H^s(R))$ is defined from the initial value u_0 to the solution u.we want to prove the following result.

Theorem 1 *For $\forall t > 0$, when $s > (6\sqrt{10} - 17)/4$ the nonlinear solution operator $K_R : H^s \to C(R; H^s(R))$ of the initial problem (7) is $(\delta_s, [\rho \to \delta_s])$-computable.*

For the proof of Theorem 1, firstly we transform the differential equation to the integral equation with Fourier transform in the Sobolev space H^s, then proof the existence and uniqueness of the solution through the contraction mapping theory, and ultimately we proof the solution is computable.

To get the equivalent integral form of equation (7), first to introduce the following definition.

Definition 2 *If $3/8 < s < 1, 1/2 < b < \min\{(2+s)/3, s + 1/8\}, 0 < \delta < 1$, for $u_0 \in H^s(R)$, define a ball $\mathbb{B} = \left\{ u \in X^\delta_{s,b} : Iu \in X^\delta_{1,b}, \|Iu\|_{X^\delta_{1,b}} \le 2C \|Iu_0\|_{H^1} \right\}$ in $X^\delta_{s,b}$.*

According to reference [10], we can obtain the equivalent integral form of equation (7)

$$S(u) = W(t)u_0 - \int_0^t W(t-\tau)B(u,u)(\tau)d\tau, \tag{8}$$

where $u \in \mathbb{B}$, $B(u,u) = \frac{1}{2}\partial_x u^2 + (1-\partial_x^2)^{-1}\partial_x \left[u^2 + \frac{1}{2}(\partial_x u)^2\right]$, $\{W(t)\}_{t\in R}$is a solution semi-group of (7), $W(t) = F_x^{-1} e^{itp(\xi)} F_x$, $p(\xi) = (-1)^{j+1}\xi^{2j+1}$ is the phase function, F represents the Fourier transform and F^{-1} represents the inverse Fourier transform.

Lemma 2 *Let $s \in R, b > 1/2$, then*

$$\|W(t)u_0\|_{X^\delta_{s,b}} \le C \|u_0\|_{H^s}, \tag{9}$$

$$\left\| \int_0^t W(t-\tau)u(\tau)d\tau \right\|_{X^\delta_{s,b}} \le C \|u\|_{X^\delta_{s,b-1}}. \tag{10}$$

Proof. Here we only prove inequality (10). For $\bar{u}$ is the extension of u, by [11, 12], we can get

$$\left\| \int_0^t W(t-\tau)u(\tau)d\tau \right\|_{X^\delta_{s,b}} \le \left\| \psi(t) \int_0^t W(t-\tau)\bar{u}(\tau)d\tau \right\|_{X^\delta_{s,b}} \le C \|\bar{u}\|_{X_{s,b-1}} = C \|u\|_{X^\delta_{s,b-1}}.$$

Where $\psi(t)$ is a truncated function for the time localization, $\psi \subseteq C_c^\infty(R)$ is a symmetrical, smooth function of radial decreasing, $\psi \equiv 1$ in $[0,1]$. ■

Lemma 3 *Let $3/8 < s < 1, 1/2 < b < \min\{(2+s)/3, s+1/8\}, 0 < \delta < 1$,then*

$$\left\| I(1-\partial_x^2)^{-1}\partial_x(\partial_x u_1 \partial_x u_2) \right\|_{X^\delta_{1,b}} \le C\left(\delta^{1-b} + \delta^{\alpha-\varepsilon}N^\beta\right) \|Iu_1\|_{X^\delta_{1,b}} \|Iu_2\|_{X^\delta_{1,b}} \tag{11}$$

where $\alpha = \min\{5/4 - b, 3/8 + s - b\}, \beta = \min\{-1/4, 3s - 23/8\}$.

Proof of Lemma 2 refers to references [3].

