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Almost Monotonicty Laws of Inelastic Interaction of Nearly Equal Solitons for the gBBM Equation

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Abstract: In this paper,we mainly prove the almost monotonicty laws of inelastic interaction of nearly equal solitons for the gBBM equation,it can be used to prove the stabliity of inelastic interaction of nearly equal solitons for the gBBM equation.

Keywords: Almost monotonicty laws; gBBM Equation

1 Introduction

We consider the gBBM equation,considering the case $\lambda = \frac{1}{2}$,

$$(1 - \frac{1}{2}\partial_x^2)\partial_t u + \partial_x(\partial_x^2 u - u + u^3) = 0, (t, x) \in \ R$$

LeVeque [4] investigated the interaction of nearly equal solitons for the KdV equation.and in [4] investigated the behavior of the explicit 2-soliton U_{c_1,c_2} satisfying

$$\lim_{t\to pm\infty} \|U_{c_1^1,c_2^-}(t,x) - Q_{c_1^-}(\cdot - c_1^- t - y_1^\pm) - Q_{c_2^-}(\cdot - c_2^- t - y_1^\pm)\|_{H^1(R)} = 0,$$

in the astmptotic $\mu_0 = \frac{c_{2-} - c_1^-}{c_1^- + c_2^-} > 0$ small.
Mizumachi [5] studied rigorously the interaction of two solitons of nearly equal speeds for gKdV for $p = 3$ and $p = 4$.
Yvan martel and Frank merle in [2]and [3] have been studied collision problem for gKdV and BBM equation.
Stability and asymptotic stability of N-solitons were studied by Maddocks and Sachs [6] in H^1 by variational techniques and in the energy space H^N by Martel, Merle and Tsai [7].

2 Almost monotonicty laws

First, by calculating,we can easy get the following estimates:
Assuming

$$\forall t \in R, \ \ (|\mu_1(t)| + |\mu_2(t)|)y(t) \le 1,$$

$$|\int \widetilde{R_1}\widetilde{R_2}| \le C(y+1)e^{-y}, \tag{1}$$

$$\|\frac{\partial V}{\partial \mu_j} - \Lambda\widetilde{R_j}\|_{H^1} + \|\frac{\partial V}{\partial y_j} + \partial_x\widetilde{R_j}\|_{L^\infty} + \frac{1}{\sqrt{y}}\|\frac{\partial V}{\partial y_j} + \partial_x\widetilde{R_j}\|_{H^1} \le Ce^{-y} \tag{2}$$

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IJNS.2014.08.15/813

$$\|V-\widetilde{R_1}-\widetilde{R_2}\|_{L^\infty}\le Ce^{-y} \tag{3}$$

$$\|V\partial_x\Phi\|_{L^\infty}\le C(|\mu_1|+|\mu_2|)e^{-2\rho y} \tag{4}$$

$$\|(\Phi-\mu_j)e^{-\frac{1}{2}|x-y_j|}\|_{L^\infty}\le C(|\mu_1|+|\mu_2|)e^{-2\rho y} \tag{5}$$

$$\|\Phi\partial_x V-\sum_{j=1,2}\mu_j\partial_x\widetilde{R_j}\|_{L^\infty}\le C(|\mu_1|+|\mu_2|)e^{-2\rho y}+Ce^{-y} \tag{6}$$

$$\|\partial_t V-\sum_{j=1,2}(\dot{\mu_j}\Lambda\widetilde{R_j}-\dot{y_j}\partial_x\widetilde{R_j})\|_{L^\infty}\le Ce^{-y} \tag{7}$$

Lemma 1 *There exist $\omega_0,\overline{y_0}>0$ and a unique C^1 map $\Gamma=(\mu_1,\mu_2,y_1,y_2):V(\omega_0,y_0)\longrightarrow(0,\infty)^2\times R^2$ such that if $u\in V(\omega_0,y_0)$ for $0<\omega<\omega_0, y_0>\overline{y_0}$ and $\varepsilon(x)=u(x)-V(x;\Gamma)$,then,for $j=1,2$,*

$$\int\varepsilon(t,x)(1-\frac{1}{2}\partial_x^2)Q_{\mu_j}(x-y_j)dx=\int\varepsilon(t,x)(1-\frac{1}{2}\partial_x^2)Q'_{\mu_j}(x-y_j)dx=0, y_1-y_2>y_0-C\omega, \|\varepsilon\|_{H^1}+|\mu_1|+|\mu_2|\le C\omega$$