Lemma 4 *Let $0 \le s < 1, 1/2 < b \le (2+s)/3, 0 < \delta < 1$, then*

$$\|I\partial_x(u_1 u_2)\|_{X^\delta_{s,b-1}} \le C\delta^{1-b} \|Iu_1\|_{X^\delta_{1,b}} \|Iu_2\|_{X^\delta_{1,b}}. \tag{12}$$

Proof of Lemma 3 is similar to Lemma 2.

Then we will prove the map S from (8) is a compressed mapping in $\mathbb{B}$. Actually from Lemma 1-3, for $u \in \mathbb{B}$, we can obtain that

$$\begin{aligned}
\|IS(u)\|_{X^{\delta}_{1,b}} &= \left\|IW(t)u_0 - I\int_0^t W(t-\tau)B(u,u)(\tau)d\tau\right\|_{X^{\delta}_{1,b}} \\
&\le \|IW(t)u_0\|_{X^{\delta}_{1,b}} + \left\|I\int_0^t W(t-\tau)B(u,u)(\tau)d\tau\right\|_{X^{\delta}_{1,b}} \\
&\le C\|Iu_0\|_{H^1} + \left\|I\int_0^t W(t-\tau)\cdot\left(\tfrac{1}{2}\partial_x u^2 + \left(1-\partial_x^2\right)^{-1}\partial_x\left(u^2+\tfrac{1}{2}(\partial_x u)^2\right)\right)d\tau\right\|_{X^{\delta}_{1,b-1}} \\
&\le C\|Iu_0\|_{H^1} + \tfrac{1}{2}\left\|I\int_0^t W(t-\tau)\cdot\partial_x u^2 d\tau\right\|_{X^{\delta}_{1,b}} \\
&\quad + \left\|I\int_0^t W(t-\tau)\left(1-\partial_x^2\right)^{-1}\cdot\partial_x\left(u^2+\tfrac{1}{2}\left(\partial_x u\right)^2\right)d\tau\right\|_{X^{\delta}_{1,b}} \\
&\le C\|Iu_0\|_{H^1} + \tfrac{1}{2}C\left\|I\partial_x u^2\right\|_{X^{\delta}_{s,b-1}} + C\left\|I\left(1-\partial_x^2\right)^{-1}\cdot\partial_x\left(u^2+\tfrac{1}{2}\left(\partial_x u\right)^2\right)\right\|_{X^{\delta}_{s,b-1}} \\
&\le C\|Iu_0\|_{H^1} + \tfrac{1}{2}C\cdot C\cdot\delta^{1-b}\|Iu\|^2_{X^{\delta}_{1,b}} + C\cdot C\cdot\left(\delta^{1-b}+\delta^{\alpha-\varepsilon}N^{\beta}\right)\|Iu\|^2_{X^{\delta}_{1,b}} \\
&\quad + \tfrac{1}{2}C\cdot\left(\delta^{1-b}+\delta^{\alpha-\varepsilon}N^{\beta}\right)\|I\partial_x u\|^2_{X^{\delta}_{1,b}} \\
&\le C\|Iu_0\|_{H^1} + \tfrac{1}{2}C\cdot C\cdot\delta^{1-b}\|Iu\|^2_{X^{\delta}_{1,b}} + C\cdot C\cdot\left(\delta^{1-b}+\delta^{\alpha-\varepsilon}N^{\beta}\right)\|Iu\|^2_{X^{\delta}_{1,b}} \\
&\quad + \tfrac{1}{2}C\cdot C\cdot C\left(\delta^{1-b}+\delta^{\alpha-\beta}N^{\beta}\right)\|Iu\|^2_{X^{\delta}_{1,b}} \\
&\le C\|Iu_0\|_{H^1} + C\left(\delta^{1-b}+\delta^{\alpha-\varepsilon}N^{\beta}\right)\|Iu\|^2_{X^{\delta}_{1,b}} \\
&\le C\|Iu_0\|_{H^1} + C\left(\delta^{1-b}+\delta^{\alpha-\varepsilon}N^{\beta}\right)\left(2C\|Iu_0\|_{H^1}\right)^2,
\end{aligned}$$

where α, β are defined in Lemma 2. By choosing a suitably large N, and let $\delta = \left(8\|Iu_0\|_{H^1}\right)^{-1/1-b}$, we can obtain

$$C\delta^{1-b}\left(2C\|Iu_0\|_{H^1}\right)^2 \le \frac{1}{2}C\|Iu_0\|_{H^1}, \tag{13}$$

$$C\delta^{\alpha-\varepsilon}N^{\beta}\left(2C\|Iu_0\|_{H^1}\right)^2 \le \frac{1}{2}C\|Iu_0\|_{H^1}. \tag{14}$$

So that $\|IS(u)\|_{X^{\delta}_{1,b}} \le 2C\|Iu_0\|_{H^1}$, i.e. $S(\mathbb{B}) \subseteq \mathbb{B}$.

Similarly, for$u, v \in \mathbb{B}$, according to inequalities (13) and (14), we can obtain

$$\begin{aligned}
&\|IS(u)-IS(v)\|_{X^{\delta}_{1,b}} \\
&= \left\|IW(t)u_0 - I\int_0^t W(t-\tau)B(u,u)(\tau)d\tau - IW(t)v_0 - IW(t)v_0 + I\int_0^t W(t-\tau)B(v,v)(\tau)d\tau\right\|_{X^{\delta}_{1,b}} \\
&\le \|IW(t)(u_0-v_0)\|_{X^{\delta}_{1,b}} + \left\|I\int_0^t W(t-\tau)\left(B\left(u,u\right)-B\left(v,v\right)\right)(\tau)d\tau\right\|_{X^{\delta}_{1,b}} \\
&\le C\left\|I\left(u_0-v_0\right)\right\|_{H^1} + C\left(\delta^{1-b}+\delta^{\alpha-\varepsilon}N^{\beta}\right)\|Iu-Iv\|_{X^{\delta}_{1,b}} \\
&\le C\left(\delta^{1-b}+\delta^{\alpha-\varepsilon}N^{\beta}\right)\left(\|Iu\|_{X^{\delta}_{1,b}}+\|Iv\|_{X^{\delta}_{1,b}}\right)\|Iu-Iv\|_{X^{\delta}_{1,b}} \\
&\le \tfrac{1}{2}\|Iu-Iv\|_{X^{\delta}_{1,b}}
\end{aligned}$$

Therefore, map S is a compressed mapping in $\mathbb{B}$. According to the Banach fixed point theorem, the point $u \in X^{\delta}_{s,b}$ is the only fixed point of the initial value problem (7), and it is the only local solution of function (7).

Lemma 5 *(type conversion) Let* (δ_i, X_i) *denotes a space where* $0 \le i \le k$*. Let* $f :\subseteq X_1 \times \cdots \times X_k \to X_0$ *and define* $F\left(x_1, \cdots, x_{k-1}\right)\left(x_k\right) := f\left(x_1, \cdots, x_k\right)$*, then* f *is* $(\delta_1, \cdots, \delta_k, \delta_0)$*-computable iff* F *is* $(\delta_1, \cdots, \delta_{k-1}, [\delta_k \to \delta_0])$*-computable.*

Lemma 6 *The evaluation function* $(f, x) \mapsto f\left(x\right)$ *is* $\left(\left[\delta \to \delta'\right], \delta, \delta'\right)$*-computable.*

Proves of Lemma 4-5 refer to reference .

For $\varphi \in H^s$, $\left(s > (6\sqrt{10} - 17)/4\right)$, we define a solution operator:

$$S(u, \phi, t) = W(t)\phi - \int_0^t W(t-\tau)B(u,u)(\tau)d\tau,$$

where $u \in \mathbb{B}$, $B(u,u) = \frac{1}{2}\partial_x u^2 + (1-\partial_x^2)^{-1}\partial_x\left[u^2 + \frac{1}{2}(\partial_x u)^2\right]$, $\{W(t)\}_{t\in R}$is a solution semi-group of (7).