Lemma 2 *There exists $\omega_0>0, C>0, \overline{y_0}>0$ such that if $u(t)$ is a solution of gBBM on some time interval I satisfying for $0<\omega<\omega_0, y_0>\overline{y_0}$,*

$$\forall t\in I, \inf_{y_1-y_1>y_0}\|u(t)-V(x;(0,0,y_1,y_2))\|_{H^1}\le\omega, \tag{8}$$

Then there exists a unique decomposition $(\Gamma(t),\varepsilon(t)$ of $u(t)$ on I,

$$u(t)=V(x,\Gamma(t))+\varepsilon(t,x), \Gamma(t)=(\mu_1(t),\mu_2(t),y_1(t),y_2(t)) of class C^1, \tag{9}$$

Such that $\forall t\in I, \int\varepsilon(t,x)(1-\frac{1}{2}\partial_x^2)\widetilde{R_j}(t,x)dx=\int\varepsilon(t,x)(1-\frac{1}{2}\partial_x^2)\partial_2\widetilde{R_j}(t,x)dx=0$

$$y(t)=y_1(t)-y_2(t)>y_0-C\omega, \|\varepsilon(t)\|_{H^1}+|\mu_1(t)|+|\mu_2(t)|\le C\omega, \tag{10}$$

$$(1-\frac{1}{2}\partial_x^2)\partial_t u+\partial_x(\partial_x^2\varepsilon-\varepsilon+(V+\varepsilon)^3-V^3)+E(V)+E(t,x)=0 \tag{11}$$

Moreover, assuming

$$\forall t\in I, (|\mu_1(t)|+|\mu_2(t)|)\le 1 \tag{12}$$

$\dot{\Gamma}(t)$satisfies the following estimates

$$|\dot{\mu_j}-M_j|\le C[\|\varepsilon\|_{L^2}^2+ye^{-y}\|\varepsilon\|_{L^2}+\int|E|(\widetilde{R_1}+\widetilde{R_2})], |\mu_j-\dot{y_j}-N_j|\le C[\|\varepsilon\|_{L^2}+\int|E|(\widetilde{R_1}+\widetilde{R_2})]. \tag{13}$$

The constant $0<\rho<\frac{1}{32}$ to be fixed later, set $\varphi(x)=\frac{2}{\pi}\arctan(exp(8\rho x)), so that \lim_{-\infty}\varphi=0, \lim_{+\infty}\varphi=1$,

$$\forall x\in R, \varphi(-x)=1-\varphi(x), \varphi'(x)=\frac{8\rho}{\pi\cosh(8\rho x)}, |\varphi''(x)|\le 8\rho|\varphi'(x)|, |\varphi'''(x)|\le(8\rho)^2|\varphi'(x)|. \tag{14}$$

Lemma 3 *Let be defined by (14). If $a(x),b(x)\in L^2$ are such that $a-\frac{1}{2}\partial_x^2 a=b$ then*

$$(1-(8\rho)^2)\int a\varphi'+\int(\partial_x a)^2\varphi'+\frac{1}{4}\int(\partial_x a)^2\varphi'\le\int b^2\varphi'. \tag{15}$$

The proof of Lemma (1),(2),(3) is omitted.

Theorem 4 *(Almost monotonicty laws). For $\rho > 0$ small enough, and under the assumptions of Lemma (2), let*

$$F_+(t) = \int [(\partial_x\varepsilon)^2 + \varepsilon^2 - \frac{1}{2}((V+\varepsilon)^4 - V^4 - 4V^3\varepsilon)] + \int [\frac{1}{2}(\partial_x\varepsilon)^2 + \varepsilon^2]\Phi(t,x) \tag{16}$$

where $\Phi(t,x) = \mu_1(t)\varphi(x) + \mu_2(t)(1-\varphi(x))$;

$$F_-(t) = \int [(\partial_x\varepsilon)^2 + \varepsilon^2 - \frac{1}{2}((V+\varepsilon)^4 - V^4 - 4V^3\varepsilon)]\Phi_1(t,x) + \int [\frac{1}{2}(\partial_x\varepsilon)^2 + \varepsilon^2]\Phi_2(t,x) \tag{17}$$

where

$$\Phi_1(t,x) = \frac{\varphi(x)}{(1+\mu_1(t))^2} + \frac{1-\varphi(x)}{(1+\mu_2(t))^2}, \Phi_2(t,x) = \frac{\mu_1(t)\varphi(x)}{(1+\mu_1(t))^2} + \frac{\mu_2(t)(1-\varphi(x))}{(1+\mu_2(t))^2}. \tag{18}$$