According to Lemma 3.2 from [1], it is easy to proof the above solution operator is $([\rho \to \delta_s], \delta_s, \rho, \delta_s)$-computable.

Corollary 7 *The function* $\bar{S} : C(R; S(R)) \times S(R) \to C(R; S(R))$,

$$\overline{S}(u, \psi)(t) := S(u, \psi, t)$$

is $([\rho \to \delta_s], \delta_s, [\rho \to \delta_s])$*-computable.*

Proof. By lemma 1 and $S(u, \varphi, t)$ is $([\rho \to \delta_s], \delta_s, \rho, \delta_s)$-computable, we can proof it. ■

Lemma 8 *The function* v *is* $(\delta_s, \gamma_N, [\rho \to \delta_s])$*-computable. Where* $v : S(R) \times N \to C(R; S(R))$,

$$v(\varphi, 0) = \overline{S}(0, \varphi),$$
$$v(\varphi, j+1) = \overline{S}(v(\varphi, j)\varphi).$$

Proof. The function v is defined by primitive recursion from computable functions. By Theorem 3.1.7 from [2], v is $(\delta_s, \gamma_N, [\rho \to \delta_s])$-computable. ■

We define functions $u_n^0, u_n^1, \ldots \in C(R; S(R))$ by

$$u_n^0 := \bar{S}(0, \varphi_n), u_n^{j+1} := \bar{S}\left(u_n^j, \varphi_n\right).$$

Then from corollary 1 and lemma 6, we can obtain $\left\{u_n^j\right\}$ is computable. Through the contraction mapping theory, we know that u_n is the fixed point of the iteration $\bar{S}$, i.e. $\lim\limits_{j\to\infty} u_n^j = u_n$. So we can select appropriate integers n_k and j_k to constitute the sequence $\left\{u_{n_k}^{j_k}\right\}_{k\in N}$, satisfying $\left\|u_{n_k}^{j_k} - u_{n_k}\right\|_{X_{s,b}} \le 2^{-k-1}$. Because appropriate numbers n_k and j_k can be computed from names of k, T and φ, we can get a δ_s-name of $u_{n_k}^{j_k}(t)$ which can be computed by lemma 4 and lemma 5, i.e. $\left\{u_{n_k}^{j_k}\right\}_{k\in N}$ is a computable sequence.

Because $S(R)$ is dense in $H^s(R)$, so there exists $\varphi_n \in S(R)$ such that $\|\varphi_n - \varphi\|_{H^s} \le 2^{-n-2}$ where $\varphi \in H^s(R)$. Then there is an appropriate number n_k such that $\|\varphi_{n_k} - \varphi\|_{H^s} \le 2^{-n_k-2} \le 2^{-k-2}$.

The following part is the proof that $\left\{u_{n_k}^{j_k}\right\}_{k\in N}$ converses fast to u uniformly. Then we can obtain u is computable.

Form $(8)-(11)$, we can get:

$$\begin{aligned}
\|u_{n_k} - u\|_{X_{s,b}} &= \left\|IW(t)(\varphi_{n_k} - \varphi) - I\int_0^t W(t-\tau)\left(B(u_{n_k}, u_{n_k}) - B(u,u)\right)(\tau)d\tau\right\|_{X_{s,b}} \\
&\le \|IW(t)(\varphi_{n_k} - \varphi)\|_{X_{s,b}} + \left\|I\int_0^t W(t-\tau)\left(B(u_{n_k}, u_{n_k}) - B(u,u)\right)(\tau)d\tau\right\|_{X_{s,b}} \\
&\le C_1\|\varphi_{n_k} - \varphi\|_{H^s} + C_2\|B(u_{n_k}, u_{n_k}) - B(u,u)\|_{X^\delta_{s,b-1}} \\
&\le C_1 \cdot 2^{-k-2} + C_2\left(\delta^{1-b} + \delta^{\alpha-\varepsilon}\cdot N^\beta\right)\left(\|u_{n_k}\|_{X^\delta_{1,b}} + \|u\|_{X^\delta_{1,b}}\right)\|u_{n_k} - u\|_{X^\delta_{1,b}} \\
&\le C_1 \cdot 2^{-k-2} + 2C_2\left(\delta^{1-b} + \delta^{\alpha-\varepsilon}\cdot N^\beta\right)\max\left(\|u_{n_k}\|_{X^\delta_{1,b}}, \|u\|_{X^\delta_{1,b}}\right)\|u_{n_k} - u\|_{X^\delta_{1,b}} \\
&\le C_1 \cdot 2^{-k-2} + 2C_2\cdot C_3\left(\delta^{1-b} + \delta^{\alpha-\varepsilon}\right)\cdot\|u_{n_k} - u\|_{X^\delta_{1,b}}.
\end{aligned}$$