There exists $C > 0$ such that

$$\|\varepsilon(t)\|_{H^1}^2 \le CF_+(t), \|\varepsilon(t)\|_{H^1}^2 \le CF_-(t). \tag{19}$$

Moreover,
(1) If $t \in I$ is such that

$$\mu_1(t) \ge \mu_2(t) and y_2(t) \le -\frac{1}{4}y(t), y_1(t) \ge \frac{1}{4}y(t), \tag{20}$$

then

$$\frac{d}{dt}F_+(t) \le C\|\varepsilon\|_{L^2}^2[e^{-\frac{3}{4}y} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2})(e^{-2\rho y + \|\varepsilon\|_{L^2}})] + C\|\varepsilon\|_{L^2}\|E\|_{L^2}. \tag{21}$$

(2) If $t \in I$ is such that

$$\mu_2(t) \ge \mu_1(t) and y_2(t) \le -\frac{1}{4}y(t), y_1(t) \ge \frac{1}{4}y(t) \tag{22}$$

then

$$\frac{d}{dt}F_-(t) \le C\|\varepsilon\|_{L^2}^2[e^{-\frac{3}{4}y} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2})(e^{-2\rho y + \|\varepsilon\|_{L^2}})] + C\|\varepsilon\|_{L^2}\|E\|_{L^2}. \tag{23}$$

Proof. Let

$$\begin{aligned}
\circleddash &= \|\varepsilon\|_{L^2}^2[e^{-\frac{3}{4}y} + (|\mu_1| + |\mu_2| + \|\varepsilon\|_{L^2})(e^{-2\rho y + \|\varepsilon\|_{L^2}})] + \|\varepsilon\|_{L^2}\|E\|_{L^2} \\
\frac{1}{2}\frac{d}{dt}F_+(t) &= \int \partial_t\varepsilon(-\partial_x^2\varepsilon + \varepsilon - ((V+\varepsilon)^3 - V^3) + (1 - \frac{1}{2}\partial_x^2)(\varepsilon\Phi) \\
&\quad - \frac{1}{2}\int (\partial_x\partial_t\varepsilon)\varepsilon\partial_x\Phi + \frac{1}{2}\int \partial_t\Phi[\frac{1}{2}(\partial_x\varepsilon)^2 + \varepsilon^2] \\
&\quad - \int \partial_t V((V+\varepsilon)^3 - V^3 - 3V^2\varepsilon) = F_1 + F_2 + F_3 + F_4
\end{aligned}$$

Observe that $\partial_x\Phi = (\mu_1 - \mu_2)\varphi' \ge 0$ in the present situation. First, using the equating of ε(i.e.(11))

$$\begin{aligned}
F_1 = &- \int (-\partial_x^2\varepsilon + \varepsilon - ((V+\varepsilon)^3 - V^3)(\Phi\partial_x\varepsilon + \varepsilon\partial_x\Phi) \\
&- \int E(1 - \frac{1}{2}\partial_x^2)^{-1}(-\partial_x^2\varepsilon + \varepsilon + (1 - \frac{1}{2}\partial_x^2)(\varepsilon\Phi) - ((V+\varepsilon)^3 - V^3) \\
&- \Sigma_{j=1,2}(\dot{\mu_j} - M_j)\int \frac{\partial V}{\partial\mu_j}(-\partial_x^2\varepsilon + \varepsilon + (1 - \frac{1}{2}\partial_x^2)(\varepsilon\Phi) - ((V+\varepsilon)^3 - V^3) \\
&+ \Sigma_{j=1,2}(\mu_j - y_j \dot{-} N_j)\int \frac{\partial V}{\partial y_j}(-\partial_x^2\varepsilon + \varepsilon + (1 - \frac{1}{2}\partial_x^2)(\varepsilon\Phi) - ((V+\varepsilon)^3 - V^3) \\
&= F_{1,1} + F_{1,2} + F_{1,3} + F_{1,4}
\end{aligned}$$

Integrating by parts and using $|\varphi'''| \leq (8\rho)^2 \leq \frac{1}{16}|\varphi'|$,

$$F_{1,1} \leq -\frac{3}{2}\int(\partial_x\varepsilon)^2\partial_x\Phi - \frac{1}{2}(1-(4\rho)^2 - C\|\varepsilon\|_{H^1})\int\varepsilon^2\partial_x\Phi$$
$$-\int\varepsilon^2(\mu_1\partial_x\widetilde{R_1} + \mu_2\partial_x\widetilde{R_2} + C(|\mu_1|+|\mu_2|)\|\varepsilon\|_{L^2}^2 e^{-\rho y} + Ce^{-\frac{3}{4}y}\|\varepsilon\|_{L^2}^2$$

Then, we decompose $F_{1,2}$ as follows

$$F_{1,2} = -\int E(1-\frac{1}{2}\partial_x^2)(-\partial_x^2\varepsilon + \varepsilon + (1-\frac{1}{2}\partial_x^2)(\varepsilon\Phi) - ((V+\varepsilon)^3 - V^3) \leq C\|E\|_{H^1}\|\varepsilon\|_{H^1}$$

For $F_{1,3}$, we use (2),$L\Lambda Q_c = -(1-\frac{1}{2}\partial_x^2)Q_c$,(10) and (5),so that

$$|(1-\frac{1}{2}\partial_x^2)\Lambda\widetilde{R_j}|\Phi - |[(1-\frac{1}{2}\partial_x^2)\Lambda\widetilde{R_j}]\mu_j| \leq Ce^{-\frac{3}{4}|x-y_j(t)|}|\Phi-\mu_j| \leq C(|\mu_1|+|\mu_2|)e^{-\frac{1}{2}|x-y_j(t)|}e^{-2\rho y}$$

and

$$F_{1,3} = \sum_{j=1,2}(\mu_j \dot{-} M_j)\int\varepsilon^2\Lambda\widetilde{R_j} + e^{-2\rho y}(|\mu_1|+|\mu_2|+\|\varepsilon\|_{L^2})O(\|\varepsilon\|_{L^2}). \tag{24}$$

Similarly,

$$F(1,4) = \sum_{j=1,2}(\mu_j - \dot{y_j} - N_j)[\int\varepsilon^2\partial_x\widetilde{R_j} + e^{-2\rho y}(|\mu_1|+|\mu_2|+\|\varepsilon\|_{L^2})O(\|\varepsilon\|_{L^2}). \tag{25}$$

Using (11), we have

$$F_1 \leq -\frac{3}{2}\int(\int\partial_x\varepsilon)^2\partial_x\Phi - \frac{3}{8}\int\varepsilon^2\partial_x\Phi - \int\varepsilon^2(\mu_1\partial_x\widetilde{R_1} + \mu_2\partial_x\widetilde{R_2})$$
$$+\sum_{j=1,2}(\dot{\mu_j} - M_j)\int\varepsilon^2\Lambda\widetilde{R_j} + \sum_{j=1,2}(\mu_j - \dot{y_j} - N_j)\int\varepsilon^2\partial_x\widetilde{R_j} + C\ominus$$

Let z_1, z_2 such that $z_1 - \frac{1}{2}\partial_x^2 z_1 = \varepsilon, z_2 - \frac{1}{2}\partial_x^2 z_2 = ((V+\varepsilon)^3 - V^3)$,Then, using the equation of ε,

$$F_2 = \frac{1}{2}\int\partial_x^2(1-\frac{1}{2}\partial_x^2)^{-1}(\partial_x^2\varepsilon - \varepsilon + (V+\varepsilon)^3 - V^3)\varepsilon\partial_x\Phi + \frac{1}{2}\int\partial_x^2(1-\frac{1}{2}\partial_x^2)^{-1}E\varepsilon\partial_x\Phi$$
$$+\frac{1}{2}\sum_{j=1,2}(\dot{\mu_j} - M_j)\int\partial_x\frac{\partial V}{\partial\mu_j}\varepsilon\partial_x\Phi - \frac{1}{2}\sum_{j=1,2}(\mu_j - \dot{y_j} - N_j)\int\partial_x\frac{\partial V}{\partial y_j}\varepsilon\partial_x\Phi = F_{2,1} + F_{2,2} + F_{2,3} + F_{2,4}$$