Selecting appropriate δ such that $0 < \frac{C_1}{1-2C_2C_3(\delta^{1-b}+\delta^{\alpha-\varepsilon})} \le 2$, we can get $\|u - u_{n_k}\|_{X_{s,b}} \le 2^{-k-1}$.

Then it holds that

$$\left\|u_{n_k}^{j_k} - u\right\|_{X_{s,b}} \le \left\|u_{n_k}^{j_k} - u_{n_k}\right\|_{X_{s,b}} + \|u - u_{n_k}\|_{X_{s,b}} \le 2^{-k-1} + 2^{-k-1} = 2^{-k}.$$

So the sequence $\left\{u_{n_k}^{j_k}\right\}$ converses fast to u uniformly.

Proof of Theorem 1 Since the sequence $\left\{u_{n_k}^{j_k}\right\}_{k\in N}$ is computable and converses fast to u uniformly, if $\delta_S(q_k) = u_{n_k}^{j_k}(t)$, then $\tilde{\delta}_{H^s}\langle q_0, q_1, \ldots\rangle = u(t)$. In other words, $\langle q_0, q_1, \ldots\rangle$ is a $\tilde{\delta}_{H^s}$ -name of $u(t)$.

So the solution of the initial value problem (7) u is computable in $t \in [0, T]$.

Then by the lemma 3.14 from [1] we have

Lemma 9 *The maps*

$$F_+ : (0, \varphi, t) \to u(t), t \in [0, T]$$

and

$$F_- : (0, \varphi, t) \to u(t), t \in [-T, 0]$$

are both $(\rho, \tilde{\delta}_{H^s}, \rho, \tilde{\delta}_{H^s})$*-computable, where* $u(t)$ *is the solution of equation (7) with* $u(x, 0) = \varphi(x)$*.*

Next we compute the solution $u(z \cdot T)$, where $z \in Z$. Define:

$$\begin{aligned} &H_+(\varphi, 0) = H_-(\varphi, 0) = \varphi \\ &H_+(\varphi, n+1) = F_+(n \cdot T, H_+(\varphi, n), (n+1) \cdot T) \\ &H_-(\varphi, n+1) = F_-(-n \cdot T, H_-(\varphi, n), -(n+1) \cdot T). \end{aligned}$$

So both $H_+(\varphi, n) = u(n \cdot T)$ and $H_-(\varphi, n) = u(-n \cdot T)$ are computable because H_+ and H_- are primitive recursions of the computable functions F_+ and F_- respectively.

Finally we show the computability of $u(t)$, let $zT \le t \le (z+1)T$, where $z \in Z$. Then we get by Lemma 7 that $F_+(z \cdot T, u(z \cdot T), t)$ is computable.

Similarly, we can proof the solution of problem (7) is Turing computable when $t < 0$.

When $t = 0$, $u(t) = \varphi$, obviously it is computable.

In conclusion, for $\forall t \in R$, $s > (6\sqrt{10} - 17)/4$, the nonlinear solution operator $K_R : H^s \to C(R; H^s(R))$ of the initial problem (7) is $(\delta_s, [\rho \to \delta_s])$-computable.

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