by using Cauchy Schwarz inequality and (14). For ρ small enough, using Lemma (3) and (4), we obtain

$$|F_{2,1}| \leq \frac{3}{2}\int(\partial_x)^2\partial_x\Phi + (\frac{1}{4} + 8\rho + C\|\varepsilon\|_{H^1} + Ce^{-y}\int\varepsilon^2\partial_x\Phi + C(|\mu_1|+|\mu_2|)\|\varepsilon\|_{L^2}^2 e^{-2\rho y}$$
$$\leq \frac{3}{2}\int(\partial_x\varepsilon)^2\partial_x\Phi + \frac{3}{8}\int\varepsilon^2\partial_x\Phi + C\ominus$$

The term $F_{2,2}$ is estimated as $F_{1,1}$ and by arguments previously used, we also obtain

$$|F_{2,3}| + |F_{2,4}| \leq C\ominus$$

Thus, we get

$$F_2 \leq \frac{3}{2}\int(\partial_x\varepsilon)^2\partial_x\Phi + \frac{3}{8}\int\varepsilon^2\partial_x\Phi + C\ominus$$

Next,so that by $F_3 = \frac{1}{2}\int(\dot{\mu_1}\varphi + \dot{\mu_2}(1-\varphi))[\frac{1}{2}(\partial_x\varepsilon)^2 + \varepsilon^2]$,$|M_j| \leq Ce^{-y}$ and (3.13)

$$|F_3| \leq C(e^{-y} + \sum_{j=1,2}(\mu_j \dot{-} M_j)\|\varepsilon\|_{H^1} \leq C\ominus. \tag{26}$$

Finally,by (7),and computing we proves

$$F_4 \leq (\varepsilon^3 + 3V^2\varepsilon)(\mu_1\partial_x\widetilde{R_1} + \mu_2\partial_x\widetilde{R_2}) - \sum_{j=1,2}(\dot{\mu_j} - M_j)\int(\varepsilon^3 + 3V^2\varepsilon)\Lambda\widetilde{R_j}$$
$$- \sum_{j=1,2}(\mu_j - \dot{y_j} - N_j)\int(\varepsilon^3 + 3V^2\varepsilon)\partial_x\widetilde{R_j} + C\circleddash$$

Since $\mu_2(t) \geq \mu_1(t)$, we have $\frac{1}{(1+\mu_1(t))^2} \geq \frac{1}{(1+\mu_2(t))^2}$,$\partial_x\Phi_1 \geq 0$ and $\partial_x\Phi_2 \leq 0$.
Note also that by explicit computations, for μ_j small enough:

$$|\partial_x\Phi_2 + \frac{1}{2}\partial_x\Phi_1| \leq C(|\mu_1| + |\mu_2|)\partial_x\Phi_1. \tag{27}$$

$$\frac{1}{2}\frac{d}{dt}F_-(t) = \int\partial_t\varepsilon(-\partial_x^2\varepsilon + \varepsilon - ((V+\varepsilon)^3 - V^3)\Phi_1 - \int\partial_t\partial_x\varepsilon\partial_x\Phi_1 + \int(1 - \frac{1}{2}\partial_x^2)\partial_t\varepsilon\varepsilon\Phi_2$$
$$- \frac{1}{2}\int\partial_x\partial_t\varepsilon\varepsilon\partial_x\Phi_2 + \frac{1}{2}\int[(\partial_x\varepsilon)^2 + \varepsilon^2 - \frac{1}{2}((\varepsilon+V)^4 - V^4 - 4V^3\varepsilon)]\partial_t\Phi_1 + [\lambda(\partial_x\varepsilon)^2 + \varepsilon^2]\partial_t\Phi_2$$
$$- \int\partial_t V((\varepsilon+V)^3 - V^3 - 3V^2\varepsilon)\Phi_1 = G_1 + G_2 + G_3 + G_4 + G_5 + G_6$$
$$F_4 \leq (\varepsilon^3 + 3V^2\varepsilon)(\mu_1\partial_x\widetilde{R_1} + \mu_2\partial_x\widetilde{R_2}) - \sum_{j=1,2}(\dot{\mu_j} - M_j)\int(\varepsilon^3 + 3V^2\varepsilon)\Lambda\widetilde{R_j}$$
$$- \sum_{j=1,2}(\mu_j - \dot{y_j} - N_j)\int(\varepsilon^3 + 3V^2\varepsilon)\partial_x\widetilde{R_j} + C\circleddash$$

Let z_3 be such that $(1 - \frac{1}{2}\partial_x^2)z_3 = \varepsilon - \partial_x^2$ and z_4 be such that $(1 - \frac{1}{2}\partial_x^2)z_4 = -((V+\varepsilon)^3 - V^3)$ so that by (11)

$$\partial_t\varepsilon = \partial_x z_3 + \partial_x z_4 - (1 - \frac{1}{2}\partial_x^2)^{-1}E - \sum_{j=1,2}[(\dot{\mu_j} - M_j)\frac{\partial V}{\partial\mu_j} - (\mu_j - \dot{y_j} - N_j)\frac{\partial V}{\partial y_j}]. \tag{28}$$

Then,

$$G_1 = \int\partial_x(z_3 + z_4)(z_3 + z_4 - \frac{1}{2}(z_3 + z_4))\Phi_1 - \int(1 - \frac{1}{2}\partial_x^2)^{-1}E(-\partial_x^2\varepsilon + \varepsilon - ((V+\varepsilon)^3 - V^3)\Phi_1$$
$$- \sum_{j=1,2}(\dot{\mu_j} - M_j)\int\frac{\partial V}{\partial\mu_j}(-\partial_x^2\varepsilon + \varepsilon - ((V+\varepsilon)^3 - V^3)\Phi_1$$
$$+ \sum_{j=1,2}(\mu_j - \dot{y_j} - N_j)\int\frac{\partial V}{\partial y_j}(-\partial_x^2\varepsilon + \varepsilon - ((V+\varepsilon)^3 - V^3)\Phi_1$$
$$= G_{1,1} + G_{1,2} + G_{1,3} + G_{1,4}$$

$$G_2 = -\int\partial_x(z_3 + z_4)\partial_x\varepsilon\partial_x\Phi_1 + \int(1 - \frac{1}{2}\partial_x^2)E\partial_x\varepsilon\partial_x\Phi_1 + \sum_{j=1,2}(\dot{\mu_j} - M_j)\int\frac{\partial V}{\partial\mu_j}\partial_x\varepsilon\partial_x\Phi_1$$
$$- \sum_{j=1,2}(\mu_j - \dot{y_j} - N_j)\int\frac{\partial V}{\partial y_j}\partial_x\varepsilon\partial_x\Phi_1 = G_{2,1} + G_{2,1} + G_{2,3} + G_{2,4}$$

$$G_3 = \int\partial_x(-\partial_x^2\varepsilon + \varepsilon - ((V+\varepsilon)^3 - V^3))\varepsilon\Phi_2 - \int E\varepsilon\Phi_2 - \sum_{j=1,2}(\dot{\mu_j} - M_j)\int(1 - \frac{1}{2}\partial_x^2)\frac{\partial V}{\partial\mu_j}\varepsilon\Phi_2$$
$$+ \sum_{j=1,2}(\mu_j - y_j \dot{-} N_j)\int(1 - \frac{1}{2}\partial_x^2)\frac{\partial V}{\partial y_j}\varepsilon\Phi_2 = G_{3,1} + G_{3,2} + G_{3,3} + G_{3,4}$$

$$G_4 = -\frac{1}{2}\int\partial_x^2(z_3 + z_4)\varepsilon\partial_x\Phi_2 + \frac{1}{2}\int(1 - \frac{1}{2}\partial_x^2)\partial_x E\varepsilon\partial_x\Phi_2 + \frac{1}{2}\sum_{j=1,2}(\mu_j \dot{-} M_j)\int\partial_x\frac{\partial V}{\partial\mu_j}\varepsilon\partial_x\Phi_2$$
$$- \frac{1}{2}\sum_{j=1,2}(\mu_j - \dot{y_j} - N_j)\int\partial_x\frac{\partial V}{\partial y_j}\varepsilon\partial_x\Phi_2 = G_{4,1} + G_{4,2} + G_{4,3} + G_{4,4}$$

Note that the terms $G_{1,2}$,$G_{2,2}$,$G_{3,2}$ and $G_{4,1}$ are readily controlled by $C\|E\|_{H^1}\|\varepsilon\|_{H^1}$.
Now, we focus on $G_{1,1}+G_{2,1}+G_{3,1}+G_{4,1}$.We denote by G_7 the quadratic parts of $G_{1,1}+G_{2,1}+G_{3,1}+G_{4,1}$.i.e. the terms coming from the liner part of the equation. We have

$$G_7=\int \partial_x z_3(z_3-\frac{1}{2}\partial_x^2 z_3)\Phi_1+\int \partial_x z_3\partial_x\varepsilon\partial_x\Phi_1+\int\partial_x(-\partial_x^2\varepsilon+\varepsilon)\varepsilon\Phi_2-\frac{1}{2}\int\partial_x^2 z_3\varepsilon\partial_x\Phi_2$$

Then, using (14), (27) and Integrating by parts, using (14),(27) and then choosing ρ small enough, we find

$$G_7\leq-\frac{1}{12}\int\varepsilon^2\partial_x\Phi_1-\frac{1}{8}\int(\partial_x\varepsilon)^2\partial_x\Phi_1$$

The nonlinear terms in $G_{1,1}+G_{2,1}+G_{3,1}+G_{4,1}$ which contain $\partial_x\Phi_1$ or $\partial_x\Phi_2$ are treated by perturbation (for ε small and y large) exactly as in the previous case, using the signed rest $-\frac{1}{12}\int\varepsilon^2\partial_x\Phi_1-\frac{1}{8}\int(\partial_x\varepsilon)^2\partial_x\Phi_1$ obtained above.
In $G_{3,1}$ we are left with one quartic $-\int 3(\varepsilon^3+2V\varepsilon^2)\partial_x V\Phi_2$ term, which cannot be controlled by $\ominus$, nor by the rest term above, since Φ_2 does not appear with a derivative. Thus, computing the main order of this term, and estimating the rest by $\ominus$, we obtain

$$G_{1,1}+G_{2,1}+G_{3,1}+G_{4,1}\leq-\sum_{j=1,2}\frac{\mu_j(t)}{(1+\mu_j(t))^2}\int 3(\varepsilon^3+2V\varepsilon^2)\partial_x\widetilde{R_j}+C\ominus. \tag{29}$$

After some computations, similarly as before, we obtain

$$|G_{2,3}|+|G_{2,4}|+|G_{3,3}|+|G_{3,4}|+|G_{4,3}|+|G_{4,4}|\leq C\ominus, \tag{30}$$

$$\begin{aligned} G_{1,3}+G_{1,4}=&\sum_{j=1,2}(\mu_j\dot{-}M_j)v_j\int(\varepsilon^3+2V\varepsilon^2)\Lambda\widetilde{R_j}\\ &-\sum_{j=1,2}(\mu_j-\dot{y_j}-N_j)v_j\int(\varepsilon^3+2V\varepsilon^2)\partial_x\widetilde{R_j}.\end{aligned} \tag{31}$$

The term G_5 is treated exactly as the F_3 so that $|G_5|\leq C\ominus$.Finally, using (7), the term G_6 writes

$$\begin{aligned} G_6=&-\int\partial_t V(\varepsilon^3+2V\varepsilon^2)\Phi_1\\ =&\sum_{j=1,2}\frac{\mu_j(t)}{(1+\mu_j(t))^2}\int(\varepsilon^3+2V\varepsilon^2)\partial_x\widetilde{R_j}-\sum_{j=1,2}(\mu_j\dot{-}M_j)\frac{1}{(1+\mu_j(t))^2}\int(\varepsilon^3+2V\varepsilon^2)\Lambda\widetilde{R_j}\\ &-\sum_{j=1,2}(\mu_j-y_j\dot{-}N_j)\frac{1}{(1+\mu_j(t))^2}\int(\varepsilon^3+2V\varepsilon^2)\partial_x\widetilde{R_j}+O(\ominus).\end{aligned} \tag{32}$$

In conclusion, combing (29)-(32), we finish the proof of the Theorem. ∎

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